Individual Events

I1	***a***	$\frac{1}{2}$	**I2**	***x***	0	**I3**	***a***	$\frac{1}{2}$	**I4**	***r***	3	**I5**	***a***	2
	b	$3\sqrt{2}$		***y***	3		***b***	8		***s***	4		***b***	2
	c	3		***z***	2		***c***	2		***t***	5		***c***	12
	d	1		***w***	$\frac{1}{6}$		***d***	120		***u***	41		***d***	$16\sqrt{3}$

Group Events

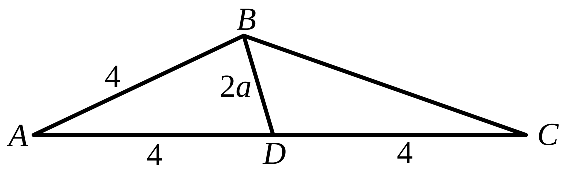
G6	***a***	5	**G7**	***a***	$\frac{1}{2}$	**G8**	***V***	1	**G9**	***A***	9	**G10**	***a***	4
	b	2		***b***	$2\sqrt{2}$		***V***	0		***B***	6		***b***	13
	c	$\frac{1}{4}$		***c***	700		***r***	3		***C***	8		***c***	16
	d	$\frac{1}{1995}$		***d***	333		***V***	35		***D***	2		***d***	$\frac{1}{10}$

Individual Event 1

I1.1 Find a, if $a=\log_{\frac{1}{4}}\frac{1}{2}$.

$$a=\log_{\frac{1}{4}}\frac{1}{2}=\log_{\frac{1}{4}}\left(\frac{1}{4}\right)^{\frac{1}{2}}=\frac{1}{2}$$

I1.2 In the figure, $AB = AD = DC = 4$, $BD = 2a$. Find b, the length of BC.



Let $\angle ADB = \theta$, $\angle CDB = 180° - \theta$ (adj. ∠s on st. line)

In ΔABD, $\cos\theta=\frac{a}{4}=\frac{1}{8}$

Apply cosine formula on ΔBCD.

$b^2 = (2a)^2 + 4^2 - 2(2a)\cdot 4\cdot\cos(180° - \theta)$

$b^2 = 1 + 16 - 2\cdot 4\cdot(-\cos\theta) = 17+8\times\frac{1}{8} = 18$

$b=3\sqrt{2}$

I1.3 It is given that $f(x) = px^3 + qx + 5$ and $f(-7) = \sqrt{2}\,b + 1$. Find c, if $c = f(7)$.

Reference: 2006 FG2.2

$p(-7)^3 + q(-7) + 5 = \sqrt{2}\cdot 3\sqrt{2}+1 = 7$

$-[p(7)^3 + q(7)] = 2$

$c = f(7)$

$= p(7)^3 + q(7) + 5$

$= -2 + 5 = 3$

I1.4 Find the least positive integer d, such that $d^c + 1000$ is divisible by $10 + c$.

$d^3 + 1000$ is divisible by 13

$13\times 77 = 1001 = 1000 + 1^3$

$d = 1$

Individual Event 2

I2.1 If $\frac{x}{(x-1)(x-4)}=\frac{x}{(x-2)(x-3)}$, find x.

Reference: 1998 HI3

$x = 0$ or $(x - 1)(x - 4) = (x - 2)(x - 3)$

$x = 0$ or $x^2 - 5x + 4 = x^2 - 5x + 6$

$x = 0$ or $4 = 6$

$x = 0$

I2.2 If $f(t) = 3\times 52^t$ and $y = f(x)$, find y.

$y = f(0) = 3\times 52^0 = 3$

I2.3 A can finish a job in y days, B can finish a job in $(y + 3)$ days. If they worked together, they can finish the job in z days, find z.

$\frac{1}{z}=\frac{1}{3}+\frac{1}{6}=\frac{1}{2}$

$z = 2$

I2.4 The probability of throwing z dice to score 7 is w, find w.

