91-92 Individual	**1**	100	**2**	3	**3**	4	**4**	60	**5**	20
	6	C	**7**	±10	**8**	4	**9**	5	**10**	16
	11	12	**12**	35	**13**	1620	**14**	32	**15**	128
	16	3	**17**	10	**18**	$\frac{9}{10}$	**19**	8	**20**	$-\frac{4}{3}$

91-92 Group	**1**	102	**2**	–1	**3**	52	**4**	191	**5**	5
	6	10	**7**	42	**8**	3	**9**	1	**10**	7

Individual Events

I1 If $(\log_{10} x)^4 - 3(\log_{10} x)^2 - 4 = 0$ and $x > 1$, find x.

$[(\log_{10} x)^2 - 4][(\log_{10} x)^2 + 1] = 0$

$\log_{10} x = 2$ or -2

$x = 100$ or $\frac{1}{100}$ (rejected)

I2 If $\begin{cases} 28x + 15y = 19xy \\ 18x - 21y = 2xy \end{cases}$ and $xy \neq 0$, find x.

$$\begin{cases} \frac{28}{y} + \frac{15}{x} = 19 & \cdots(1) \\ \frac{18}{y} - \frac{21}{x} = 2 & \cdots(2) \end{cases}$$

$7\times(1) + 5\times(2)$: $\frac{286}{y} = 143$

$y = 2$

Put $y = 2$ into (1): $14 + \frac{15}{x} = 19$

$x = 3$

I3 An integer a lying between 0 and 9 inclusive is randomly selected. It is known that the probability that the equation $x^2 - ax + 3 = 0$ has no real root is $\frac{p}{10}$, find p.

$a^2 - 12 < 0$

$0 \le a \le 2\sqrt{3} \approx 3.46$

$a = 0, 1, 2$ or 3

$p = 4$

I4 $x°$ is an acute angle satisfying $\frac{1}{2}\cos x° \ge \frac{1}{2}(5 - \cos x°) - 2$. Determine the largest possible value of x.

$\frac{1}{2}\cos x° \ge \frac{1}{2} - \frac{1}{2}\cos x°$

$\cos x° \ge \frac{1}{2}$

$x° \le 60°$

The largest value of x is 60.

I5 Let $f(x)$ be the highest common factor of $x^4 + 64$ and $x^3 + 6x^2 + 16x + 16$, find $f(2)$.

Reference: 1993 FI5.2, 2001 FI1.2, 2011 FI3.2

$x^4 + 64 = x^4 + 16x^2 + 64 - 16x^2 = (x^2 + 8)^2 - (4x)^2 = (x^2 + 4x + 8)(x^2 - 4x + 8)$

$g(x) = x^3 + 6x^2 + 16x + 16$

$g(-2) = -8 + 24 - 32 + 16 = 0$

$\Rightarrow x + 2$ is a factor of $g(x)$.

By division, $g(x) = (x + 2)(x^2 + 4x + 8)$

H.C.F. $= f(x) = x^2 + 4x + 8$

$f(2) = 2^2 + 4(2) + 8 = 20$

I6 A fruit merchant divides a large lot of oranges into four classes: A, B, C, D. The number of oranges in class A and class B doubles that in class C while the number of oranges in class B and class D doubles that in class A. If 7 oranges from class B are upgraded to class A, class A will then contain twice as many oranges as class B. It is known that one of the four classes contains 54 oranges. Determine which one class it belongs to.

$A + B = 2C \cdots\cdots (1)$

$B + D = 2A \cdots\cdots (2)$

$A + 7 = 2(B - 7)$

$\Rightarrow A = 2B - 21 \cdots\cdots (3)$

Sub. (3) into (1) and (2)

$2B - 21 + B = 2C$

$\Rightarrow 3B - 21 = 2C \cdots\cdots (4)$

$B + D = 2(2B - 21)$

$\Rightarrow 3B - 42 = D \cdots\cdots (5)$

$(4) - (5)\ 21 = 2C - D \ldots\ldots (6)$

If $A = 54$, from (3), $B = 37.5$ (reject)

If $B = 54$, from (4), $C = 70.5$ (reject)

If $D = 54$, from (6), $C = 37.5$ (reject)

If $C = 54$, from (4), $B = 43$; from (5), $D = 87$; from (3), $A = 65$

$\therefore$ Answer C

I7 Given that n is a positive integer, find **ALL** the real roots of $x^{2^n} - 10^{2^n} = 0$.

