93-94 Individual	**1**	9	**2**	$\frac{4}{9}$	**3**	256	**4**	6	**5**	30:15:10:6
	6	8	**7**	3	**8**	6.8	**9**	6	**10**	28

93-94 Group	**1**	$\frac{1}{5}$	**2**	972 cm^2	**3**	100	**4**	$\frac{16\pi}{3}$	**5**	9
	6	27	**7**	17	**8**	9	**9**	$\frac{1}{2}$	**10**	$\frac{56}{113}$

Individual Events

I1 Suppose $\log_3 p = 1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots$ to an infinite number of terms. Find p.

Reference: 2012 HI5

$\log_3 p = \frac{1}{1-\frac{1}{2}}$ = 2 (sum to infinity of a G.P. = $\frac{a}{1-r}$, a = 1, $r = \frac{1}{2}$)

$p = 3^2 = 9$

I2 Two numbers are drawn from the set of numbers 4, 5, 6, 7, 8, 9, 10, 11 and 12. Find the probability that the sum of these two numbers is even.

P(sum = even) = P(odd + odd or even + even)

= P(odd + odd) + P(even + even)

$= \frac{4}{9}\times\frac{3}{8}+\frac{5}{9}\times\frac{4}{8}$

$= \frac{32}{72} = \frac{4}{9}$

I3 Given $a*b = a^b$, find the value of $\frac{2*(2*(2*2))}{((2*2)*2)*2}$.

$2*2 = 2^2 = 4$

$2*(2*2) = 2^4 = 16$; $(2*2)*2 = 4^2 = 16$

$2*(2*(2*2)) = 2^{16}$; $((2*2)*2)*2 = 16^2 = 2^8$

$\frac{2*(2*(2*2))}{((2*2)*2)*2} = \frac{2^{16}}{2^8} = 2^8 = 256$

I4 If $\log_a x = 2$ and $2a + x = 8$, find $a + x$.

$x = a^2$.

$2a + a^2 = 8$

$\Rightarrow a^2 + 2a - 8 = 0$

$(a + 4)(a - 2) = 0$

$a = -4$ (rejected, because –4 cannot be the base) or $a = 2$

$x = a^2 = 4$

$a + x = 2 + 4 = 6$

I5 If $a : b = 2 : 1$, $b : c = 3 : 2$ and $c : d = 5 : 3$, find $a : b : c : d$.

$a : b = 30 : 15$, $b : c = 15 : 10$, $c : d = 10 : 6$

$a : b : c : d = 30 : 15 : 10 : 6$

I6 A, B, C, D are different integers ranging from 0 to 9 and

$$\begin{array}{r} A\ \ B\ \ C\ \ D \\ \times \qquad\quad 9 \\ \hline D\ \ C\ \ B\ \ A \end{array}$$

Find C.

Reference: 1987 FG9

Consider the thousands digit.

$9A = D$

$\Rightarrow A = 1, D = 9$

$\because$ there is no carry digit in the hundreds digit

$\therefore B = 0$

Consider the tens digit.

$9C + 8 \equiv 0 \pmod{10}$

$\Rightarrow 9C \equiv 2 \pmod{10}$

$C = 8$

I7 Find the last digit of the number 3^{1993}.

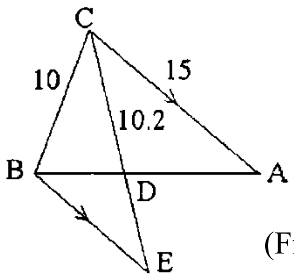
$3^1 = 3, 3^2 = 9, 3^3 = 27, 3^4 = 81$

The pattern of units digit repeat for every multiples of 4.

$3^{1993} = (3^4)^{498} \times 3 \equiv 1 \times 3 \pmod{10}$

The units digit is 3.

I8 In figure 1, CD bisects $\angle BCA$, $BE // CA$, $BC = 10$, $CA = 15$ and $CD = 10.2$. Find the length of DE.



(Figure 1)

Let $\angle BCD = \theta = \angle ACD$ (angle bisector)

$\angle BED = \theta$ (alt. $\angle$s, $AC // EB$)

$BE = BC = 10$ (sides opp. equal angles)

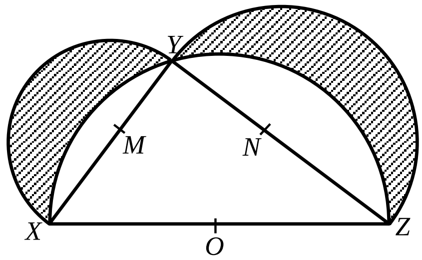
Let $\angle CAD = \alpha = \angle EBD$ (alt. $\angle$s, $AC // EB$)

$\therefore \Delta ACD \sim \Delta BED$ (equiangular)

$\frac{DE}{10.2} = \frac{10}{15}$ (ratio of sides, $\sim\Delta$)

$DE = 6.8$

I9 In figure 2, $XY = 3$, $YZ = 4$ and $ZX = 5$. Semi-circles are constructed with M, N, O as centres as shown where M, N, O are mid-points of XY, YZ and ZX respectively. Find the sum of the shaded areas. **Reference 2009 FG4.2, 2019 FG4.1**