P(sum of 2 dice = 7) = P((1,6), (2,5), (3,4), (4,3), (5,2), (6, 1)) $=\frac{6}{36}=\frac{1}{6}$

$w=\frac{1}{6}$

Individual Event 3

I3.1 If a = sin 30° + sin 300° + sin 3000°, find a.

$a=\frac{1}{2}-\frac{\sqrt{3}}{2}+\sin(360°\times 8+120°)=\frac{1}{2}-\frac{\sqrt{3}}{2}+\frac{\sqrt{3}}{2}=\frac{1}{2}$

I3.2 It is given that $\frac{x+y}{2}=\frac{z+x}{3}=\frac{y+z}{4}$ and $x+y+z=36a$. Find the value of b, if $b=x+y$.

$x+y=2k$ …… (1)

$z+x=3k$ …… (2)

$y+z=4k$ …… (3)

(1) + (2) + (3): $2(x+y+z)=9k$

$2(36)(\frac{1}{2})=9k$

$k=4$

$b=x+y$

$=2k$

$=2(4)=8$

I3.3 It is given that the equation $x+6+8k=k(x+b)$ has positive integral solution. Find c, the least value of k.

$x+6+8k=k(x+8)$

$(k-1)x=6$

If $k=1$, the equation has no solution

If $k\neq 1$, $x=\frac{6}{k-1}$

The positive integral solution, 6 must be divisible by $k-1$.

The least positive factor of 6 is 1, $c=2$

I3.4 A car has already travelled 40% of its journey at an average speed of $40c$ km/h. In order to make the average speed of the whole journey become 100 km/h, the speed of the car is adjusted to d km/h to complete the rest of the journey. Find d.

Let the total distance be s.

$$\frac{s}{\frac{0.4s}{40(2)}+\frac{0.6s}{d}}=100$$

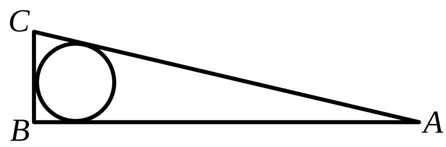
$$\Rightarrow \frac{1}{200}+\frac{3}{5d}=\frac{1}{100}$$

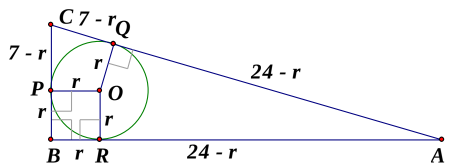
$$\Rightarrow \frac{120}{200d}=\frac{1}{200}$$

$\Rightarrow d=120$

Individual Event 4

I4.1 In triangle ABC, $\angle B = 90°$, $BC = 7$ and $AB = 24$. If r is the radius of the inscribed circle, find r.





Let O be the centre of the inscribed circle, which touches BC, CA, AB at P, Q, R respectively.

$OP \perp BC$, $OQ \perp AC$, $OR \perp AB$ (tangent $\perp$ radius)

$ORBP$ is a rectangle (it has 3 right angles)

$BR = r$, $BP = r$ (opp. sides of rectangle)

$CP = 7 - r$, $AR = 24 - r$

$AC^2 = AB^2 + BC^2$ (Pythagoras' Theorem)

$= 24^2 + 7^2 = 625$

$AC = 25$

$CQ = 7 - r$, $AQ = 24 - r$ (tangent from ext. point)

$CQ + AQ = AC$

$7 - r + 24 - r = 25$

$r = 3$

I4.2 If $x^2 + x - 1 = 0$ and $s = x^3 + 2x^2 + r$, find s.

By division, $s = x^3 + 2x^2 + 3 = (x + 1)(x^2 + x - 1) + 4 = 4$

I4.3 It is given that $F_1 = F_2 = 1$ and $F_n = F_{n-1} + F_{n-2}$, where $n \geq 3$. If $F_t = s + 1$, find t.