$\left(x^{2^{n-1}}\right)^2 - \left(10^{2^{n-1}}\right)^2 = 0$

$\left(x^{2^{n-1}} + 10^{2^{n-1}}\right)\left(x^{2^{n-1}} - 10^{2^{n-1}}\right) = 0$

$\left(x^{2^{n-1}} + 10^{2^{n-1}}\right)\left(x^{2^{n-2}} + 10^{2^{n-2}}\right)\cdots\left(x^2 + 10^2\right)(x + 10)(x - 10) = 0$

$x = \pm 10$

I8 If n is an integer randomly selected from 1 to 100, and the probability that the unit digit of 5678^n is greater than 3 is $\dfrac{3}{x}$, find x.

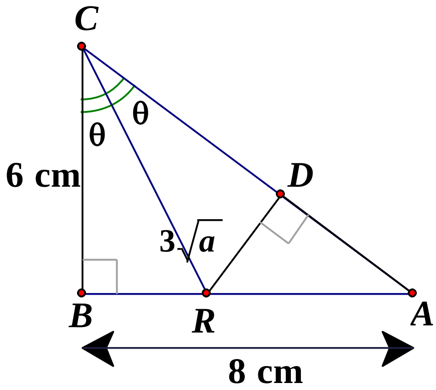
$8^1 = 8$, $8^2 = 64$, $8^3 = 512$, $8^4 = 4096$, $8^5 = 32768$

The pattern of unit digit repeats for every multiples of 4 .

$$\begin{aligned} P(\text{unit digit} > 3) &= 1 - P(\text{unit digit} \le 3) \\ &= 1 - P(n = 3, 7, 11, \cdots, 99) \\ &= \frac{3}{4} \end{aligned}$$

$x = 4$

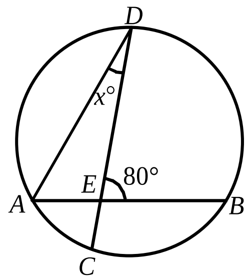
I9 In ΔABC, $AB = 8$ cm, $BC = 6$ cm and $\angle ABC = 90°$. If the bisector of $\angle ACB$ cuts AB at R and $CR = 3\sqrt{a}$ cm, find a.



Let $BR = x$ cm, then $AR = (8 - x)$ cm.
Let D be the foot of perpendicular drawn from R onto AC
$CR = CR$ (common sides)
$\angle BCR = \angle DCR = \theta$ (given)
$\angle CBR = \angle CDR = 90°$ (by construction)
$\therefore \Delta BCR \cong \Delta DCR$ (A.A.S.)
$DR = x$ cm (corr. sides, $\cong \Delta$s)
$CD = BC = 6$ cm (corr. sides, $\cong \Delta$s)
$AC = 10$ cm (Pythagoras' theorem)
$AD = (10 - 6)$ cm $= 4$ cm
In ΔADR, $x^2 + 4^2 = (8 - x)^2$ (Pythagoras' theorem)
$16 = 64 - 16x$
$x = 3$
$CR = \sqrt{3^2 + 6^2}$ cm $= \sqrt{45}$ cm $= 3\sqrt{5}$ cm (Pythagoras' theorem)
$a = 5$

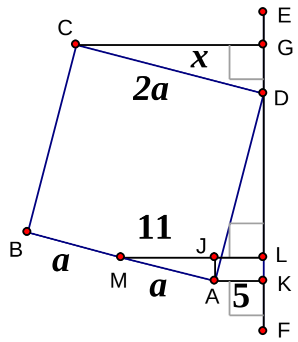
I10 In figure 1, arc BD is 4 times the arc AC, $\angle DEB = 80°$ and $\angle ADC = x°$, find x.
$\angle BAD = 4x°$ ($\angle$s $\propto$ arcs)
$x° + 4x° = 80°$ (ext. $\angle$ of ΔADE)
$x = 16$

(Figure 1) (圖一)



