94-95 Individual	**1**	1111111	**2**	$\frac{2}{3}$	**3**	0, 2	**4**	$\frac{11}{450}$	**5**	$-\frac{4}{3}$
	6	5	**7**	1	**8**	7	**9**	$\frac{1}{6}$	**10**	12

94-95 Group	**1**	1	**2**	132	**3**	$\frac{45}{2}$	**4**	45	**5**	24
	6	5130	**7**	$2\sqrt{3}-3$	**8**	8	**9**	124	**10**	$\frac{2}{3}$

Individual Events

I1 Find the square root of 1234567654321.

Observe the pattern $11^2 = 121$; $111^2 = 12321$, $1111^2 = 1234321$, ……

$1234567654321 = 1111111^2$

$\Rightarrow \sqrt{1234567654321} = 1111111$ (7 digits)

I2 Given that $f\left(\frac{1}{x}\right) = \frac{x}{1-x^2}$, find the value of f(2).

$$f(2) = f\left(\frac{1}{\frac{1}{2}}\right) = \frac{\frac{1}{2}}{1-\left(\frac{1}{2}\right)^2} = \frac{2}{3}$$

I3 Solve $3^{2x} + 9 = 10(3^x)$.

Let $y = 3^x$, then $y^2 = 3^{2x}$

$y^2 + 9 = 10y$

$y^2 - 10y + 9 = 0$

$(y - 1)(y - 9) = 0$

$y = 1$ or $y = 9$

$3^x = 1$ or $3^x = 9$

$x = 0$ or 2

I4 A three-digit number is selected at random. Find the probability that the number selected is a perfect square.

Reference: 1997 FG1.4

The three-digit numbers consists of {100, 101, ⋯ , 999}, altogether 900 numbers.

Favourable outcomes = {100, 121, … , 961} = $\{10^2, 11^2, \cdots, 31^2\}$, 22 outcomes

$$\text{P(perfect squares)} = \frac{22}{900} = \frac{11}{450}$$

I5 Given that $\sin x + \cos x = \frac{1}{5}$ and $0 \le x \le \pi$, find $\tan x$.

Reference: 1992 HI20, 1993 G10, 2007 HI7, 2007 FI1.4, 2014 HG3

$(\sin \alpha + \cos \alpha)^2 = \frac{1}{25}$

$\sin^2 \alpha + 2 \sin \alpha \cos \alpha + \cos^2 \alpha = \frac{1}{25}$

$1 + 2 \sin \alpha \cos \alpha = \frac{1}{25}$

$\sin \alpha \cos \alpha = -\frac{12}{25}$

$25 \sin \alpha \cos \alpha = -12(\sin^2 \alpha + \cos^2 \alpha)$

$12\sin^2 \alpha + 25 \sin \alpha \cos \alpha + 12\cos^2 \alpha = 0$

$(3 \sin \alpha + 4 \cos \alpha)(4 \sin \alpha + 3 \cos \alpha) = 0$

$\tan \alpha = -\frac{4}{3}$ or $-\frac{3}{4}$

Check when $\tan \alpha = -\frac{4}{3}$, then $\sin \alpha = \frac{4}{5}$, $\cos \alpha = -\frac{3}{5}$

LHS $= \sin \alpha + \cos \alpha = \frac{4}{5} + \left(-\frac{3}{5}\right) = \frac{1}{5} =$ RHS

When $\tan \alpha = -\frac{3}{4}$, then $\sin \alpha = \frac{3}{5}$, $\cos \alpha = -\frac{4}{5}$

LHS $= \sin \alpha + \cos \alpha = \frac{3}{5} - \frac{4}{5} = -\frac{1}{5} \ne$ RHS

$\therefore B = \tan \alpha = -\frac{4}{3}$

I6 How many pairs of positive integers x, y are there satisfying $xy - 3x - 2y = 10$?

$xy - 3x - 2y + 6 = 10 + 6$

$(x - 3)(y - 2) = 16$

$x - 3$	$y - 2$	16	x	y
1	16		4	18
2	8		5	10
4	4		7	6
8	2		11	4
16	1		19	3

$\therefore$ There are 5 pairs of positive integers.

I7 x, y are positive integers and $3x + 5y = 123$. Find the least value of $|x - y|$.

