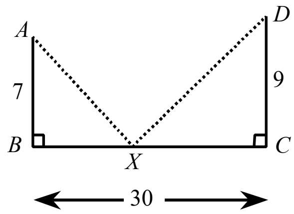
92-93 Individual	**1**	34	**2**	121	**3**	2	**4**	$\frac{3}{5}$	**5**	720
	6	11260	**7**	11	**8**	80	**9**	13	**10**	9

92-93 Group	**1**	2 km	**2**	45	**3**	6	**4**	211	**5**	9
	6	–7	**7**	26	**8**	$2+\sqrt{3}$	**9**	70	**10**	$\frac{\sqrt{13}}{3}$

Individual Events

I1 X is a point on the line segment BC as shown in figure 1.
If $AB = 7$, $CD = 9$ and $BC = 30$, find the minimum value of $AX + XD$.



(Figure 1)

Reference: 1983 FG8.1, 1991 HG9, 1996 HG9

Reflect point A along BC to A'.

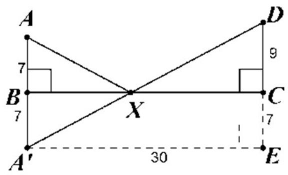
By the property of reflection.

$A'B \perp BC$ and $A'B = 7$

Join $A'D$, which cuts BC at X.

$\Delta ABX \cong \Delta A'BX$ (S.A.S.)

$AX + XD = A'X + XD$



This is the minimum when A', X, D are collinear.

Draw AE // BC which intersects DC produced at E.

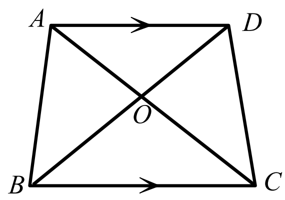
Then $A'E \perp DE$ (corr. $\angle$s, BC // $A'E$)

$A'E = 30$ and $CE = 7$ (opp. sides, rectangle)

$A'D^2 = 30^2 + (7 + 9)^2 = 1156 \Rightarrow A'D = 34$

The minimum value of $AX + XD = 34$

I2 In quadrilateral $ABCD$, AD // BC, and AC, BD intersect at O (as shown in figure 2). Given that area of $\Delta BOC = 36$, area of $\Delta AOD = 25$, determine the area of the quadrilateral $ABCD$.



(Figure 2)

Reference: 1997 HG3, 2000 FI2.2, 2002 FI1.3, 2004 HG7, 2010HG4, 2013 HG2

$\Delta AOD \sim \Delta COB$ (equiangular)

$$\frac{AO^2}{OC^2} = \frac{\text{area of } \Delta AOD}{\text{area of } \Delta BOC} = \frac{25}{36}$$

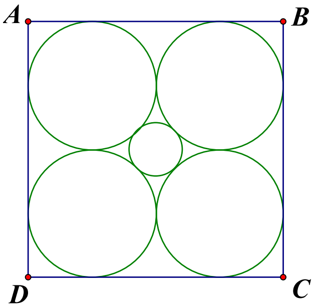
$$\frac{AO}{OC} = \frac{5}{6}$$

Area of $\Delta AOB = \frac{5}{6} \times$area of $\Delta BOC = \frac{5}{6} \times 36 = 30$

Area of $\Delta COD = \frac{6}{5} \times$area of $\Delta AOD = \frac{6}{5} \times 25 = 30$

Area of quadrilateral $ABCD = 25 + 30 + 36 + 30 = 121$

I3 In figure 3, *ABCD* is a square of side $8(\sqrt{2}+1)$. Find the radius of the small circle at the centre of the square.

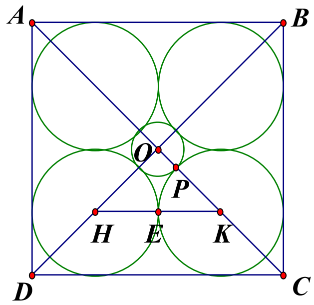


(Figure 3)

Let *AC* and *BD* intersect at *O*. $AC \perp BD$.

Let *H*, *K* be the centres of two adjacent circles touch each other at *E*.

The small circle touches one of the other circles at *P*.