$F_t = 4 + 1 = 5$

$F_3 = 1 + 1 = 2$, $F_4 = 2 + 1 = 3$, $F_5 = 3 + 2 = 5$

$t = 5$

I4.4 If $u = \sqrt{t(t+1)(t+2)(t+3)+1}$, find u.

Reference: 1993 HG6, 1996 FG10.1, 2000 FG3.1, 2004 FG3.1, 2012 FI2.3

$u = \sqrt{5\times6\times7\times8+1} = \sqrt{40\times42+1}$

$= \sqrt{(41-1)\times(41+1)+1} = \sqrt{41^2-1+1}$

$u = 41$

Individual Event 5

I5.1 It is given that $\log_7(\log_3(\log_2 x)) = 0$. Find a, if $a = x^{\frac{1}{3}}$.

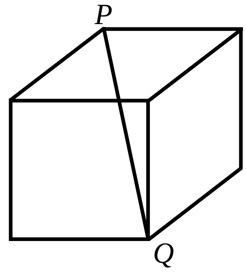
$\log_3(\log_2 x) = 1$

$\log_2 x = 3$

$x = 2^3 = 8$

$a = x^{\frac{1}{3}} = 2$

I5.2 In the figure, PQ is a diagonal of the cube and $PQ = \frac{a}{2}$.



Find b, if b is the total surface area of the cube.

Reference: 1992 HI14, 2003 HI7

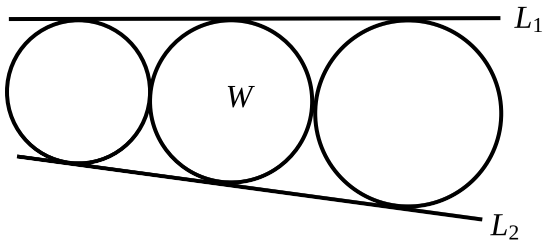
Let the length of the cube be x. $PQ = 1$

$x^2 + x^2 + x^2 = 1$ (Pythagoras' Theorem)

$3x^2 = 1$

The total surface area $= b = 6x^2 = 2$

I5.3 In the figure, L_1 and L_2 are tangents to the three circles. If the radius of the largest circle is 18 and the radius of the smallest circle is $4b$, find c, where c is the radius of the circle W.



Let the centres of the 3 circles be A, B, C as shown in the figure.

L_1 touches the circles at D, E, F as shown.

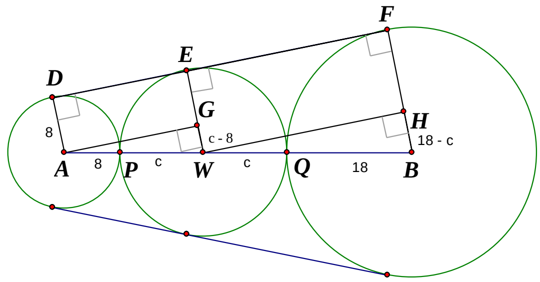
$AD \perp L_1$, $WE \perp L_1$, $BF \perp L_1$ (tangent $\perp$ radius)

Let AB intersects the circle W at P and Q.

$AD = AP = 4b = 8$, $EW = WQ = PW = c$

$QB = BF = 18$ (radii of the circle)

Draw AG // DE, WH // EF as shown



EW // FB (int. $\angle$ supp.)