$x = 41$, $y = 0$ is a particular solution of the equation.

The general solution is $x = 41 - 5t$, $y = 3t$, where t is any integer.

$|x - y| = |41 - 5t - 3t| = |41 - 8t|$

The least value is $|41 - 8\times5| = 1$.

I8 Find the remainder when 1997^{913} is divided by 10.

Note that $7^1 = 7$, $7^2 = 49$, $7^3 = 343$, $7^4 = 2401$.

Also, $7^{4n+1} \equiv 7 \pmod{10}$, $7^{4n+2} \equiv 9 \pmod{10}$, $7^{4n+3} \equiv 3 \pmod{10}$, $7^{4n} \equiv 1 \pmod{10}$

$1997^{913} \equiv 7^{913} \pmod{10} \equiv 7^{912+1} \equiv 7^{4(228)+1} \equiv 7 \pmod{10}$

The remainder is 7.

I9 In figure 1, if $BC = 3DE$, find the value of r, where $r = \dfrac{\text{Area of } \Delta ADE}{\text{Area of } \Delta BDC}$.

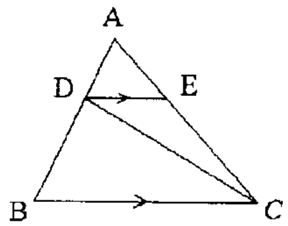


figure 1

$\Delta ADE \sim \Delta ABC$

$$\Rightarrow \frac{\text{Area of } \Delta ADE}{\text{Area of } \Delta ABC} = \left(\frac{1}{3}\right)^2 = \frac{1}{9}$$

$$\frac{\text{Area of } \Delta ADE}{\text{Area of } BCED} = \frac{1}{9-1} = \frac{1}{8} \quad \ldots\ldots (1)$$

$AE : AC = DE : BC = 1 : 3$ (ratio of sides, $\sim\Delta$)

$AE : EC = 1 : 2$

ΔADE and ΔCDE have the same height with base ratio 1 : 2

$$\frac{\text{Area of } \Delta ADE}{\text{Area of } \Delta CDE} = \frac{AE}{CE} = \frac{1}{2} \quad \ldots\ldots (2)$$

$$r = \frac{\text{Area of } \Delta ADE}{\text{Area of } \Delta BDC} = \frac{1}{8-2} = \frac{1}{6} \quad \text{by (1) and (2)}$$

I10 A, B, C, D are points on the sides of the right-angled triangle PQR as shown in figure 2. If $ABCD$ is a square, $QA = 8$ and $BR = 18$, find AB.

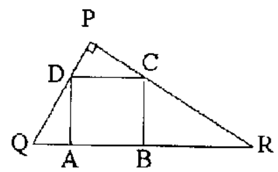


figure 2

Let $\angle BRC = \theta$, then $\angle DQA = 90° - \theta$ ($\angle$s sum of Δ)

$\angle DAQ = 90°$ ($\angle$ of a square), $\angle QDA = \theta$ ($\angle$s sum of Δ)

$BC = BR \tan \theta = 18 \tan \theta = AD$ (opp. sides of square)

$QA = 8 = AD \tan \theta = 18 \tan^2 \theta$

$$\Rightarrow \tan \theta = \frac{2}{3}$$

$$AB = BC = 18 \tan \theta = 18 \times \frac{2}{3} = 12$$

Method 2

It is easy to show that $\Delta PDC \sim \Delta AQD \sim \Delta BCR$ (equiangular)

Let $AB = AD = BC = CD = x$

$PD : PC : x = 8 : x : QD = x : 18 : CR$ (cor. sides, $\sim\Delta$s)

$x^2 = 8 \times 18$

$AB = x = 12$

Group Events

G1 Find the number of positive integral solutions of the equation $x^3 + (x + 1)^3 + (x + 2)^3 = (x + 3)^3$

Expand: $x^3 + x^3 + 3x^2 + 3x + 1 + x^3 + 6x^2 + 12x + 8 = x^3 + 9x^2 + 27x + 27$

$2x^3 - 12x - 18 = 0$

$x^3 - 6x - 9 = 0$; let $f(x) = x^3 - 6x - 9$

$f(3) = 27 - 18 - 9 = 0$ $\therefore$ $x - 3$ is a factor.