$HE = EK = \frac{CD}{4} = 2(\sqrt{2}+1) = KP$, $HK = 4(\sqrt{2}+1)$

$OH = OK = HK \cos 45° = 2(2+\sqrt{2})$

$OP = OK - KP = 2(2+\sqrt{2}) - 2(\sqrt{2}+1) = 2$

$\therefore$ The radius = 2

I4 Thirty cards are marked from 1 to 30 and one is drawn at random. Find the probability of getting a multiple of 2 or a multiple of 5.

Let *A* be the event that the number drawn is a multiple of 2.

B be the event that the number drawn is a multiple of 5.

$A \cap B$ is the event that the number drawn is a multiple of 10.

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$= \frac{15}{30} + \frac{6}{30} - \frac{3}{30}$

$= \frac{18}{30} = \frac{3}{5}$

I5 The areas of three different faces of a rectangular box are 120, 72 and 60 respectively. Find its volume.

Let the lengths of sides of the box be *a*, *b*, *c*, where $a > b > c$.

$ab = 120$ ⋯⋯ (1)

$bc = 60$ ⋯⋯ (2)

$ca = 72$ ⋯⋯ (3)

(1)×(2) ×(3): $(abc)^2 = (60\times 6\times 2)^2$

$abc = 720$

The volume is 720 .

I6 For any positive integer n , it is known that $1^2 + 2^2 + \dots + n^2 = \frac{n(n+1)(2n+1)}{6}$. Find the value of $12^2 + 14^2 + 16^2 + \dots + 40^2$. **(Reference: 1989 HG3)**

$12^2 + 14^2 + 16^2 + \dots + 40^2 = 4\times(6^2 + 7^2 + 8^2 + \cdots + 20^2)$

$= 4\times[1^2 + \cdots + 20^2 - (1^2 + \cdots + 5^2)]$

$= 4\times\left[\frac{20(21)(41)}{6} - \frac{5(6)(11)}{6}\right]$

$= 4(2870 - 55) = 11260$

I7 If x and y are prime numbers such that $x^2 - y^2 = 117$, find the value of x.

Reference: 1995 HG4, 1997 HI1

$(x + y)(x - y) = 117 = 3^2 \times 13$

Without loss of generality, assume $x \geq y$.

$x + y = 117, x - y = 1 \cdots\cdots (1)$

or $x + y = 39, x - y = 3 \cdots\cdots (2)$

or $x + y = 13, x - y = 9 \cdots\cdots (3)$

From (1), $x = 59$, $y = 58$, not a prime, rejected

From (2), $x = 21$, $y = 18$, not a prime, rejected

From (3), $x = 11$, $y = 2 \Rightarrow x = 11$

I8 If m is the total number of positive divisors of 54000, find the value of m.

Reference 1994 FI3.2, 1997 HI3, 1998 HI10, 1998 FI1.4, 2002 FG4.1, 2005 FI4.4

$54000 = 2^4 \times 3^3 \times 5^3$

Positive divisors are in the form $2^x \times 3^y \times 5^z$ where x, y, z are integers and $0 \leq x \leq 4$, $0 \leq y \leq 3$, $0 \leq z \leq 3$

Total number of positive factors = $5 \times 4 \times 4 = 80$

I9 If a is a real number such that $a^2 - a - 1 = 0$, find the value of $a^4 - 2a^3 + 3a^2 - 2a + 10$.

Reference: 2000 HG1, 2001 FG2.1, 2007 HG3, 2009 HG2

By division algorithm,

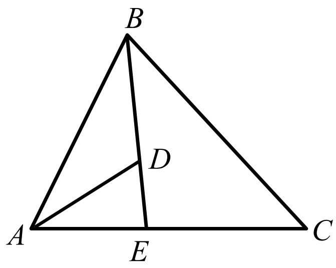
$a^4 - 2a^3 + 3a^2 - 2a + 10$

$= (a^2 - a - 1)(a^2 - a + 3) + 13$

$= 13$

$$\begin{array}{r} a^2 - a + 3 \\ a^2 - a - 1 \overline{) a^4 - 2a^3 + 3a^2 - 2a + 10} \\ \underline{a^4 - a^3 - a^2} \\ -a^3 + 4a^2 - 2a \\ \underline{-a^3 + a^2 + a} \\ 3a^2 - 3a + 10 \\ \underline{3a^2 - 3a - 3} \\ 13 \end{array}$$

I10 In figure 4, BDE and AEC are straight lines, $AB = 2$, $BC = 3$, $\angle ABC = 60°$, $AE : EC = 1 : 2$. If $BD : DE = 9 : 1$ and area of $\Delta DBA = \frac{a\sqrt{3}}{20}$, find the value of a.