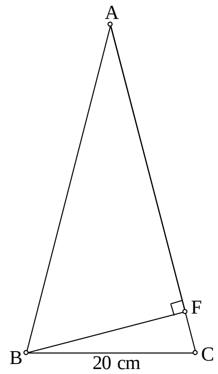
I11 In figure 2, $ABCD$ is a square. EDF is a straight line. M is the mid-point of AB. If the distances of A, M and C from the line EF are 5 cm, 11 cm and x cm respectively, find x.
Let K, L and G be the feet of perpendiculars drawn from A, M, C onto EF respectively. $AK = 5$ cm, $ML = 11$ cm, $CG = x$ cm
Let $CD = 2a$ cm, $AM = a$ cm, $BM = a$ cm.
From A, draw $AJ \perp ML$, then $AKLJ$ is a rectangle.
$JL = 5$ cm (opp. sides of rectangle)
$MJ = (11 - 5)$ cm $= 6$ cm
It is easy to show that $\Delta AMJ \sim \Delta DCG$

(Figure 2) (圖二)



$CG : MJ = CD : AM$ (ratio of sides, $\sim\Delta$s)
$x : 6 = 2a : a$
$x = 12$

I12 In the figure, $AB = AC = 2BC$ and $BC = 20$ cm. If BF is perpendicular to AC and $AF = x$ cm, find x.



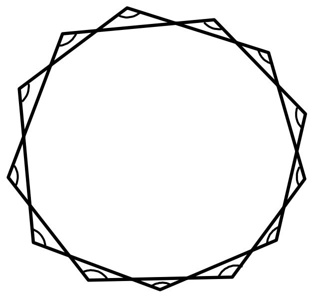
Let $\angle ABC = \theta = \angle ACB$ (base $\angle$s isosceles Δ)
$AB = AC = 40$

$$\cos\theta = \frac{\frac{1}{2}BC}{AC} = \frac{10}{40} = \frac{1}{4}$$

$$CF = BC\cos\theta = 20 \times \frac{1}{4} = 5$$

$AF = AC - CF = 40 - 5 = 35$ cm
$x = 35$

I13 Figure 4 shows a figure obtained by producing the sides of a 13-sided polygon. If the sum of the marked angles is $n°$, find n.



Reference: 2000 HI5, 2012 FG3.2

Consider the 13 small triangles outside.

Let the marked angles be $x_1°, x_2°, \cdots, x_{13}°$.

angle sum of 13 triangles = $13\times180° = 2340°$

$x_1° + x_2° + \cdots + x_{13}° + 2$(sum of ext. $\angle$ of polygon) $= 2340°$

$x_1° + x_2° + \cdots + x_{13}° = 2340° - 720° = 1620°$

$n = 1620$

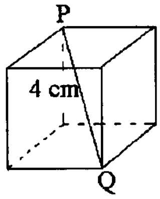
I14 In figure 5, PQ is a diagonal of the cube. If $PQ = 4$ cm and the total surface area of the cube is x cm^2, find x. (**Reference: 1995 FI5.2, 2003 HI7)**

Let the length of one side = a cm

$a^2 + a^2 + a^2 = 4^2$ Pythagoras' theorem

$a^2 = \frac{16}{3}$

$x = 6a^2 = 32$



(Figure 5)

I15 If $(3x-1)^7 = a_1x^7 + a_2x^6 + a_3x^5 + \cdots + a_8$, find the value of $a_1 + a_2 + a_3 + \cdots + a_8$.

Put $x = 1$, $2^7 = a_1 + a_2 + a_3 + \cdots + a_8$

$a_1 + a_2 + a_3 + \cdots + a_8 = 128$

I16 $A(1, 1)$, $B(a, 0)$ and $C(1, a)$ are the vertices of the triangle ABC. Find the value of a if the area of ΔABC is 2 square units and $a > 0$.

$$\frac{1}{2}\left|\begin{vmatrix}1 & 1\\ a & 0\\ 1 & a\\ 1 & 1\end{vmatrix}\right| = 2$$

$|a^2 + 1 - a - a| = 4$

$a^2 - 2a + 1 = 4$ or $a^2 - 2a + 1 = -4$

$a^2 - 2a - 3 = 0$ or $a^2 - 2a + 5 = 0$

$(a - 3)(a + 1) = 0$ or no solution

$a = 3$ ($\because a > 0$)

I17 If $N = 2^{12} \times 5^8$, find the number of digits of N. (**Reference: 1982 FG10.1, 2012 HI4**)

$N = 2^4 \times 10^8 = 16\times10^8$

Number of digits = 10

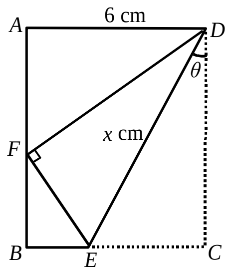
I18 If $a : b = 3 : 4$ and $a : c = 2 : 5$, find the value of $\frac{ac}{a^2 + b^2}$.

