Individual Events

SI	***a***	2	**I1**	***a***	6	**I2**	***A***	$\frac{5}{16}$	**I3**	***a***	6	**I4**	***a***	8	**I5**	***A***	1
	b	54		***b***	12		***B***	60		***b***	*9 see the remarks		***b***	10		***B***	36
	c	2		***c***	10		***C***	15		***c***	16		***c***	3		***C***	$\frac{1}{2}$
	d	1		***d***	20		***D***	68		***d***	48		***d***	203		***D***	50

Group Events

SG	***a***	19	**G6**	***a***	4, $\frac{1}{2}$	**G7**	***a***	24	**G8**	***a***	56	**G9**	***A***	2	**G10**	***a***	10
	b	8		***b***	16		***b***	1024		***b***	83		***B***	1		***b***	$\sqrt{37}$
	c	$\frac{1}{50}$		***c***	$\frac{3}{7}$		***c***	2		***c***	256		***C***	4		***c***	$\frac{1}{16}$
	d	200		***d***	186		***d***	–1		***d***	711040		***D***	9		***d***	4

Sample Individual Event (1985 Final Sample Individual Event)

SI.1 The sum of two numbers is 40, their product is 20 .

If the sum of their reciprocals is a, find the value of a .

Let the two numbers be x, y .

$x + y = 40$; $xy = 20$

$$a = \frac{1}{x} + \frac{1}{y}$$

$$= \frac{x+y}{xy} = \frac{40}{20}$$

$= 2$

SI.2 If b cm^2 is the total surface area of a cube of side $(a+1)$ cm, find the value of b .

$b = 6(2 + 1)^2 = 54$

SI.3 One ball is taken at random from a bag containing $(b - 4)$ white balls and $(b + 46)$ red balls.

If $\frac{c}{6}$ is the probability that the ball is white, find the value of c .

There are 50 white balls and 100 red balls.

$$\text{P(white ball)} = \frac{50}{150} = \frac{1}{3} = \frac{2}{6} = \frac{c}{6}$$

$\Rightarrow c = 2$

SI.4 The length of a side of an equilateral triangle is c cm.

If its area is $d\sqrt{3}$ cm^2, find the value of d .

$$\frac{1}{2}(2)^2 \sin 60^\circ = d\sqrt{3}$$

$$\sqrt{3} = d\sqrt{3}$$

$\Rightarrow d = 1$

Individual Event 1

I1.1 The equation $x^2 - ax + (a + 3) = 0$ has equal roots. Find a, if a is a positive integer.

$\Delta = (-a)^2 - 4(a + 3) = 0$

$a^2 - 4a - 12 = 0$

$(a - 6)(a + 2) = 0$

$a = 6$ or $a = -2$ (rejected)

I1.2 In a test, there are 20 questions. a marks will be given to a correct answer and 3 marks will be deducted for each wrong answer. A student has done all the 20 questions and scored 48 marks. Find b, the number of questions that he has answered correctly.

Reference: 1998 HG10

$6b - 3(20 - b) = 48$

$9b = 108$

$\Rightarrow b = 12$

I1.3 If $x : y = 2 : 3$, $x : z = 4 : 5$, $y : z = b : c$, find the value of c .

$x : y : z = 4 : 6 : 5$

$y : z = 6 : 5 = 12 : 10$

$\Rightarrow c = 10$

I1.4 Let $P(x, d)$ be a point on the straight line $x + y = 22$ such that the slope of OP equals to c (O is the origin). Determine the value of d .

Reference: 1993 FI3.2

$x + d = 22$

$\Rightarrow x = 22 - d$

$m_{OP} = \frac{d}{x} = c$

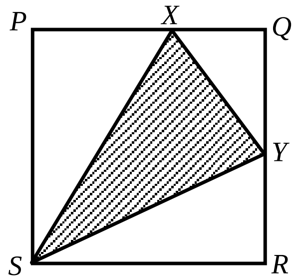
$\Rightarrow \frac{d}{22 - d} = 10$

$d = 220 - 10d$

$\Rightarrow d = 20$

Individual Event 2

I2.1 In square *PQRS*, *Y* is the mid-point of the side *QR* and $PX = \frac{3}{4}PQ$.



If *A* is the ratio of the area of the shaded triangle to the area of the square, find *A*.

Let *PQ* = 4*x*, *PX* = 3*x*, *QX* = *x*, *QY* = *YR* = 2*x*

$$A = \frac{(4x)^2 - \frac{1}{2}\cdot 4x(3x) - \frac{1}{2}\cdot x(2x) - \frac{1}{2}\cdot 4x(2x)}{(4x)^2}$$

$$= \frac{5}{16}$$

I2.2 A man bought a number of ping-pong balls where a 16*A*% sales tax is added. If he did not have to pay tax he could have bought 3 more balls for the same amount of money. If *B* is the total number of balls that he bought, find *B*.