(Figure 4)

Area of $\Delta ABC = \frac{1}{2} AB \cdot BC \cdot \sin 60^\circ = \frac{2}{2} \cdot 3 \cdot \frac{\sqrt{3}}{2} = \frac{3\sqrt{3}}{2}$

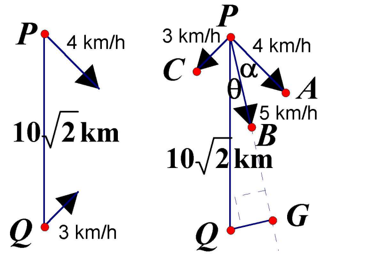
Area of $\Delta ABE = \frac{1}{3} \cdot$area of $\Delta ABC = \frac{\sqrt{3}}{2}$

Area of $\Delta ABD = \frac{9}{10} \cdot$area of $\Delta ABE = \frac{9\sqrt{3}}{20}$

$\Rightarrow a = 9$

Group Events

G1 A car P is $10\sqrt{2}$ km north of another car Q. The two cars start to move at the same time with P moving south-east at 4 km/h and Q moving north-east at 3 km/h. Find their smallest distance of separation in km.



distance diagram **velocity diagram**

Consider the relative velocity.

Keep Q fixed, the velocity of P relative to Q is 5 km/h in the direction of PB, where $\angle BPQ = \theta$.

Let $\angle APB = \alpha$, $\angle APQ = 45°$

$\sin\alpha = \frac{3}{5}$, $\cos\alpha = \frac{4}{5}$

$\sin\theta = \sin(45° - \alpha) = \sin 45° \cos\alpha - \cos 45° \sin\alpha$

$= \frac{1}{\sqrt{2}}\cdot\frac{4}{5} - \frac{1}{\sqrt{2}}\cdot\frac{3}{5} = \frac{1}{5\sqrt{2}}$

When the course of PB is nearest to Q (i.e at G),

The shortest distance is $GQ = PQ \sin\theta = 10\sqrt{2}\times\frac{1}{5\sqrt{2}} = 2$ km

G2 If α, β are the roots of the equation $x^2 - 3x - 3 = 0$, find $\alpha^3 + 12\beta$.

Reference: 2010 HI2, 2013 HG4

$\alpha^2 - 3\alpha - 3 = 0$

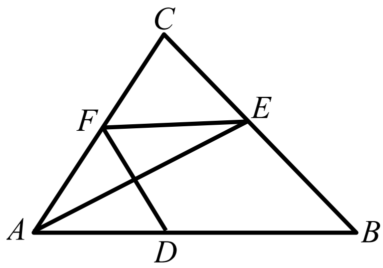
$\Rightarrow \alpha^3 = 3\alpha^2 + 3\alpha = 3(3\alpha + 3) + 3\alpha = 12\alpha + 9$

$\alpha + \beta = 3$, $\alpha\beta = -3$

$\alpha^3 + 12\beta = 12\alpha + 9 + 12\beta$

$= 12\times 3 + 9 = 45$

G3 As shown in figure 1, the area of ΔABC is 10. D, E, F are points on AB, BC and CA respectively such that $AD : DB = 2 : 3$, and area of ΔABE = area of quadrilateral $BEFD$. Find the area of ΔABE.



(Figure 1)

Join DE. Area of ΔADE = area of ΔDEF

$\because$ ΔADE and ΔDEF have the same base and the same height

$\therefore DE // AC$

$BE : EC = BD : DA = 3 : 2$ (theorem of equal ratio)

Area of ΔABE = Area of $\Delta ABC \times \frac{BE}{BC} = 10\times\frac{3}{3+2} = 6$

G4 What is the maximum number of regions produced by drawing 20 straight lines on a plane?

2 lines: maximum number of regions = 4 = 2 + 2

3 lines: maximum number of regions = 7 = 2 + 2 + 3

4 lines: maximum number of regions = 11 = 2 + 2 + 3 + 4

…………………………………………………………………………………

20 lines, maximum number of regions = 2 + 2 + 3 + … + 20 $= 1 + \frac{21}{2}\times 20 = 211$

G5 The product of 4 consecutive positive integers is 3024. Find the largest integer among the four.