$a : b : c = 6 : 8 : 15$

$a = 6k$, $b = 8k$, $c = 15k$

$$\frac{ac}{a^2+b^2}=\frac{6k\cdot 15k}{(6k)^2+(8k)^2}=\frac{90}{100}=\frac{9}{10}$$

I19 A rectangular piece of paper of width 6 cm is folded such that one corner touches the opposite side as shown in figure 6. If $\theta = 30°$ and $DE = x$ cm, find x. **Reference American High School Mathematics Examination 1972 Q30**



$DE = DE = x$ cm common sides

$\angle DFE = \angle DCE = 90°$ by fold paper

$\angle EDF = \angle EDC = \theta$ by fold paper

$\therefore \Delta DEF \cong \Delta DEC$ (A.A.S.)

$CE = EF = x \sin\theta$ cm $= x \sin 30°$ cm $= \frac{x}{2}$ cm

$\angle CED = \angle FED = 60°$ (corr. $\angle$s, $\cong \Delta$s)

$\angle BEF = 180° - 2\times 60° = 60°$ (adj. $\angle$s on st. line)

$BE = EF \cos 60° = \frac{x}{2}\cdot\frac{1}{2}=\frac{x}{4}$ cm

$BE + EC = BC = AD$ (opp. sides of rectangle)

$\frac{x}{4}+\frac{x}{2}=6$

$x = 8$

Method 2

Let $BE = a$ cm

$\Delta DEF \cong \Delta DEC$ (A.A.S.)

$\angle CED = \angle FED = 60°$ (corr. $\angle$s, $\cong \Delta$s)

$\angle BEF = 60°$ (adj. $\angle$s on st. line)

$EF = a \div \cos 60° = 2a = CE = 6 - a$

$a = 2$

$x \sin 30° = 6 - a$

$x = 8$

I20 If $\sin x + \cos x = \frac{1}{5}$ and $0 \le x \le \pi$, find $\tan x$.

Reference: 1993 HG10, 1995 HI5, 2007 HI7, 2007 FI1.4, 2014 HG3

$(\sin x + \cos x)^2 = \frac{1}{25}$

$1 + 2 \sin x \cos x = \frac{1}{25}$

$\frac{24}{25} + 2 \sin x \cos x = 0$

$12 + 25 \sin x \cos x = 0$

$12(\sin^2 x + \cos^2 x) + 25 \sin x \cos x = 0$

$(3 \sin x + 4 \cos x)(4 \sin x + 3 \cos x) = 0$

$\tan x = -\frac{4}{3}$ or $-\frac{3}{4}$

When $\tan x = -\frac{4}{3}$, $\sin x = \frac{4}{5}$, $\cos x = -\frac{3}{5}$; original equation LHS = $\sin x + \cos x = \frac{1}{5}$

When $\tan x = -\frac{3}{4}$, $\sin x = \frac{3}{5}$, $\cos x = -\frac{4}{5}$; original equation LHS = $\sin x + \cos x = -\frac{1}{5}$ (reject)

$\therefore \tan x = -\frac{4}{3}$

Group Events

G1 A, B, C are three men in a team. The age of A is greater than the sum of the ages of B and C by 16. The square of the age of A is greater than the square of the sum of the ages of B and C by 1632. Find the sum of the ages of A, B and C.

$A = B + C + 16$ ⋯⋯ (1)

$A^2 = (B + C)^2 + 1632$ ⋯⋯ (2)

From (1), sub. $B + C = A - 16$ into (2):

$A^2 = A^2 - 32A + 256 + 1632$

$A = 59$

$B + C = 59 - 16 = 43$

$A + B + C = 59 + 43 = 102$

G2 a, b, c are non-zero real numbers such that $\frac{a+b-c}{c} = \frac{a-b+c}{b} = \frac{-a+b+c}{a}$.

