Given a triangle *ABC*. To draw an equilateral triangle *DEF* with minimum perimeter on Δ*ABC*.

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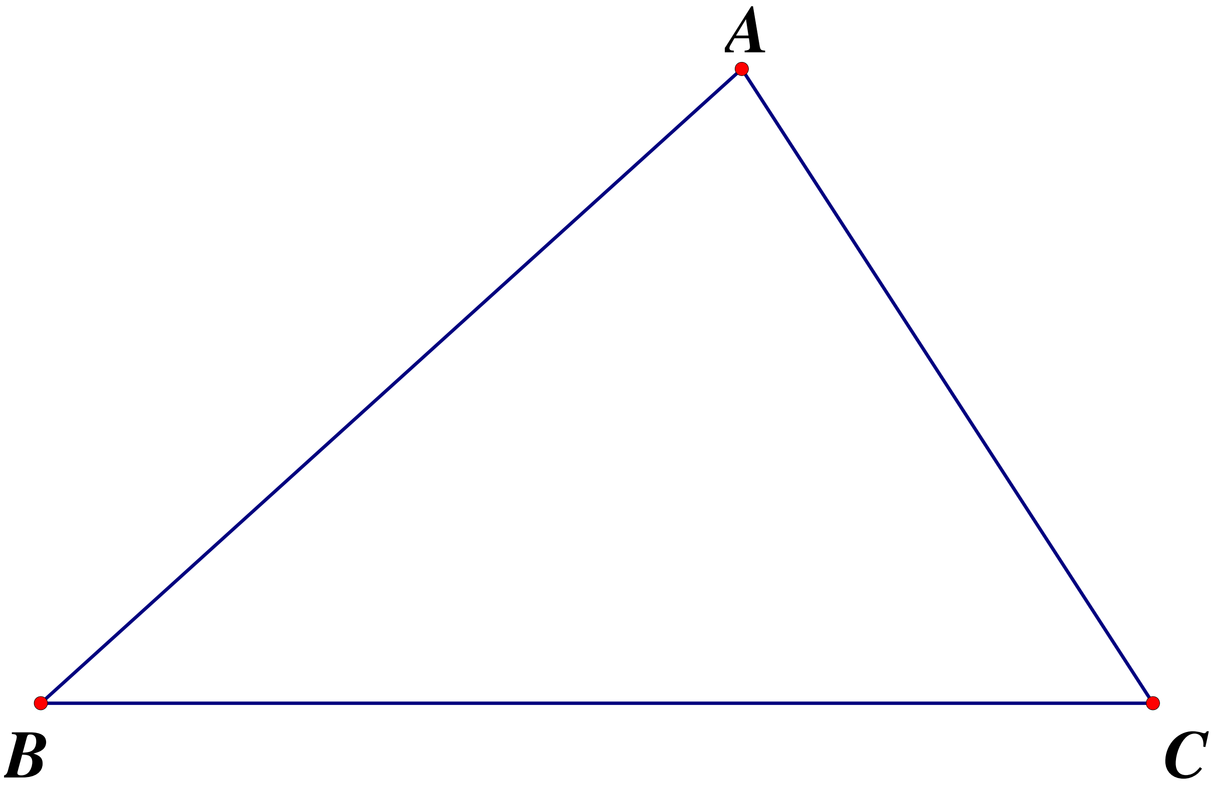


Theorem 1 Given a triangle *ABC*. To draw an equilateral triangle *DEF* on ∆*ABC*.

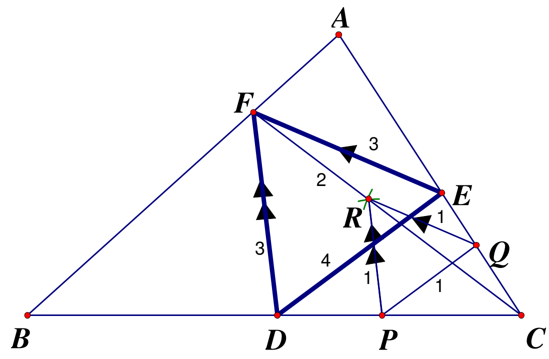
(1) Choose any point *P* on *BC*, any point *Q* on *AC*.
Construct an equilateral ∆*PQR* inside ∆*ABC*.

(2) Join *CR*. Produce *CR* to cut *AB* at *F*.

(3) Draw *EF* // *QR*, cutting *AC* at *E*.
Draw *DF* // *PR*, cutting *BC* at *D*.