$\angle AWG = \angle WBH$ (corr. $\angle$s EW // FB)

$AG \perp GW$, $WH \perp HB$ (by construction)

$\Delta AGW \sim \Delta WHB$ (equiangular)

$GW = c - 8$, $BH = c + 18$ (opp. sides of rectangle)

$\frac{c-8}{c+8} = \frac{18-c}{c+18}$ (ratio of sides, $\sim \Delta$)

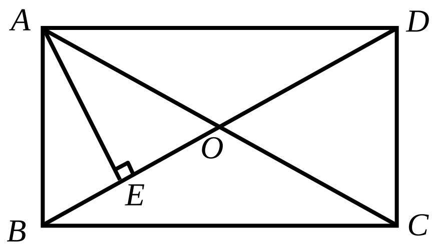
$(c - 8)(c + 18) = (c + 8)(18 - c)$

$c^2 + 10c - 144 = -c^2 + 10c + 144$

$2c^2 = 2(144)$

$c = 12$

I5.4 Refer to the figure, $ABCD$ is a rectangle. $AE \perp BD$ and $BE = EO = \frac{c}{6}$. Find d, the area of the rectangle $ABCD$.



$BO = 4 = OD = AO = OC$ (diagonal of rectangle)

$AE^2 = OA^2 - OE^2 = 4^2 - 2^2 = 12$ (Pythagoras' Theorem)

$AE = 2\sqrt{3}$

$\Delta ABD \cong \Delta CDB$ (R.H.S.)

$d = 2\times$area of ΔABD

$= \frac{2\times(4+4)\cdot 2\sqrt{3}}{2}$

$= 16\sqrt{3}$

Group Event 6

G6.1 $2^a \cdot 9^b$ is a four digit number and its thousands digit is 2, its hundreds digit is a, its tens digit is 9 and its units digit is b, find a, b.

$2^a \cdot 9^b = 2000 + 100a + 90 + b$

If $a = 0$, $9^b = 2090 + b$

$9^3 = 729$, $9^4 = 6561$

$\Rightarrow$ No solution for a

$\therefore$ $a > 0$ and $0 \le b \le 3$, $2000 + 100a + 90 + b$ is divisible by 2

$b = 0$ or 2

If $b = 0$, $2^a = 2090 + 100a$

$2^{10} = 1024$, $2^{11} = 2048$, $2^{12} = 4096$ and $0 \le a \le 9$

$\Rightarrow$ No solution for a

$\therefore$ $b = 2$, $2000 + 100a + 92$ is divisible by 9

$2 + a + 9 + 2 = 9m$, where m is a positive integer

$a = 5$, $b = 2$

Check: $2^5 \cdot 9^2 = 32\times 81 = 2592 = 2000 + 100(5) + 90 + 2$

G6.2 Find c, if $c = \left(1+\frac{1}{2}+\frac{1}{3}\right)\left(\frac{1}{2}+\frac{1}{3}+\frac{1}{4}\right) - \left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}\right)\left(\frac{1}{2}+\frac{1}{3}\right)$.

Reference: 2006 FI4.1

Let $x = 1+\frac{1}{2}+\frac{1}{3}$, $y = 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}$, then $c = x(y-1) - y(x-1) = -x + y = \frac{1}{4}$

G6.3 Find d, if

$$d = \left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{1994}\right)\left(\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{1995}\right) - \left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{1995}\right)\left(\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{1994}\right)$$

$x = 1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{1994}$, $y = 1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{1995}$

$\Rightarrow d = x(y-1) - y(x-1)$

$= -x + y = \frac{1}{1995}$

Group Event 7

G7.1 Let p, q, r be the three sides of triangle PQR. If $p^4 + q^4 + r^4 = 2r^2(p^2 + q^2)$, find a, where $a = \cos^2 R$ and R denotes the angle opposite r.

$\cos R = \dfrac{p^2+q^2-r^2}{2pq}$

$a = \cos^2 R$

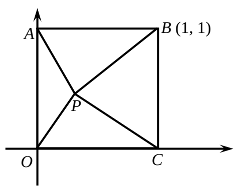
$= \dfrac{(p^2+q^2-r^2)^2}{4p^2q^2}$

$= \dfrac{p^4+q^4+r^4+2p^2q^2-2p^2r^2-2q^2r^2}{4p^2q^2}$

$= \dfrac{2r^2(p^2+q^2)+2p^2q^2-2p^2r^2-2q^2r^2}{4p^2q^2}$

$= \dfrac{2p^2q^2}{4p^2q^2} = \dfrac{1}{2}$

G7.2 Refer to the diagram, P is any point inside the square $OABC$ and b is the minimum value of $PO + PA + PB + PC$, find b.