Reference 1993 HG6, 1995 FI4.4, 1996 FG10.1, 2000 FG3.1, 2004 FG3.1, 2012 FI2.3, 2013HI5

Let the four integers be x, $x + 1$, $x + 2$, $x + 3$.

$x(x + 1)(x + 2)(x + 3) = 3024$

$(x^2 + 3x)(x^2 + 3x + 2) = 3024$

$(x^2 + 3x)^2 + 2(x^2 + 3x) + 1 = 3025$

$(x^2 + 3x + 1)^2 = 55^2$

$x^2 + 3x + 1 = 55$ or $x^2 + 3x + 1 = -55$

$x^2 + 3x - 54 = 0$ or $x^2 + 3x + 56 = 0$

$(x - 6)(x + 9) = 0$ or no real solution

$\because x > 0 \therefore x = 6$

The largest integer = 9

Method 2

$3024 + 1 = 3025 = 55^2$

$3024 = 55^2 - 1^2 = (55 - 1)(55 + 1)$

$3024 = 54\times 56 = 6\times 9\times 7\times 8$

The largest integer is 9.

G6 Find the sum of all real roots of the equation $(x + 2)(x + 3)(x + 4)(x + 5) = 3$.

Reference 1993 HG5, 1995 FI4.4, 1996 FG10.1, 2000 FG3.1, 2004 FG3.1, 2012 FI2.3, 2013 HI5

Let $t = x + 3.5$

$(t - 1.5)(t - 0.5)(t + 0.5)(t + 1.5) = 3$

$t^4 - \frac{5}{2}t^2 + \frac{9}{16} - 3 = 0$

$\left(t^2 - \frac{5}{4}\right)^2 - 4 = 0$

$\left(t^2 - \frac{5}{4} + 2\right)\left(t^2 - \frac{5}{4} - 2\right) = 0$

$t^2 = \frac{13}{4} \Rightarrow t = \pm\frac{\sqrt{13}}{2}$

$x = t - 3.5 = \frac{-7 \pm \sqrt{13}}{2}$

Sum of real roots = –7

Method 2

$(x + 2)(x + 5)(x + 3)(x + 4) = 3$

$(x^2 + 7x + 10)(x^2 + 7x + 12) = 3$

Let $y = x^2 + 7x$

$(y + 10)(y + 12) = 3$

$y^2 + 22y + 117 = 0$

$(y + 9)(y + 13) = 0$

When $y = -9 = x^2 + 7x$

$x^2 + 7x + 9 = 0$

When $y = -13 = x^2 + 7x$

$x^2 + 7x + 13 = 0$

$\Delta = 49 - 52 < 0$, no solution

$\therefore$ Sum of roots = –7

G7 If a is an integer and $a^7 = 8031810176$, find the value of a.

$1280000000 = 20^7 < 8031810176 < 30^7 = 21870000000$

Clearly a is an even integer.

$2^7 \equiv 8, 4^7 \equiv 4, 6^7 \equiv 6, 8^7 \equiv 2 \pmod{10}$

$\therefore a = 26$

G8 If x and y are real numbers satisfying $\begin{cases} x^2 - xy + y^2 - 3x - 3y = 1 \\ xy = 1 \end{cases}$ and $x > y > 0$,

find the value of x. **Reference: 2010 FI1.3, 2013 FI4.4**

Let $t = x + y$, (1) becomes $(x + y)^2 - 3 - 3(x + y) = 1$

$t^2 - 3t - 4 = 0$

$(t + 1)(t - 4) = 0$

$t = -1$ (rejected) or $t = 4$

$x + y = 4$ and $xy = 1$

x and y are the roots of $u^2 - 4u + 1 = 0$

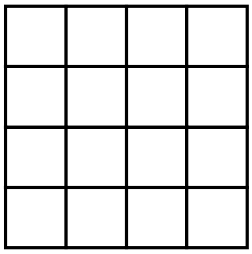
$x = 2 + \sqrt{3}$

G9 Each side of a square is divided into four equal parts and straight lines are joined as shown in figure 2. Find the number of rectangles which are not squares. **(Reference: 2013 FI1.1)**

Number of rectangles including squares = $C_2^5 \times C_2^5 = 100$

Number of squares = 16 + 9 + 4 + 1 = 30

Total number of rectangles which are not squares = 100 – 30 = 70



(Figure 2)

G10 If 0° ≤ θ ≤ 90° and $\cos\theta - \sin\theta = \frac{\sqrt{5}}{3}$, find the value of cos θ + sin θ.