(4) Join *DE*.

Then ∆*DEF* is the required equilateral triangle.



Proof: It is easy to prove that ∆*CQR* ~ ∆*CEF* and ∆*CPR* ~ ∆*CDF* (equiangular).

$\frac{EF}{QR}=\frac{CF}{CR}$ …… (1), $\frac{DF}{PR}=\frac{CF}{CR}$ …… (2) (corr. sides, ~∆s)

(1) = (2): $\frac{EF}{QR}=\frac{DF}{PR}$

∵ *QR* = *PR* (sides of an equilateral triangle)

∴ *EF* = *DF*

∠*DFE* = ∠*CFD* + ∠*CFE*

= ∠*CRP* + ∠*CRQ* (corr. ∠s, *EF* // *QR*, corr. ∠s, *DF* // *PR*)

= ∠*PRQ* = 60° (property of equilateral triangle)

∴ ∆*DEF* is an isosceles triangle with ∠*DFE* = 60°

∠*EDF* = ∠*DEF* (base ∠s isos. ∆)

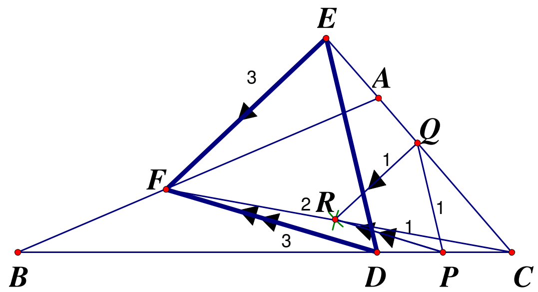
$=\frac{180^\circ-60^\circ}{2}$ (∠s sum of ∆)

= 60°

∴ ∆*DEF* is the required equilateral triangle.

Remark:

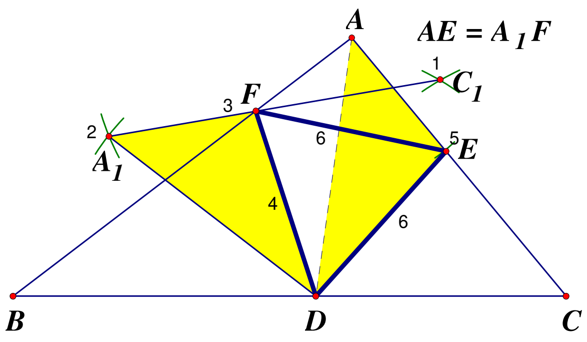
It is possible that the vertices of ∆*DEF* lies outside ∆*ABC*.
The following figure indicates that *E* lies on *CA* produced.



Theorem 2 Given a triangle *ABC*. *D* is a point on *BC*.
To draw an equilateral triangle *DEF* on ΔABC.

(1) Construct an equilateral ΔC_1CD.
(2) Construct an equilateral ΔA_1AD.
(3) Join A_1C_1, cutting *AB* at *F*.
(4) Join *DF*.
(5) Use *A* as centre, A_1F as radius to draw an arc, cutting *AC* at *E*.
(6) Join *EF* and *DE*.

Then ΔDEF is the required equilateral triangle.