Let the price of 1 ping-pong ball be *x*. Sales tax = 5%

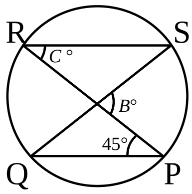
$Bx(1 + 5\%) = (B + 3)x$

$\frac{21}{20}B = B + 3$

$\Rightarrow 21B = 20B + 60$

$\Rightarrow B = 60$

I2.3 Refer to the diagram, find *C*.



$\angle PQS = C°$ ($\angle$s in the same segment)

$C + 45 = B$ (ext. $\angle$ of Δ)

$C = 60 - 45 = 15$

I2.4 The sum of 2*C* consecutive even numbers is 1170. If *D* is the largest of them, find *D*.

$$\frac{30}{2}[2D + (30-1)\cdot(-2)] = 1170$$

$\Rightarrow D = 68$

Individual Event 3

I3.1 If 183*a*8 is a mu1tiple of 287, find the value of a .

$287 = 7\times 41$ and $2614\times 7 = 18298$

$183a8 - 18298 = 10(a + 1)$, a multiple of 7

$a = 6$

Method 2 The quotient $\frac{183a8}{287}$ should be a two digit number.

The first digit of $\frac{183}{3}$ (approximate value) is 6. The last digit must be 4 ($\because$ $7\times 4 = 28$)

$\therefore$ $287\times 64 = 18368$

$\Rightarrow a = 6$

I3.2 The number of positive factors of a^2 is b, find the value of b .

Reference 1993 HI8, 1997 HI3, 1998 HI10, 1998 FI1.4, 2002 FG4.1, 2005 FI4.4

Remark: The original question is: The number of factors of a^2 ⋯⋯, which may include negative factors.

$6^2 = 2^2\times 3^2$

Factors of 36 are in the form $2^x\times 3^y$, where $0 \le x \le 2$, $0 \le y \le 2$.

The number of factors = (1 + 2)(1 + 2) = 9

I3.3 In an urn, there are c balls, b of them are either black or red, $(b + 2)$ of them are either red or white and 12 of them are either black or white. Find the value of c .

Suppose there are x black balls, y red balls, z white balls.

$x + y = 9$ …… (1)

$y + z = 11$ …… (2)

$z + x = 12$ …… (3)

(1) + (2) + (3): $2(x + y + z) = 32$

$c = x + y + z = 16$

I3.4 Given $f(3 + x) = f(3 - x)$ for all values of x, and the equation $f(x) = 0$ has exactly c distinct roots. Find d, the sum of these roots.

Reference: 2010 FG3.4

Let one root be $3 + \alpha$.

$f(3 + \alpha) = 0 = f(3 - \alpha)$

$\Rightarrow 3 - \alpha$ is also a root.

$3 + \alpha + 3 - \alpha = 6$

$\therefore$ Sum of a pair of roots = 6

There are 16 roots, i.e. 8 pairs of roots

Sum of all roots = $8\times 6 = 48$

Individual Event 4

I4.1 The remainder when $x^6 - 8x^3 + 6$ is divided by $(x - 1)(x - 2)$ is $7x - a$, find a.

Let $f(x) = x^6 - 8x^3 + 6$

$f(1) = 1 - 8 + 6 = 7 - a$

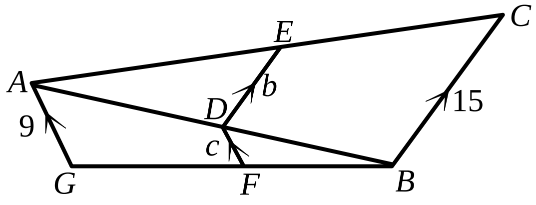
$a = 8$

I4.2 If $x^2 - x + 1 = 0$ and $b = x^3 - 3x^2 + 3x + a$, find b.

$b = x(x^2 - x + 1) - 2(x^2 - x + 1) + 10$

$= 10$

I4.3 Refer to the diagram, find c.



Reference: 1989 FG10.2

$\Delta ADE \sim \Delta ABC$, $AD : AB = 10 : 15 = 2 : 3$

$AD : DB = 2 : 1$

$BD : AB = 1 : 3$

$\Delta BDF \sim \Delta BAG$, $c : 9 = 1 : 3$

$c = 3$

I4.4 If c boys were all born in June 1990 and the probability that their birthdays are all different is $\frac{d}{225}$, find the value of d.