If $x = \frac{(a+b)(b+c)(c+a)}{abc}$ and $x < 0$, find the value of x. **(Reference: 1999 FI2.1)**

$\frac{a+b}{c} - 1 = \frac{a+c}{b} - 1 = \frac{b+c}{a} - 1$

$\frac{a+b}{c} = \frac{a+c}{b} = \frac{b+c}{a} = k$

$a + b = ck$ ⋯⋯ (1)

$a + c = bk$ ⋯⋯ (2)

$b + c = ak$ ⋯⋯ (3)

(1) + (2) + (3): $2(a + b + c) = (a + b + c)k$

$a + b + c = 0$ or $k = 2$

$x = \frac{a+b}{c} \cdot \frac{a+c}{b} \cdot \frac{b+c}{a} = k^3 < 0$ (given)

$\therefore k = 2$ is rejected

$a + b + c = 0$

$\Rightarrow a + b = -c$

$\Rightarrow \frac{a+b}{c} = -1$

$\Rightarrow k = -1$

$\Rightarrow x = (-1)^3 = -1$

G3 An interior angle of an n-sided convex polygon is $x°$. The sum of the other interior angles is 2468°. Find x.

Reference: 1989 HG2, 1990 FG10.3-4, 2002 FI3.4, 2013HI6

$2468 = 180 \times 14 - 52$

$180 \times 14 - 52 + x = 180(n - 2)$ ∠s sum of polygon

$x = 180(n - 2) - 180 \times 14 + 52$

$x = 180(n - 16) + 52$

$\because x < 180$

$\therefore x = 52$

G4 When a positive integer N is divided by 4, 7, 9, the remainders are 3, 2, 2 respectively. Find the least value of N.

Reference: 1990 HG2

$N = 4a + 3 \cdots\cdots (1)$

$N = 7b + 2 \cdots\cdots (2)$

$N = 9c + 2 \cdots\cdots (3)$, where a, b, c are integers

$7b + 2 = 9c = 2$

$\Rightarrow b = 9k, c = 7k$ for some integer k

(1) = (2): $4a + 3 = 7b + 2$

$7b - 4a = 1$

$b = 3$, $a = 5$ is a particular solution

The general solution is $b = 3 + 4t$, $a = 5 + 7t$ for all real numbers t

$\therefore\ 3 + 4t = 9k$

$k = 3$, $t = 6$ is the smallest set of integral solution

$N = 4(5 + 7\times6) + 3 = 191$

G5 Find the remainder when 10^{1991} is divided by 7.

$1001 = 7\times143$

$10^3 = 7\times143 - 1$

$10^{1991} = (10^3)^{663}\times10^2$

$= (7\times143 - 1)^{663}\times100$

$= (7m - 1)\times(98 + 2)$

$\equiv -2 \equiv 5 \mod 7$

Method 2

$10 \div 7 \cdots 3$; $10^2 \div 7 \cdots 2$

$10^3 \div 7 \cdots 6$; $10^4 \div 7 \cdots 4$

$10^5 \div 7 \cdots 5$; $10^6 \div 7 \cdots 1$

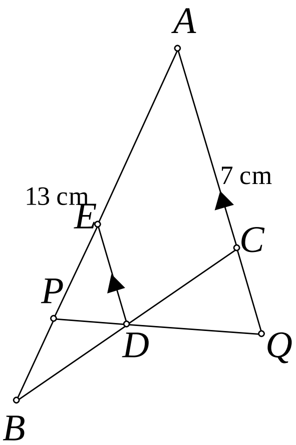
The remainders pattern repeats for every multiples of 6.

$10^{1991} = (10^6)^{331}\times10^5$

$\therefore$ The remainder is 5.

G6 In the figure, $BD = DC$, $AP = AQ$.

If $AB = 13$ cm, $AC = 7$ cm and $AP = x$ cm, find x.



Reference: 1999 FI3.3

From D, draw a parallel line DE // QA

$\because$ D is the mid-point of BC.

