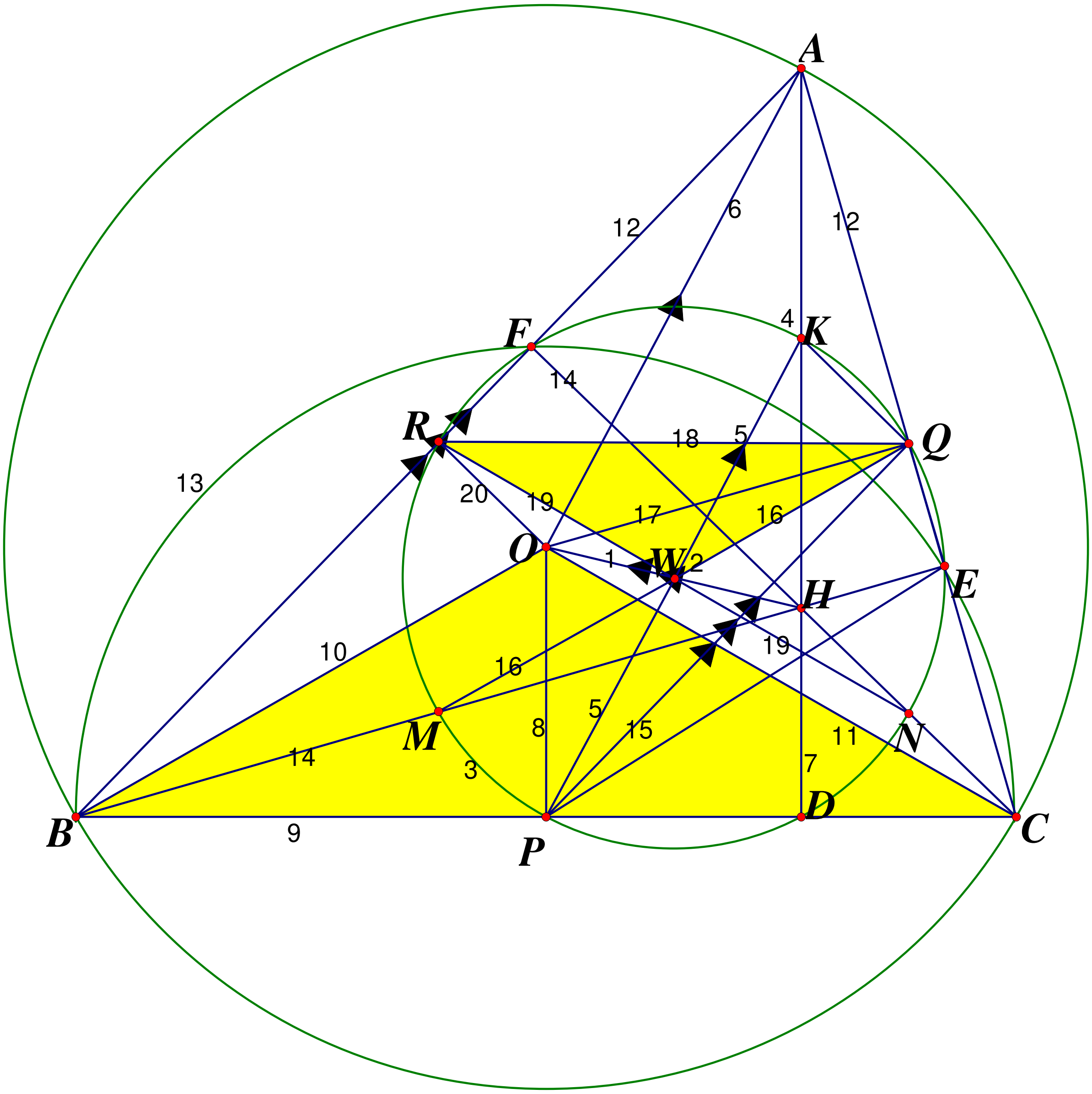
Given a circle centre O, radius r and a point H inside the circle.
To construct a triangle ABC on the circle and H as the orthocentre.

Created by Mr. Francis Hung on 20140901 Last updated: 29/09/2021



(1) Join OH.

(2) Locate the mid-point of OH as W.

(3) Use W as centre, radius $\frac{1}{2}r$ to draw a circle (nine-point circle).

(4) Let K be any point on the nine-point circle.

(5) Join KW and produce it to the other end P of the nine-point circle.

(6) Draw a line segment through O and parallel to WK cutting the circumcircle at A.
(A, K lies on the same side of OH)

$\Delta WHK \sim \Delta OHA$ (S.A.S.)

$\angle WKH = \angle OAH$ (corr. $\angle$s, ~ Δs)

KH // AH (corr. $\angle$s eq.)