P(3 boys were born in different days) $= 1 \times \frac{29}{30} \times \frac{28}{30} = \frac{d}{225}$

$d = 203$

Individual Event 5

I5.1 Given $1-\frac{4}{x}+\frac{4}{x^2}=0$. If $A=\frac{2}{x}$, find the value of A.

Reference: 1999 FI5.2

$$\left(1-\frac{2}{x}\right)^2=0$$

$$\Rightarrow A=\frac{2}{x}=1$$

I5.2 If B circular pipes each with an internal diameter of A cm carry the same amount of water as a pipe with an internal diameter 6 cm, find the value of B.

$\pi(1)^2 \cdot B=\pi(6)^2$

$\Rightarrow B = 36$

I5.3 If C is the area of the triangle formed by x-axis, y-axis and the line $Bx + 9y = 18$, find the value of C.

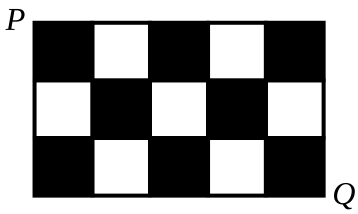
Reference: 1990 FI3.3

$36x + 9y = 18$

x-intercept $=\frac{1}{2}$, y-intercept = 2

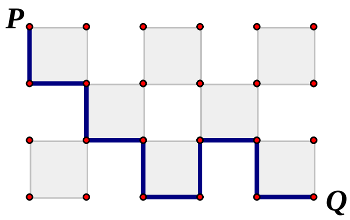
$C=\frac{1}{2}\cdot\frac{1}{2}\cdot 2=\frac{1}{2}$

I5.4 Fifteen square tiles with side $10C$ units long are arranged as shown. An ant walks along the edges of the tiles, always keeping a black tile on its left. Find the shortest distance D that the ant would walk in going from P to Q.



Length of a square = $10C = 5$

As shown in the figure, $D = 10(10C) = 50$



Sample Group Event (1985 Sample Group Event)

SG.1 If $x*y = xy + 1$ and $a = (2*4)*2$, find the value of a .

$2*4 = 2(4) + 1 = 9$

$(2 * 4)*2 = 9*2 = 9(2) + 1 = 19$

SG.2 If the b^{th} prime number is a, find the value of b .

List the prime number in ascending order: 2, 3, 5, 7, 11, 13, 17, 19 .

$b = 8$

SG.3 If $c = \left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1-\frac{1}{4}\right)\cdots\left(1-\frac{1}{50}\right)$, find the value of c in the simplest fractional form.

$c = \frac{1}{2}\cdot\frac{2}{3}\cdot\frac{3}{4}\cdot\cdots\cdot\frac{49}{50} = \frac{1}{50}$

SG.4 If d is the area of a square inscribed in a circle of radius 10 , find the value of d .

Diameter = 20 = diagonal of the square

Let the side of the square be x .

By Pythagoras' Theorem, $2x^2 = 20^2 = 400$

$d = x^2 = 200$

Group Event 6

G6.1 If $\log_2 a - 2\log_a 2 = 1$, find the value of a .

$$\frac{\log a}{\log 2} - \frac{2\log 2}{\log a} = 1$$

$(\log a)^2 - 2(\log 2)^2 = \log 2 \log a$

$(\log a)^2 - \log 2 \log a - 2(\log 2)^2 = 0$

$(\log a - 2\log 2)(\log a + \log 2) = 0$

$\log a = 2\log 2$ or $-\log 2$

$a = 4$ or $\frac{1}{2}$

G6.2 If $b = \log_3[2(3+1)(3^2+1)(3^4+1)(3^8+1)+1]$, find the value of b .

Reference: 2016 FG1.4

$b = \log_3[(3-1)(3+1)(3^2+1)(3^4+1)(3^8+1)+1]$

$= \log_3[(3^2-1)(3^2+1)(3^4+1)(3^8+1)+1]$

$= \log_3[(3^4-1)(3^4+1)(3^8+1)+1]$

$= \log_3[(3^8-1)(3^8+1)+1]$

$= \log_3(3^{16}-1+1) = 16$

G6.3 If a 31-day month is taken at random, find c, the probability that there are 5 Sundays in the month.

1^{st} day = Sunday $\Rightarrow$ 29^{th} day = 5^{th} Sunday

1^{st} day = Saturday $\Rightarrow$ 30^{th} day = 5^{th} Sunday

1^{st} day = Friday $\Rightarrow$ 31^{st} day = 5^{th} Sunday

Probability $= \frac{3}{7}$

G6.4 A group of 5 people is to be selected from 6 men and 4 women. Find d, the number of ways that there are always more men than women.