$\therefore$ $BE = EA$ (intercept theorem)

$= 13 \div 2 = 6.5$ cm

$DE = 7 \div 2 = 3.5$ cm (mid-point theorem on ΔABC)

$\angle APQ = \angle AQP$ (base $\angle$s. isos. Δ, $AP = AQ$)

$= \angle EDP$ (corr. $\angle$s, AQ // ED)

$\therefore$ $PE = DE$ (side opp. equal $\angle$s)

$= 3.5$ cm

$AP = AE + EP = 6.5 + 35 = 10$ cm

$x = 10$

G7 In the figure, $BL=\frac{1}{3}BC$, $CM=\frac{1}{3}CA$ and $AN=\frac{1}{3}AB$. If the areas of ΔPQR and ΔABC are 6 cm^2 and x cm^2 respectively, find x .

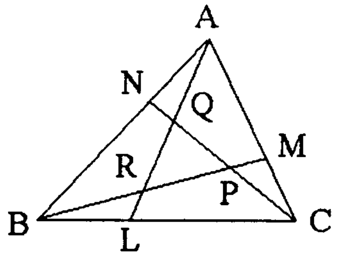


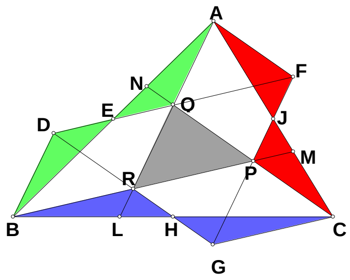
figure 2

Reference American High School Mathematics Examination 1952 Q49

Reference 2021 P1Q10

Denote [ABC] = area of triangle *ABC*.

Draw *BEQF* // *BM* // *GC*, *GHRD* // *CN* // *FA*, *FJPG* // *AL* // *DB* as shown. *AC* intersects *GF* at *J*, *BC* intersects *DG* at *H*, *AB* intersects *DF* at *E*.



Then *AQPF*, *QRGP* are congruent parallelograms.

BDQR, *RQFP* are congruent parallelograms. *CGRP*, *PRDQ* are congruent parallelograms.

AQF, *PFQ*, *QRP*, *RQD*, *DBR*, *GPR*, *PGC* are congruent Δs.

Consider triangles *AFJ* and *CPJ*:

$AF = QP$ (opp. sides of //-gram)

$= RG$ (opp. sides of //-gram)

$= PC$ (opp. sides of //-gram)

$AF \parallel PC$ (by construction)

AFCP is a parallelogram (Two sides are eq. and //)

$AJ = JC$ diagonal of a //-gram

$\angle AJF = \angle CJP$ vert. opp. $\angle$s

$\angle AFJ = \angle CPJ$ alt. $\angle$s $AF \parallel PC$

$\therefore\ \Delta AFJ \cong \Delta CPJ$ (AAS)

Areas $[CPJ] = [AFJ]$

In a similar manner, $[BRH] = [CGH]$, $[AQE] = [BDE]$

$[ABC] = [PQR] + [AQC] + [CPQ] + [BRA]$

$= [PQR] + [AQPF] + [CPRG] + [BRQD]$

$= 7\,[PQR]$ ($\because$ they are congruent triangles, so areas equal)

$= 7 \times 6 = 42$

Method 2

By considering the areas of ΔACL and ΔABL

$$\frac{\frac{1}{2}AC\cdot AL\sin\angle CAL}{\frac{1}{2}AB\cdot AL\sin\angle BAL}=\frac{2}{1}$$

$$\Rightarrow \frac{AC\sin\angle CAL}{AB\sin\angle BAL}=2 \cdots\cdots (1)$$

By considering the areas of ΔAMR and ΔABR

$$\frac{\frac{1}{2}AM\cdot AR\sin\angle CAL}{\frac{1}{2}AB\cdot AR\sin\angle BAL}=\frac{MR}{BR}$$

$$\frac{AM\sin\angle CAL}{AB\sin\angle BAL}=\frac{MR}{BR}$$

$$\frac{\frac{2}{3}AC\sin\angle CAL}{AB\sin\angle BAL}=\frac{MR}{BR}$$

By (1), $\frac{2}{3}\times 2=\frac{MR}{BR}\Rightarrow\frac{MR}{BR}=\frac{4}{3}\cdots\cdots (2)$