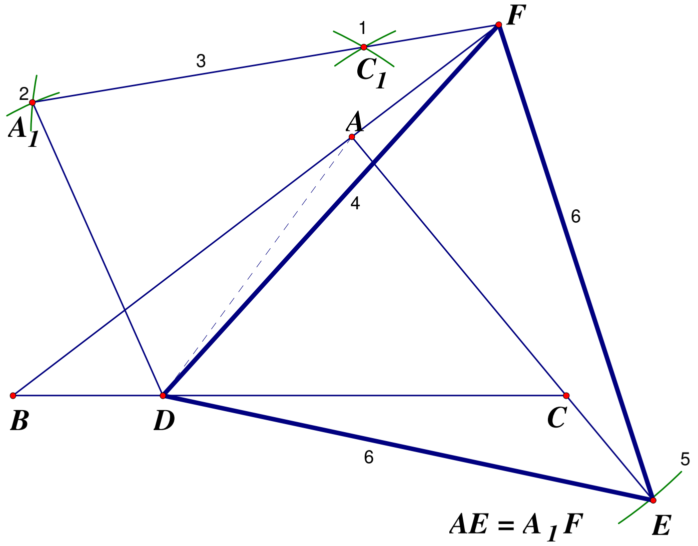


Proof: Let $\angle ADC_1 = \theta$

$\angle ADC = \theta + 60°$ (property of equilateral triangle)
$= \angle A_1DC_1$
$A_1D = AD$, $DC_1 = DC$ (property of equilateral triangle)
$\therefore \Delta ACD \cong \Delta A_1DC_1$ (S.A.S.)
$\angle DA_1F = \angle DAE$ (cor. $\angle$s $\cong \Delta$s)
$DA_1 = DA$ (property of equilateral triangle)
$A_1F = AE$ (by construction step (5))
$\therefore \Delta ADE \cong \Delta A_1DF$ (S.A.S.)
$DF = DE$ (cor. sides $\cong \Delta$s)
$\angle EDF = \angle ADF + \angle ADE$
$= \angle ADF + \angle A_1DF$ (cor. $\angle$s $\cong \Delta$s)
$= \angle ADA_1$
$= 60°$ (property of equilateral triangle)
$\therefore \Delta DEF$ is an isosceles triangle with $\angle EDF = 60°$
$\angle DFE = \angle DEF$ (base $\angle$s isos. Δ)
$= \dfrac{180° - 60°}{2}$ ($\angle$s sum of Δ)
$= 60°$
$\therefore \Delta DEF$ is the required equilateral triangle.

Remark:

It is possible that the vertices of ΔDEF lies outside ΔABC.
The following figure indicates that *E* lies on *AC* produced and *F* lies on *BA* produced.



Theorem 3 Given a triangle *ABC* with $AB > AC$. *AH* is the internal angle bisector of $\angle A$, cutting *BC* at *H*. *AK* is the external angle bisector of $\angle A$, cutting *BC* produced at *K*. Then

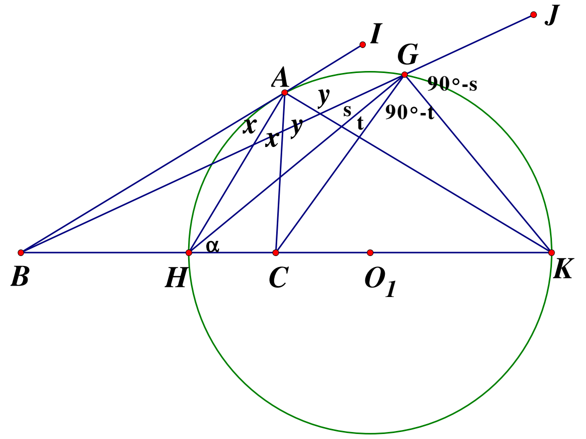
I. $BH : CH = AB : AC = BK : CK$

Let O_1 be the mid-point of *HK*. Use O_1 as centre, O_1H as radius to draw a circle. This circle is called the circle of Apollonius. Then

II. *A* lies on the circle.

Let *G* be any point on the semi-circular arc *HAK*. Then

III. *GH* is the internal angle bisector of $\angle BGC$ and *GK* is the external angle bisector of $\angle BGC$.



Proof: Produce *BA* to *I*. Let $\angle BAH = \angle CAH = x$, $\angle CAK = \angle KAI = y$

By Internal Angle Bisector theorem and External Angle Bisector theorem,

$BH : CH = AB : AC = BK : CK$

I is proved

$x + x + y + y = 180°$ (adj. ∠s on st. line)

$\angle HAK = x + y = 90°$

A lies on the circle (converse, ∠ in semi-circle)

II is proved

$\because HB : HC = KB : KC$ and $\angle HGK = 90°$ (∠ in semi-circle)

Let $\angle GHC = \alpha$, $\angle GHB = 180° - \alpha$, $\angle BGH = s$, $\angle CGH = t$, $\angle CGK = 90° - t$

Produce *BG* to *J*, $\angle KGJ = 90° - s$ (adj. ∠s on st. line)

Apply sine rules on ΔGHB and ΔGHC

$$\frac{BG}{\sin\left(180^\circ - \alpha\right)} = \frac{BH}{\sin s} \quad \cdots (1), \quad \frac{CG}{\sin\alpha} = \frac{CH}{\sin t} \quad \cdots (2)$$