3 men and 2 women, number of combinations $= C_3^6 \cdot C_2^4 = 120$

4 men and 1 woman, number of combinations $= C_4^6 \cdot C_1^4 = 60$

5 men, number of combinations $= C_5^6 = 6$

Total number of ways = 120 + 60 + 6 = 186

Group Event 7

G7.1 There are a zeros at the end of the product $1\times2\times3\times\ldots\times100$. Find the value of a .

Reference: 1990 HG6, 1996 HI3, 2004 FG1.1, 2011 HG7, 2012 FI1.4, 2012 FG1.3

When each factor of 5 is multiplied by 2, a trailing zero will appear in the product.

The number of factors of 2 is clearly more than the number of factors of 5 in 100!

It is sufficient to find the number of factors of 5.

5, 10, 15, … , 100; altogether 20 numbers, have at least one factor of 5.

25, 50, 75, 100; altogether 4 numbers, have two factors of 5.

$\therefore$ Total number of factors of 5 is 20 + 4 = 24

There are 24 trailing zeros of 100!

$\Rightarrow a = 24$

G7.2 Find b, if b is the remainder when 1998^{10} is divided by 10^4 .

$1998^{10} = (2000 - 2)^{10}$

$= \sum_{k=0}^{10} C_k^{10} \cdot 2000^{10-k} \cdot 2^k$

$= \sum_{k=0}^{9} C_k^{10} \cdot 2000^{10-k} \cdot 2^k + 2^{10}$

$\equiv 2^{10} \bmod 10^4$ ($\because C_9^{10} \cdot 2000^{10-9} \cdot 2^1 = 10000m$, where m is an integer)

$\equiv 1024 \bmod 10^4$

$\Rightarrow b = 1024$

G7.3 Find the largest value of c, if $c = 2 - x + 2\sqrt{x-1}$ and $x > 1$.

$(c + x - 2)^2 = 4(x - 1)$

$c^2 + x^2 + 4 + 2cx - 4x - 4c = 4x - 4$

$x^2 + 2(c - 4)x + (c^2 - 4c + 8) = 0$

For real values of x, $\Delta \geq 0$

$4(c - 4)^2 - 4(c^2 - 4c + 8) \geq 0$

$c^2 - 8c + 16 - c^2 + 4c - 8 \geq 0$

$8 \geq 4c$

$\Rightarrow c \leq 2$

$\Rightarrow$ The largest value $c = 2$

Method 2

Let $y = \sqrt{x-1}$, then $y^2 = x - 1$

$\Rightarrow x = y^2 + 1$

$c = 2 - (y^2 + 1) + 2y = 2 - (1 - y)^2 \leq 2$

The largest value of $c = 2$.

G7.4 Find the least value of d, if $\left|\frac{3-2d}{5}+2\right| \leq 3$.

$-3 \leq \frac{3-2d}{5} + 2 \leq 3$

$\Leftrightarrow -5 \leq \frac{3-2d}{5} \leq 1$

$\Leftrightarrow -25 \leq 3 - 2d \leq 5$

$\Leftrightarrow -28 \leq -2d \leq 2$

$\Leftrightarrow 14 \geq d \geq -1$

The least value of $d = -1$

Group Event 8

G8.1 From 1 to 121, there are a numbers which are multiplies of 3 or 5. Find the value of a .

Reference: 1993 FG8.3-4, 1998 HI6, 2015 FI3.1

Number of multiples of 3 = 40 (120 = 3×40)

Number of multiples of 5 = 24 (120 = 5×24)

Number of multiples of 15 = 8 (120 = 15×8)

Number of multiples of 3 or 5 = a = 40 + 24 – 8 = 56

G8.2 From 1 to 121, there are b numbers which are not divisible by 5 nor 7. Find the value of b .

Number of multiples of 5 = 24 (120 = 5×24)

Number of multiples of 7 = 17 (119 = 7×17)

Number of multiples of 35 = 3 (105 = 35×3)

Number of multiples of 5 or 7 = 24 + 17 – 3 = 38

Number which are not divisible by 5 nor 7 = 121 – 38 = 83

From the digits 1, 2, 3, 4, when each digit can be used repeatedly, 4-digit numbers are formed. Find

G8.3 c, the number of 4-digit numbers that can be formed.

$c = 4^4 = 256$

G8.4 d, the sum of all these 4-digit numbers.

Reference: 2002 HI4

∵ There are 256 different numbers

∴ 1, 2, 3, 4 each appears 64 times in the thousands, hundreds, tens and units digit.