A, K, H are collinear. (Playfair's axiom)

(7) Join AH and produce AH to meet the nine-point circle at D.

(8) Join OP.

(9) Extend PD on both sides to cut the circumcircle at B and C.

(10) Join OB.

(11) Join OC.

$\Delta WHK \cong \Delta WOP$ (S.A.S.)
$\angle WHK = \angle WOP$ (corr. $\angle$s, $\cong$ Δs)
AD // OP (alt. $\angle$s eq.)
$\angle KDP = 90°$ ($\angle$ in semi-circle)
$\angle OPD = 90°$ (int. $\angle$s, AD // OP)
$BP = PC$ ($\perp$ from centre bisect chord)
$\therefore$ OP is the $\perp$ bisector of BC and AD is an altitude of ΔABC.
(12) Join AB, AC.
(13) Use P as centre, $PB = PC$ as radius to draw a semi-circle, cutting AB at F and AC at E.
(14) Join BE and CF.
$\angle BEC = 90° = \angle BFC$ ($\angle$ in semi-circle)
$\therefore$ AD, BE and CF are the altitudes of ΔABC $\cdots\cdots$ (*)

$\because$ The 3 altitudes are concurrent at H

$\therefore$ B, H, E and C, H, F are collinear respectively.
(15) Through P draw a line segment parallel to BA, cutting AC at Q.
(16) Join QW and produce it to cut BH at M.
$\because$ $BP = PC$ and BA // PQ $\therefore$ $AQ = QC$ (intercept theorem)
$PQ = \frac{1}{2}AB$ (mid-point theorem)
$\angle QPW = \angle BAO$ (corr. $\angle$s, AB // QP and AO // WP)
$WP = \frac{1}{2}OA$ (by step (3))
$\Delta WPQ \sim \Delta OAB$ (S.A.S.)
$\therefore$ $WQ = \frac{1}{2}OB = \frac{1}{2}r$ (ratio of sides, ~Δs)
$\Rightarrow$ Q lies on the nine-point circle.
$\angle WQP = \angle OBA$ (corr. $\angle$s, $\Delta WQP \sim \Delta OBA$)
$\because$ PQ // BA $\therefore$ BO // MQ (alt. $\angle$s eq.) $\cdots\cdots$ (**)
$\therefore$ $BM = MH$ (intercept theorem)
i.e. M is the mid-point of BH.
Also, $MW = \frac{1}{2}OB = \frac{1}{2}r$ (mid-point theorem)
$\therefore$ M lies on the nine-point circle.
(17) Join OQ.
MQ = diameter of the nine-point circle = OB (by step (3))
$\therefore$ $OBMQ$ is a //-gram (opp. sides are eq. and //)
OQ // BE (property of //-gram)
$\angle OQC = \angle BEC = 90°$ (by (*), corr. $\angle$s, OQ // BE)
$\therefore$ $AQ = QC$ ($\perp$ from centre bisects chord)
$\Rightarrow$ OQ is the $\perp$ bisector of AC.
(18) Through Q draw a line segment parallel to BC, cutting AB at R.
(19) Join RW and produce it to cut CF at N.
$\because$ $AQ = QC$ and BC // RQ $\therefore$ $AR = RB$ (intercept theorem)
$RQ = \frac{1}{2}BC$ (mid-point theorem)
$\angle RQW = \angle OBC$ (alt. $\angle$s, RQ // BC and by (**), BO // WQ)

$WQ = \frac{1}{2} OB$ (by step (3))
$\Delta WQR \sim \Delta OBC$ (S.A.S.)
$\therefore WR = \frac{1}{2} OC = \frac{1}{2} r$ (ratio of sides, ~Δs)
$\Rightarrow R$ lies on the nine-point circle.
$\angle WRQ = \angle OCB$ (corr. ∠s, $\Delta WQR \sim \Delta OBC$)
$\because RQ$ // BC $\therefore OC$ // RN (alt. ∠s eq.)
$\therefore HN = NC$ (intercept theorem)
i.e. N is the mid-point of CH.
Also, $WN = \frac{1}{2} OC = \frac{1}{2} r$ (mid-point theorem)
$\therefore N$ lies on the nine-point circle.
(20) Join OR.
RN = diameter of the nine-point circle = OC (by step (3))
$\therefore ORNC$ is a //-gram (opp. sides are eq. and //)
OR // CF (property of //-gram)
$\angle ORB = \angle CFB = 90°$ (corr. ∠s, OR // CF)
$\therefore AR = RB$ (⊥ from centre bisects chord)
$\Rightarrow OR$ is the ⊥ bisector of AC.
Since K is arbitrary, there are infinite many different triangles satisfying the given conditions.