(Figure 2)

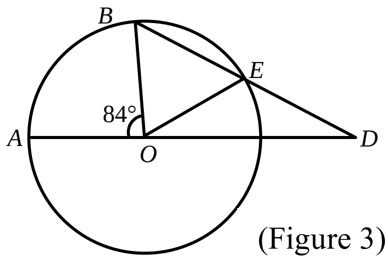
Sum of the shaded areas

$= S_{\text{semi-circle } XMY} + S_{\text{semi-circle } YNZ} + S_{\Delta XYZ} - S_{\text{semi-circle } XYZ}$

$= \frac{\pi}{2}\cdot\left(\frac{3}{2}\right)^2 + \frac{\pi}{2}\cdot\left(\frac{4}{2}\right)^2 + \frac{1}{2}\cdot 3\times 4 - \frac{\pi}{2}\cdot\left(\frac{5}{2}\right)^2$

$= 6$

I10 In figure 3, O is the centre of the circle, $OE = DE$ and $\angle AOB = 84°$. Find a if $\angle ADE = a°$.



(Figure 3)

$\angle DOE = a°$ (base $\angle$s isos. ΔODE)

$\angle OEB = 2a°$ (ext. $\angle$ of ΔODE)

$\angle OBE = 2a°$ (base $\angle$s isos. ΔOBE)

$\angle BOE = 180 - 4a°$ ($\angle$s sum of ΔOBE)

$84° + 180 - 4a° + a° = 180°$ (adj. $\angle$s on st. line)

$a = 28$

G1 Find the least value of x so that $|1-2x|+|1-3x|+|1-5x|=1$.

Reference: 2001 HG9, 2004 FG4.2, 2008 HI8, 2008 FI1.3, 2010 HG6, 2011 FGS.1, 2012 FG2.3

When $x<\frac{1}{5}$, $1-2x+1-3x+1-5x=1$

$\Rightarrow x=\frac{1}{5}$ (rejected)

When $\frac{1}{5}\le x\le\frac{1}{3}$, $1-2x+1-3x+5x-1=1$

$\Rightarrow 0=0\Rightarrow\frac{1}{5}\le x\le\frac{1}{3}$

When $\frac{1}{3}<x\le\frac{1}{2}$, $1-2x+3x-1+5x-1=1$

$\Rightarrow 6x=2\Rightarrow x=\frac{1}{3}$ (rejected)

When $\frac{1}{2}<x$, $2x-1+3x-1+5x-1=1$

$\Rightarrow x=\frac{2}{5}$ (rejected)

$\therefore$ The least value of $x=\frac{1}{5}$.

Method 2

By triangle inequality $|a|+|b|\ge|a+b|$

$|1-2x|+|1-3x|+|1-5x|$

$\ge|1-2x+1-3x+5x-1|=1$

Equality holds when $1\ge 2x$, $1\ge 3x$ and $5x\ge 1$

i.e. $\frac{1}{2}\ge x$ and $\frac{1}{3}\ge x$ and $x\ge\frac{1}{5}$

i.e $\frac{1}{3}\ge x\ge\frac{1}{5}$

The minimum value of $x=\frac{1}{5}$.

G2 A solid cube with edges of length 9 cm is painted completely on the outside. It is then cut into 27 congruent little cubes with edges 3 cm. Find the total area of the unpainted faces of these cubes.

Reference: 1991 FI3.2

There are 8 cubes each painted with 3 sides. Number of unpainted surfaces = 3/cube

There are 12 cubes each painted with 2 sides. Number of unpainted surfaces = 4/cube

There are 6 cubes each painted with 1 side. Number of unpainted surfaces = 5/cube

There is 1 cube which is unpainted.

Total area of the unpainted faces = $(8\times3+12\times4+6\times5+1\times6)\times3^2$ cm^2 = 972 cm^2

Method 2 Total surface area of the 27 little cubes = $27\times6\times3^2$ cm^2 = 1458 cm^2

Total area of painted surface = 6×9^2 cm^2 = 486 cm^2

$\therefore$ The total area of the unpainted faces of these cubes = (1458 – 486) cm^2 = 972 cm^2

G3 In a race of 2000 m, A finishes 200 m ahead of B and 290 m ahead of C. If B and C continue to run at their previous average speeds, then B will finish x metres ahead of C. Find x.

Let the speeds of A, B C be a m/s, b m/s, c m/s respectively.

Suppose A, B, C finishes the race in t_1 s, t_2 s, t_3 s.

$at_1 = 2000$

$bt_1 + 200 = 2000 \Rightarrow bt_1 = 1800$ …… (1)

$ct_1 + 290 = 2000 \Rightarrow ct_1 = 1710$ …… (2)

(2) ÷ (1): $\frac{c}{b} = \frac{1710}{1800} = \frac{19}{20}$ …… (3)

$t_2 = \frac{2000}{b}$ …… (4)

$x = 2000 - ct_2$

$= 2000 - 2000 \cdot \frac{c}{b}$ by (4)

$= 2000 \cdot \left(1 - \frac{c}{b}\right) = 2000 \cdot \left(1 - \frac{19}{20}\right)$ by (3)

$= 2000 \cdot \frac{1}{20} = 100$

G4 Given that the perimeter of an equilateral triangle inscribed in a circle is 12. Find the area of the circle in terms of π.