Apply sine rules on ΔGKB and ΔGKC

$$\frac{BG}{\sin\angle CKG} = \frac{BK}{\sin\left(180^\circ - 90^\circ + s\right)} \quad \cdots (3), \quad \frac{CG}{\sin\angle CKG} = \frac{CK}{\sin\left(90^\circ - t\right)} \quad \cdots (4)$$

$$\frac{(1)\times(4)}{(2)\times(3)}: \frac{BG}{\sin\alpha}\times\frac{CG}{\sin\angle CKG}\times\frac{\sin\alpha}{CG}\times\frac{\sin\angle CKG}{BG} = \frac{BH}{\sin s}\times\frac{CK}{\cos t}\times\frac{\sin t}{CH}\times\frac{\cos s}{BK}$$

$$\because \text{By I, } \frac{BH}{CH} = \frac{BK}{CK} \quad \therefore \quad 1 = \frac{\tan t}{\tan s}$$

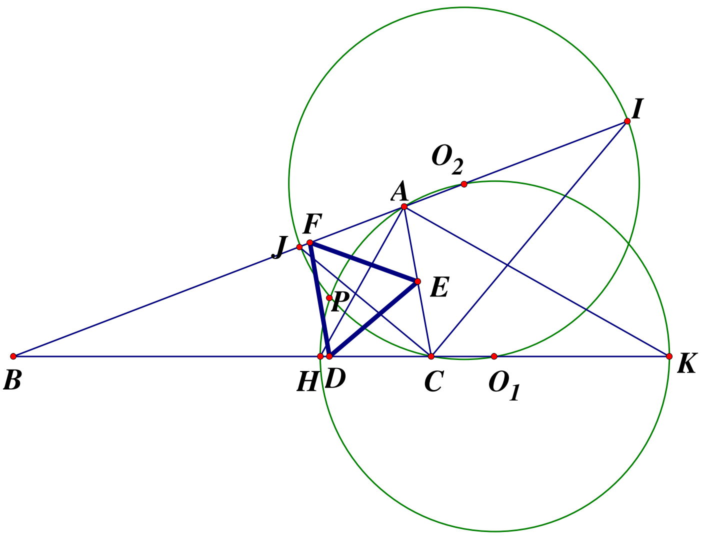
$\tan s = \tan t$

$s = t$

III is proved.

Theorem 4 Use the same notation as in theorem 3.

CJ is the internal angle bisector of $\angle C$, cutting *AB* at *J*. *CI* is the external angle bisector of $\angle C$. Let O_2 be the mid-point of *IJ*. Use O_2 as centre, O_2J as radius to draw another Apollonius circle.
Suppose the two circles intersect at *P* inside ΔABC.
Let the feet of perpendiculars from *P* to *BC*, *CA* and *AB* be *D*, *E* and *F* respectively. Join *DE*, *EF* and *DF*.
Then ΔDEF is the equilateral triangle with minimum perimeter on ΔABC.
First we shall show that ΔDEF is an equilateral triangle



Proof: $\because$ *P* lies on the intersection of two circles.

By theorem 3.1 and 3.3, $\frac{BH}{CH}=\frac{BP}{CP}$ $\cdots\cdots$ (5) and $\frac{BH}{CH}=\frac{AB}{AC}$ $\cdots\cdots$ (6)

(5) = (6): $\frac{BP}{CP}=\frac{AB}{AC}$

$\Rightarrow AC\cdot BP=AB\cdot CP$ $\cdots\cdots$ (7)

$\frac{BJ}{AJ}=\frac{BP}{AP}$ $\cdots\cdots$ (8) and $\frac{BJ}{AJ}=\frac{BC}{AC}$ $\cdots\cdots$ (9)

(8) = (9): $\frac{BP}{AP}=\frac{BC}{AC}$

$\Rightarrow AC\cdot BP=BC\cdot AP$ $\cdots\cdots$ (10)

(7) = (10): $AC\cdot BP=AB\cdot CP=BC\cdot AP$ $\cdots\cdots$ (11)

$\because \angle CDP+\angle CEP=90°+90°$ (by construction)

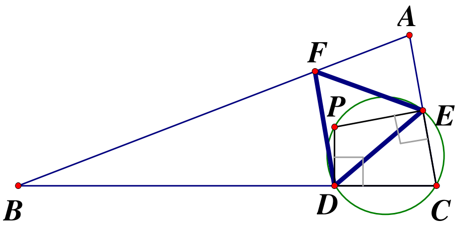
$\therefore$ *CDPE* is a cyclic quadrilateral (opp. $\angle$s supp.) $\cdots\cdots$ (*)

CP is the diameter of the circle *CDE* (converse, $\angle$ in semi-circle)

In ΔCDE, $\frac{DE}{\sin C}=CP$ (sine formula)