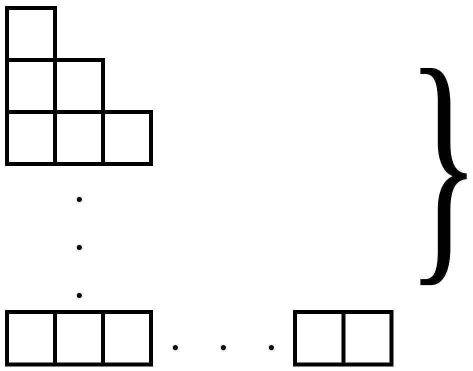
$PO + PA + PB + PC \ge OB + AC$ (triangle inequality)

$= 2\ OB$

$= 2\sqrt{1^2+1^2}$

$\Rightarrow b = 2\sqrt{2}$

G7.3 Identical matches of length 1 are used to arrange the following pattern, if c denotes the total length of matches used, find c.



25 rows

1^{st} row = 4

1^{st} row + 2^{nd} row = 4 + 6 = 10

$1^{st} + 2^{nd} + 3^{rd} = 4 + 6 + 8 = 18$

....

$c = 1^{st} + \ldots + 25^{th}$ row

$= 4 + 6 + 8 + \ldots + [4 + (25 - 1)\cdot 2]$

$= \dfrac{n[2a+(n-1)d]}{2}$

$= \dfrac{25[2(4)+(24)(2)]}{2}$

$= 700$

G7.4 Find d, where $d = \sqrt{111111-222}$.

Reference: 2000 FI2.4

$111111 - 222 = 111(1001 - 2)$

$= 111\times 999$

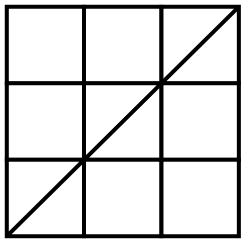
$= 3^2 \times 111^2$

$= 333^2$

$\Rightarrow d = 333$

Group Event 8

Rectangles of length ℓ and breadth b where ℓ, b are positive integers, are drawn on square grid paper. For each of these rectangles, a diagonal is drawn and the number of vertices V intersected (excluding the two end points) is counted (see figure).



$\ell = b = 3$

$V = 2$

G8.1 Find V, when $\ell = 6$, $b = 4$.

Intersection point = (3, 2)

$V = 1$

G.8.2 Find V, when $\ell = 5$, $b = 3$

As 3 and 5 are relatively prime, there is no intersection $\Rightarrow V = 0$

G8.3 When $\ell = 12$ and $1 < b < 12$, find r, the number of different values of b that makes $V = 0$?

$b = 5, 7, 11$ are relatively prime to 12 .

The number of different values of $b = 3$

G8.4 Find V, when $\ell = 108$, $b = 72$.

H.C.F. (108, 72) = 36, $108 = 36\times3$, $72 = 36\times2$

Intersection points = (3, 2), (6, 4), (9, 6), $\cdots$, (105, 70)

$\Rightarrow V = 35$

Group Event 9

A, B, C, D are different integers ranging from 0 to 9 and

Find A, B, C and D.

$$\begin{array}{ccccc} & A & A & B & C \\ - & B & A & C & B \\ \hline & D & A & C & D \end{array}$$

If $A = 0$, then $B \geq 1$, $(AABC) - (BACB) < 0$ rejected

$\therefore A > 0$, consider the hundreds digit:

If there is no borrow digit in the hundreds digit, then $A - A = A$

$\Rightarrow A = 0$ rejected

$\therefore$ There is a borrow digit in the hundreds digit. Also, there is a borrow digit in the thousands digit