The length of the triangle = 4

Let the radius of the circumscribed circle be r.

$2r \cos 30° = 4$

$\Rightarrow r = \frac{4}{\sqrt{3}}$

The area of the circle $= \pi\left(\frac{4}{\sqrt{3}}\right)^2 = \frac{16\pi}{3}$

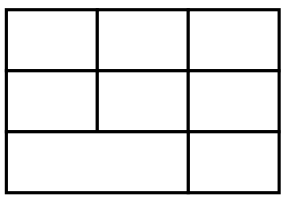
G5 Given that $x > 0$ and $y > 0$, find the value of y if $(\log_3 x)(\log_x 2x)(\log_{2x} y) = \log_x x^2$.

By the change of base formula, $\frac{\log x}{\log 3} \cdot \frac{\log 2x}{\log x} \cdot \frac{\log y}{\log 2x} = \frac{\log x^2}{\log x} = 2$

$\log y = 2 \log 3$

$\Rightarrow y = 9$

G6 There are n rectangles in figure 1. Find n.



(Figure 1)

Let the length and the width of the smallest rectangle be a and b respectively.

Number of the smallest rectangles = 7

Number of rectangles with dimension $a \times 2b = 4$

Number of rectangles with dimension $a \times 3b = 1$

Number of rectangles with dimension $2a \times b = 5$

Number of rectangles with dimension $2a \times 2b = 2$

Number of rectangles with dimension $2a \times 3b = 2$

Number of rectangles with dimension $3a \times b = 3$

Number of rectangles with dimension $3a \times 2b = 2$

Number of rectangles with dimension $3a \times 3b = 1$

Total number of rectangles = 7 + 4 + 1 + 5 + 2 + 2 + 3 + 2 + 1 = 27

G7 The base of a triangle is 80 cm and one of the base angles is 60°. The sum of the lengths of the other two sides is 90 cm. The length of the shortest side of this triangle is a cm. Find a.

Let $\angle B = 60°$, $BC = 80$ cm, $AB = c$ cm, $AC = b$ cm, then $b + c = 90$.

By cosine rule, $b^2 = (90 - c)^2 = 80^2 + c^2 - 2c\times 80 \cos 60°$

$8100 - 180c = 6400 - 80c$

$1700 = 100c$

$\Rightarrow c = 17, b = 90 - 17 = 73$

The shortest side is 17 cm ($c = 17$).

G8 A student on a vacation of d days observed that:

(*i*) it rained 7 times, morning or afternoons;

(*ii*) when it rained in the afternoon, it was clear in the morning;

(*iii*) there were 5 clear afternoons;

(*iv*) there were 6 clear mornings.

What is the value of d ?

Suppose it rained in the morning for x days, rained in the afternoon in y days and the number of clear days (both in the morning and the afternoon) be z .

$x + y = 7$ …… (1)

$x + z = 5$ …… (2)

$y + z = 6$ …… (3)

(1) + (2) + (3): $2(x + y + z) = 18$

$d = x + y + z = 9$

G9 $[a]$ denotes the greatest integer not greater than a . For example, $[1] = 1$, $[\sqrt{2}] = 1$, $[-\sqrt{2}] = -2$.

If $[5x] = 3x + \frac{1}{2}$, find the value of x .

Reference: 2001 FI2.4

$[5x] = 3x + \frac{1}{2} \Rightarrow 5x = 3x + \frac{1}{2} + a$, where $0 \le a < 1$

$a = 2x - \frac{1}{2} \Rightarrow 0 \le 2x - \frac{1}{2} < 1$

$\Rightarrow \frac{1}{2} \le 2x < \frac{3}{2}$

$\Rightarrow \frac{3}{4} \le 3x < \frac{9}{4}$

$\Rightarrow \frac{5}{4} \le 3x + \frac{1}{2} < \frac{11}{4}$

$\because 3x + \frac{1}{2}$ is an integer $\therefore 3x + \frac{1}{2} = 2$

$x = \frac{1}{2}$

G10 Given that $\frac{1}{n}-\frac{1}{n+2}=\frac{2}{n(n+2)}$.

Find the value of *a* if $a=\frac{1}{1\cdot 3}+\frac{1}{3\cdot 5}+\frac{1}{5\cdot 7}+\cdots+\frac{1}{n(n+2)}+\cdots+\frac{1}{111\cdot 113}$.

$$a=\frac{1}{2}\left(1-\frac{1}{3}+\frac{1}{3}-\frac{1}{5}+\frac{1}{5}-\frac{1}{7}+\cdots+\frac{1}{111}-\frac{1}{113}\right)$$

$$=\frac{1}{2}\left(1-\frac{1}{113}\right)$$

$$=\frac{56}{113}$$