By division, $(x - 3)(x^2 + 3x + 3) = 0$

$x = 3$ or $\dfrac{-3\pm\sqrt{-3}}{2}$ (rejected)

$\therefore$ There is **one** positive integral solution $x = 3$.

G2 In figure 1, *ABCD* is a quadrilateral whose diagonals intersect at *O*.

If $\angle AOB = 30°$, $AC = 24$ and $BD = 22$,

find the area of the quadrilateral *ABCD*.

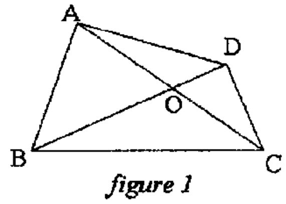


figure 1

The area $= \dfrac{1}{2} 24\times 22\times \sin 30^\circ$

$= 132$

G3 Given that $\dfrac{1}{n}+\dfrac{2}{n}+\dfrac{3}{n}+\cdots+\dfrac{n-1}{n}=\dfrac{n-1}{2}$,

find the value of $\dfrac{1}{2}+\left(\dfrac{1}{3}+\dfrac{2}{3}\right)+\left(\dfrac{1}{4}+\dfrac{2}{4}+\dfrac{3}{4}\right)+\ldots+\left(\dfrac{1}{10}+\cdots+\dfrac{9}{10}\right)$.

Reference: 1996 FG9.4, 2004HG1, 2018 HG9

$\dfrac{1}{2}+\left(\dfrac{1}{3}+\dfrac{2}{3}\right)+\left(\dfrac{1}{4}+\dfrac{2}{4}+\dfrac{3}{4}\right)+\cdots+\left(\dfrac{1}{10}+\cdots+\dfrac{9}{10}\right)$

$=\dfrac{1}{2}+\dfrac{2}{2}+\dfrac{3}{2}+\dfrac{4}{2}+\cdots+\dfrac{9}{2}$

$=\dfrac{1+2+\cdots+9}{2}=\dfrac{45}{2}$

G4 Suppose x and y are positive integers such that $x^2 = y^2 + 2000$, find the least value of x.

Reference: 1993 HI7, 1997 HI1

$x^2 - y^2 = 2000 = 1\times 2000$

$= 2\times 1000 = 4\times 500$

$= 5\times 400 = 8\times 250$

$= 10\times 200 = 16\times 125$

$= 20\times 100 = 25\times 80$

$= 40\times 50$

$(x + y)(x - y) = 2000$

$\because$ x and y are positive integers

$\therefore$ $x + y$ and $x - y$ are also positive integers

$x > y$

x is the least when y is the largest

$\therefore$ The difference between x and y is the largest

$x + y = 50$, $x - y = 40$

Solving, $x = 45$

G5 Given that 37^{100} is a 157-digit number, and 37^{15} is an n-digit number. Find n.

Reference: 2003 FI2.1

Let $y = 37^{100}$, then $\log y = \log 37^{100} = 156 + a$, where $0 \le a < 1$

$100 \log 37 = 156 + a$

$15 \log 37 = \frac{15}{100}(156 + a)$

$\log 37^{15} = 23.4 + 0.15a$

$23 < \log 37^{15} < 24$

37^{15} is a 24 digit number.

$n = 24$.

G6 Given that $1^2 + 2^2 + 3^2 + \dots + n^2 = \frac{n}{6}(n+1)(2n+1)$,

find the value of $19\times21 + 18\times22 + 17\times23 + \dots + 1\times39$.

$19\times21 + 18\times22 + 17\times23 + \dots + 1\times39$

$= (20 - 1)(20 + 1) + (20 - 2)(20 + 2) + (20 - 3)(20 + 3) + \dots + (20 - 19)(20 + 19)$

$= (20^2 - 1^2) + (20^2 - 2^2) + (20^2 - 3^2) + \dots + (20^2 - 19^2)$

$= 20^2 + \cdots + 20^2$ (19 times) $- (1^2 + 2^2 + 3^2 + \cdots + 19^2)$

$= 19\times400 - \frac{19}{6}(20)(39)$

$= 7600 - 2470$

$= 5130$

G7 In figure 2, $ABCD$ is a square where $AB = 1$ and CPQ is an equilateral triangle. Find the area of ΔCPQ.