$d = [1000(1 + 2 + 3 + 4) + 100(1 + 2 + 3 + 4) + 10(1 + 2 + 3 + 4) + 1 + 2 + 3 + 4]\times 64$

$= 1111(10)\times 64 = 711040$

Group Event 9

A, B, C, D are different integers ranging from 0 to 9 and $(ABA)^2 = (CCDCC) < 100000$.

Find the values of A, B, C and D.

$$\begin{array}{r} A\ B\ A \\ \times \quad A\ B\ A \\ \hline C\ C\ D\ C\ C \end{array}$$

$(ABA) < \sqrt{100000} < 316$

$A = 1, 2$ or 3

When $A = 1$, then $A^2 = 1 = C$ contradict that A and C must be different $\therefore$ rejected

When $A = 2$, $C = 4$

$(202 + 10B)^2 = 44044 + 100D$

$40804 + 4040B + 100B^2 = 44044 + 100D$

$4040B + 100B^2 = 3240 + 100D$

$404B + 10B^2 = 324 + 10D$

$\therefore B = 1, 414 = 324 + 10D$

$\Rightarrow D = 9$

When $A = 3$, $C = 9$

$(303 + 10B)^2 = 99099 + 100D$

$91809 + 6060B + 100B^2 = 99099 + 100D$

$606B + 10B^2 = 729 + 10D$

$\therefore B = 1, 616 = 729 + 10D$

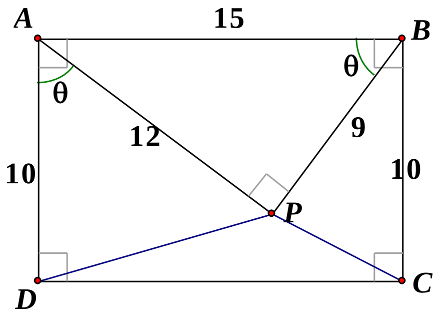
$\Rightarrow$ no solution for D

$\therefore A = 2, B = 1, C = 4, D = 9$

Group Event 10

In rectangle *ABCD*, *AD* = 10, *CD* = 15, *P* is a point inside the rectangle such that *PB* = 9, *PA* = 12. Find **(Reference: 2001 FG2.2, 2003 FI3.4, 2018 HI7)**

G10.1 *a*, the length of *PD* and



$AP^2 + BP^2 = 12^2 + 9^2 = 144 + 81 = 225 = 15^2 = AB^2$

$\therefore \angle APB = 90°$ (Converse, Pythagoras' theorem)

Let $\angle ABP = \theta$, then $\cos\theta = \frac{9}{15} = \frac{3}{5}$, $\sin\theta = \frac{4}{5}$

$\angle BAP = 90° - \theta$ ($\angle$s sum of Δ)

$\angle DAP = \theta$

$\angle PBC = 90° - \theta$

$a = PD = \sqrt{10^2 + 12^2 - 2\cdot 10\cdot 12\cdot \frac{3}{5}}$ (Cosine rule on ΔADP)

$a = 10$

G10.2 *b*, the length of *PC*.

$b = CP = \sqrt{10^2 + 9^2 - 2\cdot 10\cdot 9\cdot \frac{4}{5}} = \sqrt{37}$

G10.3 It is given that $\sin 2\theta = 2\sin\theta\cos\theta$. Find *c*, if $c = \frac{\sin 20^\circ \cos 20^\circ \cos 40^\circ \cos 60^\circ \cos 80^\circ}{\sin 160^\circ}$.

$$C = \frac{2\sin 20^\circ \cos 20^\circ \cos 40^\circ \cos 60^\circ \cos 80^\circ}{2\sin 160^\circ} = \frac{2\sin 40^\circ \cos 40^\circ \cos 60^\circ \cos 80^\circ}{4\sin 160^\circ}$$

$$= \frac{2\sin 80^\circ \cos 60^\circ \cos 80^\circ}{8\sin 160^\circ} = \frac{\sin 160^\circ \times \frac{1}{2}}{8\sin 160^\circ} = \frac{1}{16}$$

G10.4 It is given that $\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A\tan B}$. Find *d*, if

$d = (1 + \tan 21°)(1 + \tan 22°)(1 + \tan 23°)(1 + \tan 24°)$.

If $A + B = 45°$, $1 = \tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A\tan B}$

$1 - \tan A\tan B = \tan A + \tan B$

$2 = 1 + \tan A + \tan B + \tan A\tan B$

$(1 + \tan A)(1 + \tan B) = 2$

$d = (1 + \tan 21°)(1 + \tan 24°)(1 + \tan 22°)(1 + \tan 23°)$

$= 2\times 2 = 4$