Reference: 1992 HI20, 1995 HI5, 2007 HI7, 2007 FI1.4, 2014 HG3

$$(\cos\theta - \sin\theta)^2 = \frac{5}{9}$$

$$1 - 2\sin\theta\cos\theta = \frac{5}{9}$$

$$\frac{4}{9} - 2\sin\theta\cos\theta = 0$$

$$2 - 9\sin\theta\cos\theta = 0$$

$$2(\sin^2\theta + \cos^2\theta) - 9\sin\theta\cos\theta = 0$$

$$2\tan^2\theta - 9\tan\theta + 2 = 0$$

$$\tan\theta = \frac{9+\sqrt{65}}{4} \text{ or } \frac{9-\sqrt{65}}{4}$$

When $\tan\theta = \frac{9+\sqrt{65}}{4}$, $\sin\theta = \frac{9+\sqrt{65}}{3\left(\sqrt{13}+\sqrt{5}\right)}$, $\cos\theta = \frac{4}{3\left(\sqrt{13}+\sqrt{5}\right)}$,

Original equation LHS = $\cos\theta - \sin\theta = -\frac{5+\sqrt{65}}{3\left(\sqrt{13}+\sqrt{5}\right)} = -\frac{\sqrt{5}}{3}$ (reject)

When $\tan\theta = \frac{9-\sqrt{65}}{4}$, $\sin\theta = \frac{9-\sqrt{65}}{3\left(\sqrt{13}-\sqrt{5}\right)}$, $\cos\theta = \frac{4}{3\left(\sqrt{13}-\sqrt{5}\right)}$,

Original equation LHS = $\cos\theta - \sin\theta = \frac{\sqrt{5}}{3}$

$$\therefore \cos\theta + \sin\theta = \frac{9-\sqrt{65}}{3\left(\sqrt{13}-\sqrt{5}\right)} + \frac{4}{3\left(\sqrt{13}-\sqrt{5}\right)} = \frac{\sqrt{13}}{3}$$

Method 2

$\cos\theta > \sin\theta \Rightarrow \theta < 45° \Rightarrow 2\theta < 90°$

$$(\cos\theta - \sin\theta)^2 = \frac{5}{9}$$

$$1 - 2\sin\theta\cos\theta = \frac{5}{9}$$

$$\sin 2\theta = \frac{4}{9} \Rightarrow \cos 2\theta = \frac{\sqrt{65}}{9} \quad \because 2\theta < 90°$$

$$(\cos\theta - \sin\theta)(\cos\theta + \sin\theta) = \frac{\sqrt{5}}{3}(\cos\theta + \sin\theta)$$

$\cos^2\theta - \sin^2\theta = \frac{\sqrt{5}}{3}(\cos\theta + \sin\theta)$

$\frac{\sqrt{65}}{9} = \cos 2\theta = \frac{\sqrt{5}}{3}(\cos\theta + \sin\theta)$

$\cos\theta + \sin\theta = \frac{\sqrt{65}}{9} \div \frac{\sqrt{5}}{3} = \frac{\sqrt{13}}{3}$

Method 3

$(\cos\theta - \sin\theta)^2 = \frac{5}{9}$

$1 - 2\sin\theta\cos\theta = \frac{5}{9}$

$2\sin\theta\cos\theta = \frac{4}{9}$

$1 + 2\sin\theta\cos\theta = \frac{13}{9}$

$\cos^2\theta + 2\sin\theta\cos\theta + \sin^2\theta = \frac{13}{9}$

$(\cos\theta + \sin\theta)^2 = \frac{13}{9}$

$\cos\theta + \sin\theta = \frac{\sqrt{13}}{3}$ or $-\frac{\sqrt{13}}{3}$

$\because 0° \le \theta \le 90°$

$\therefore \cos\theta + \sin\theta > 0$

$\cos\theta + \sin\theta = \frac{\sqrt{13}}{3}$