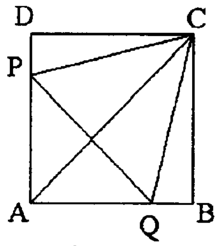


figure 2

Reference: 2008 FI4.4

Let $AQ = AP = x$.

Then $BQ = DP = (1 - x)$

By Pythagoras' Theorem,

$CP = CQ \Rightarrow 1 + (1 - x)^2 = x^2 + x^2$

$2 - 2x + x^2 = 2x^2$

$x^2 + 2x - 2 = 0 \Rightarrow x^2 = 2 - 2x$

$x = -1 + \sqrt{3}$

Area of ΔCPQ = Area of square – area of ΔAPQ – 2 area of ΔCDP

$= 1 - \frac{x^2}{2} - 2\times\frac{1\times(1-x)}{2} = x - \frac{x^2}{2} = x - \frac{2-2x}{2} = 2x - 1$

$= 2\left(-1+\sqrt{3}\right) - 1 = 2\sqrt{3} - 3$

Method 2 Area of $\Delta CPQ = \frac{1}{2}PQ^2 \sin 60^\circ = \frac{1}{2}\left(x^2 + x^2\right)\cdot\frac{\sqrt{3}}{2}$

$= \frac{\sqrt{3}x^2}{2} = \frac{\sqrt{3}\left(1+3-2\sqrt{3}\right)}{2} = \left(2\sqrt{3} - 3\right)$

G8 The number of ways to pay a sum of \$17 by using \$1 coins, \$2 coins and \$5 coins is n. Find n. (Assume that all types of coins must be used each time.)

Suppose we used $x + 1$ \$1 coins, $y + 1$ \$2 coins, $z + 1$ \$5 coins, where x, y, z are non-negative integers. Then $(x + 1) + 2(y + 1) + 5(z + 1) = 17$

$x + 2y + 5z = 9$

$(x, y, z) = (9, 0, 0), (7, 1, 0), (5, 2, 0), (3, 3, 0), (1, 4, 0), (4, 0, 1), (2, 1, 1), (0, 2, 1)$.

Altogether 8 ways.

G9 In figure 3, find the total number of triangles in the 3×3 square.

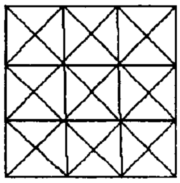
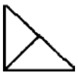


figure 3

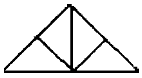
Reference: 1998 HG9

There are 36 smallest triangles with length = 1

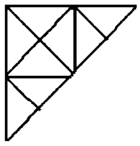
There are 36 triangles with length = $\sqrt{2}$



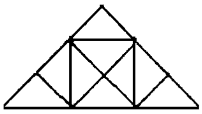
There are 24 triangles with length = 2



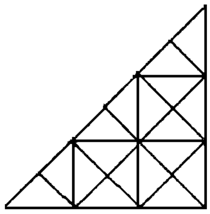
There are 16 triangles with length = $2\sqrt{2}$



There are 8 triangles with length = 3



There are 4 triangles with length = $3\sqrt{2}$



Altogether 124 triangles.

G10 In figure 4, the radius of the quadrant and the diameter of the large semi-circle is 2. Find the radius of the small semi-circle.

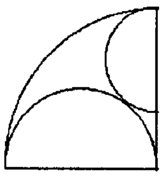


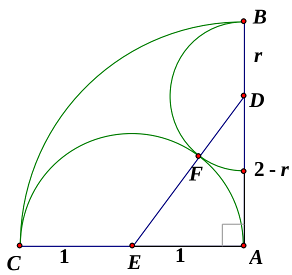
figure 4

Let the radius of the smaller semi-circle be r cm.

Let A, D, E be the centres of the quadrant, the larger and the smaller semi-circles respectively.

$\angle BAC = 90°$

DE intersects the two semicircles at F.



$AE = EC = 1$ cm

$BD = DF = r$ cm

$AC = AB = 2$ cm

$AD = (2 - r)$ cm, $DE = (1 + r)$ cm

$AD^2 + AE^2 = DE^2$ (Pythagoras' theorem)

$1^2 + (2 - r)^2 = (1 + r)^2$

$1 + 4 - 4r + r^2 = 1 + 2r + r^2$

$\Rightarrow r = \frac{2}{3}$