$DE=CP\cdot\sin C$ $\cdots\cdots$ (12)

Similarly, $EF=AP\cdot\sin A$, $DF=BP\cdot\sin B$ $\cdots\cdots$ (12)



In ΔABC, $\frac{AB}{\sin C}=\frac{BC}{\sin A}=\frac{CA}{\sin B}=2r$

where r is the circumradius

$\Rightarrow \sin C=\frac{AB}{2r}$, $\sin A=\frac{BC}{2r}$, $\sin B=\frac{CA}{2r}$ $\cdots\cdots$ (13)

Sub. (13) into (12): $DE=\frac{AB\cdot CP}{2r}$, $EF=\frac{BC\cdot AP}{2r}$, $DF=\frac{AC\cdot BP}{2r}$ $\cdots\cdots$ (14)

Sub. (11) into (14), we have $DE=EF=FD$

$\therefore$ ΔDEF is an equilateral triangle.

Next, we are going to show that the length of *DE* is the least.

Let *S* be any point on *BC* other than *D*.

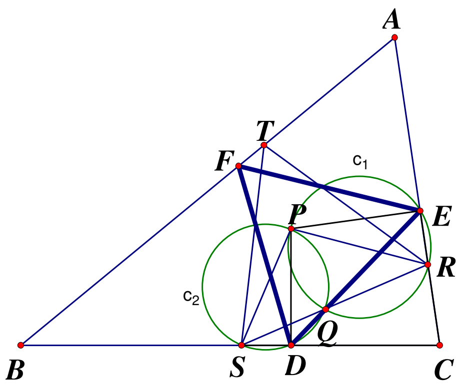
Use theorem 2 to construct an equilateral ΔRST on ΔABC

Join *PE*, *PR*, *PD*, *PS*.

Use *PR* as diameter to draw a circle C_1.

Use *PD* as diameter to draw another circle C_2.

The two circles intersect again at *Q*.



$\angle PER = 90°$ (by construction in theorem 4 on Page 5)

$\angle PDS = 90°$ (by construction in theorem 4 on Page 5)

E lies on C_1 and *D* lies on C_2. (converse, $\angle$ in semi-circle)

$\angle PQR = 90°$, $\angle PQS = 90°$ ($\angle$ in semi-circle)

$\angle PQR + \angle PQS = 180°$

$\therefore RQS$ is a straight line

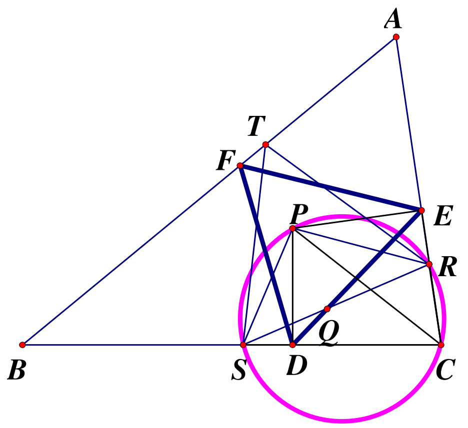
Let $\angle DPS = \alpha$

$\angle DQS = \alpha$ ($\angle$s in the same segment)

$\angle EQR = \alpha$ (vert. opp. $\angle$s)

$\angle EPR = \alpha$ ($\angle$s in the same segment)

$\angle RPS = \alpha + \angle DPR = \angle DPE$ $\cdots\cdots$ (15)



$\because$ *CDPE* is a cyclic quadrilateral (by (*))

$\therefore$ $\angle DPE + \angle DCE = 180°$ (opp. $\angle$s cyclic quad.)

$\angle RPS + \angle DCE = 180°$ (by (15))

$\therefore$ *CRPS* is a cyclic quadrilateral (opp. $\angle$s supp.)

Construct the circumcircle of *CRPS*.

In ΔCSR, $\dfrac{RS}{\sin C} = 2k$, where *k* is the circumradius

Claim: $CP < 2k$

Proof: Otherwise, $CP = 2k$ = diameter of the circumcircle

$\angle PRC = 90°$ ($\angle$ in semi-circle)

$\angle PER + \angle EPR = \angle PRC$ (ext. $\angle$ of Δ)

$90° + \angle EPR = 90°$

which is a contradiction

$\therefore$ $SR = 2k \sin C$ $\cdots\cdots$ (16)

$> CP \sin C$

$= DE$ (by (12))

$\therefore$ The equilateral triangle *DEF* has the minimum perimeter.