$10 + A - 1 - A = A$

$\Rightarrow A = 9$

Consider the thousands digit: $A - 1 - B = D$

$\Rightarrow B + D = 8$ …… (1)

Consider the units digit:

If $C < B$, then $10 + C - B = D$

$\Rightarrow 10 + C = B + D$

$\Rightarrow 10 + C = 8$ by (1)

$\Rightarrow C = -2$ (rejected)

$\therefore C > B$ and there is no borrow digit in the tens digit

Consider the tens digit: $10 + B - C = C$

$10 + B = 2C$ …… (2)

Consider the units digit, $\because C > B \therefore C - B = D$

$C = B + D$

$\Rightarrow C = 8$ by (1)

Sub. $C = 8$ into (2)

$10 + B = 16$

$\Rightarrow B = 6$

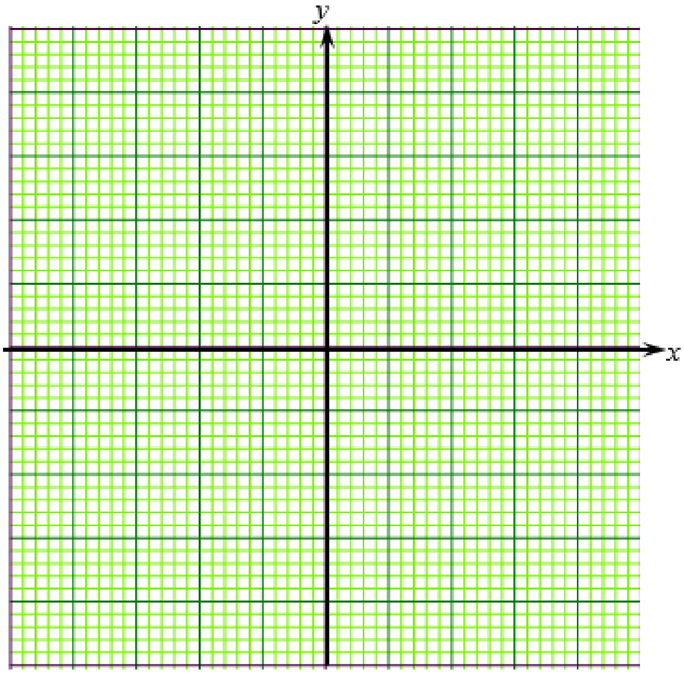
Sub. $B = 6$ into (1), $6 + D = 8$

$\Rightarrow D = 2$

$\therefore A = 9$, $B = 6$, $C = 8$, $D = 2$

Group Event 10

Lattice points are points on a rectangular coordinate plane having both x- and y-coordinates being integers. A moving point P is initially located at (0, 0). It moves 1 unit along the coordinate lines (in either directions) in a single step.



G10.1 If P moves 1 step then P can reach a different lattice points, find a .

(1, 0), (–1, 0), (0, 1), (0, –1)

$a = 4$

G10.2 If P moves not more than 2 steps then P can reach b different lattice points, find b.

(1, 0), (–1, 0), (0, 1), (0, –1),

(1, 1), (1, (1, –1), (–1, 1), (–1, –1)

(2, 0), (–2, 0), (0, 2), (0, –2), (0, 0)

$b = 13$

G10.3 If P moves 3 steps then P can reach c different lattice points, find c.

(1, 0), (–1, 0), (0, 1), (0, –1), (3, 0), (2, 1), (1, 2), (0, 3), (–1, 2), (–2, 1), (–3, 0), (–2, –1), (–1, –2), (0, –3), (1, –2), (2, –1); $c = 4 + 12 = 16$

G10.4 If d is the probability that P lies on the straight line $x + y = 9$ when P advances 9 steps, find d.

Total number of outcomes = 4 + 12 + 20 + 28 + 36 = 100

Favourable outcomes = {(0,9), (1,8), (2,7), $\cdots$, (9,0)}, number = 10

Probability $= \frac{1}{10}$