By considering the areas of ΔACN and ΔBCN

$$\frac{\frac{1}{2}AC\cdot CN\sin\angle ACN}{\frac{1}{2}BC\cdot CN\sin\angle BCN}=\frac{1}{2}\Rightarrow\frac{AC\sin\angle ACN}{BC\sin\angle BCN}=\frac{1}{2}\quad\cdots\cdots (3)$$

By considering the areas of ΔMCP and ΔBCP

$$\frac{\frac{1}{2}CM\cdot CP\sin\angle ACN}{\frac{1}{2}BC\cdot CP\sin\angle BCN}=\frac{MP}{BP}$$

$$\frac{CM\sin\angle ACN}{BC\sin\angle BCN}=\frac{MP}{BP}$$

$$\frac{\frac{1}{3}AC\sin\angle ACN}{BC\sin\angle BCN}=\frac{MP}{BP}$$

By (3), $\frac{1}{3}\times\frac{1}{2}=\frac{MP}{BP}$

$\Rightarrow\frac{MP}{BP}=\frac{1}{6}$ .... (4)

By (2) and (4), $MP : PR : RB = 1 : 3 : 3$

By symmetry $NQ : QP : PC = 1 : 3 : 3$ and $NR : RQ : QA = 1 : 3 : 3$

Let s stands for the area, x = area of ΔABC.

$s_{\Delta ABL}=s_{\Delta BCM}=s_{\Delta ACN}=\frac{x}{3}$

and $s_{\Delta ANQ}=s_{\Delta BLR}=s_{\Delta CMP}=\frac{1}{7}\times\frac{x}{3}=\frac{x}{21}$ ($\because NQ = QC = 1 : 6\Rightarrow NQ=\frac{1}{7}CN$)

The total area of ΔABC: $x = s_{\Delta ABL}+s_{\Delta BCM}+s_{\Delta ACN}+s_{\Delta PQR}-3\,s_{\Delta ANQ}$

$x=\frac{x}{3}+\frac{x}{3}+\frac{x}{3}+6-3\times\frac{x}{21}$

$0=6-\frac{1}{7}x$

$x = 42$

Method 3 (Vector method)

Let $\overrightarrow{AC}=\vec{c}$, $\overrightarrow{AB}=\vec{b}$

Suppose $BR : RM = r : s$

By ratio formula, $\overrightarrow{AR}=\frac{r\left(\frac{2}{3}\vec{c}\right)+s\vec{b}}{r+s}$; $\overrightarrow{AL}=\frac{\vec{c}+2\vec{b}}{3}$

$\because AR \,//\, AL$ $\therefore$ $\frac{\frac{s}{r+s}}{\frac{2}{3}}=\frac{\frac{2r}{3(r+s)}}{\frac{1}{3}}$ (their coefficients are in proportional)

$3s = 4r$

$r : s = 3 : 4$

Suppose $BP : PM = m : n$, let $\overrightarrow{CB}=\vec{a}$

By ratio formula, $\overrightarrow{CP}=\frac{n\vec{a}+m\left(-\frac{1}{3}\vec{c}\right)}{m+n}$; $\overrightarrow{CN}=\frac{\vec{a}+2(-\vec{c})}{3}$

$\because CP \,//\, CN$ $\therefore$ $\frac{\frac{n}{m+n}}{\frac{1}{3}}=\frac{-\frac{m}{3(m+n)}}{-\frac{2}{3}}$ (their coefficients are in proportional)

$6n = m$

$m : n = 6 : 1$

$\because r : s = 3 : 4$ and $m : n = 6 : 1$

$\therefore$ $MP : PR : RB = 1 : 3 : 3$

By symmetry $NQ : QP : PC = 1 : 3 : 3$ and $NR : RQ : QA = 1 : 3 : 3$

The remaining steps are similar, so is omitted.

G8 *ABC* is an equilateral triangle of side $\sqrt{12}$ cm, and *P* is any point inside the triangle (as shown in figure 3). If the sum of the perpendicular distances from *P* to the three sides *AB*, *BC* and *CA* is *x* cm, find *x*.

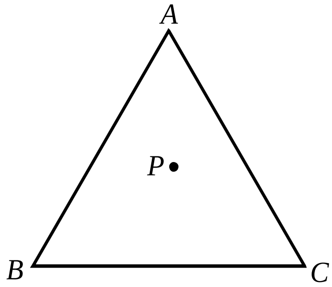


Figure 3

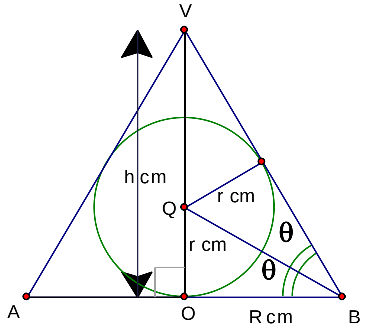
Reference 2005 HG9, 2015 HG2

Let the distance from *P* to *AB*, *BC*, *CA* be h_1, h_2, h_3 respectively.

$\frac{1}{2}\sqrt{12}h_1+\frac{1}{2}\sqrt{12}h_2+\frac{1}{2}\sqrt{12}h_3=\text{area of }\Delta ABC=\frac{1}{2}\left(\sqrt{12}\right)^2\sin 60^\circ=3\sqrt{3}$

$x = h_1 + h_2 + h_3 = 3$

G9 A sphere of radius *r* cm can just be covered on a table by a conical vessel of volume $\frac{8\pi r^2}{3}$ cm^3 (as shown in figure 4). Determine the largest possible value of *r*.



Let the vertex of the cone be *V*, *Q* is the centre of the sphere, *O* is the centre of the base, *AOB* is the diameter of the base. *VQO* are collinear and $VQO \perp AOB$.

Let $\angle OBQ = \theta$, the height be *h* cm and the base radius be *R* cm

$R = r\cot\theta$

$h = R\tan 2\theta = \frac{r\tan 2\theta}{\tan\theta}$

$\frac{1}{3}\pi[r\cot\theta]^2\cdot\frac{r\tan 2\theta}{\tan\theta}=\frac{8\pi r^2}{3}$

$r=\frac{8\tan^3\theta}{\tan 2\theta}=\frac{8\tan^3\theta}{\frac{2\tan\theta}{1-\tan^2\theta}}=4\tan^2\theta(1-\tan^2\theta)$

$r=1-4\left(\frac{1}{4}-\tan^2\theta+\tan^4\theta\right)=1-4\left(\frac{1}{2}-\tan^2\theta\right)^2$

$r \le 1$

r is the maximum when $\tan^2\theta=\frac{1}{2}$

In this case $\theta < 45°$, which is possible.

G10 a, b, c, d are four numbers. The arithmetic means of (i) a, b, c; (ii) b, c, d; (iii) a, b, d are respectively 13, 15 and 17. If the median of a, b, c and d is $c + 9$, find the largest possible value of c .

$a + b + c = 3\times 13 = 39 \cdots\cdots (1)$

$b + c + d = 3\times 15 = 45 \cdots\cdots (2)$

$a + b + d = 3\times 17 = 51 \cdots\cdots (3)$

(2) – (1): $d - a = 6 \cdots\cdots (4) \Rightarrow d > a$

(3) – (1): $d - c = 12 \cdots\cdots (5) \Rightarrow d > c$

$\therefore a = d - 6$
and $c = d - 12$
$\therefore$ The three numbers are $d - 12$, $d - 6$ and d in ascending order.
If $b \le d - 12$, then the median is $c + 9$
$\Rightarrow 2(d - 12 + 9) = d - 12 + d - 6$
$\Rightarrow -6 = -18$ reject
If $d - 12 < b \le d - 6$, then the median is $c + 9$
$\Rightarrow 2(d - 3) = b + d - 6 \Rightarrow b = d$ reject
If $d - 6 < b < d$, then the median is $c + 9$
$\Rightarrow 2(d - 3) = b + d - 6$
$\Rightarrow b = d$ reject
If $d \le b$, then the median $= c + 9$
$\Rightarrow 2(d - 3) = d - 6 + d$ accept
From (1), $b = 39 - a - c$
$= 39 - (d - 6) - (d - 12)$
$= 45 - d - d + 12$
$= 57 - 2d$
$b \ge d$
$\Rightarrow 57 - 2d \ge d$
$\Rightarrow 19 \ge d$
$c = d - 12 \le 19 - 12 = 7$
The largest possible value of c is 7.