Individual Events

I1	***a***	3	**I2**	***a***	9	**I3**	***a***	5	**I4**	***a***	$\frac{7}{2}$	**I5**	***a***	$\frac{11}{16}$	**ISpare**	***a***	17
	b	18		***b***	3		***b***	3		***b***	2		***b***	2		***b***	9
	c	36		***c***	4		***c***	2		***c***	10		***c***	$\frac{10}{21}$		***c***	1
	d	$\frac{5}{36}$		***d***	15		***d***	16		***d***	$\frac{9}{10}$		***d***	$\frac{23}{28}$		***d***	258

Group Events

G6	***a***	7	**G7**	***a***	2 *see the remark	**G8**	***a***	8	**G9**	***a***	2	**G10**	***a***	1003001
	b	7		***b***	333		***b***	4		***b***	2		***b***	10
	c	12		***c***	1		***c***	7		***c***	39923992		***c***	35
	d	757		***d***	9		***d***	$-\frac{3}{2}$		***d***	885		***d***	92

Individual Event 1

I1.1 The perimeter of an equilateral triangle is exactly the same in length as the perimeter of a regular hexagon. The ratio of the areas of the triangle and the hexagon is 2 : a, find the value of a.

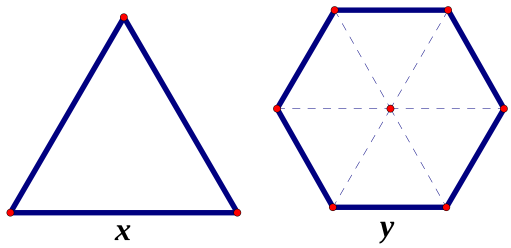
Reference: 2014 FI4.3, 2016 FI2.1

Let the length of the equilateral triangle be x, and that of the regular hexagon be y.

Since they have equal perimeter, $3x = 6y$

$\therefore x = 2y$

The hexagon can be divided into 6 identical equilateral triangles.



Ratio of areas $= \frac{1}{2}x^2 \sin 60^\circ : 6\times\frac{1}{2}y^2 \sin 60^\circ = 2 : a$

$x^2 : 6y^2 = 2 : a$

$(2y)^2 : 6y^2 = 2 : a$

$\Rightarrow a = 3$

I1.2 If $5^x + 5^{-x} = a$ and $5^{3x} + 5^{-3x} = b$, find the value of b.

Reference: 1983 FG7.3, 1998 FG5.2, 2010 FI3.2

$(5^x + 5^{-x})^2 = 3^2$

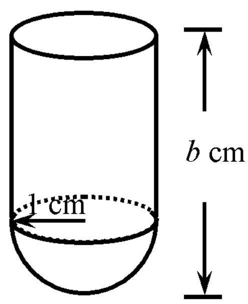
$5^{2x} + 2 + 5^{-2x} = 9$

$5^{2x} + 5^{-2x} = 7$

$b = (5^x + 5^{-x})(5^{2x} - 1 + 5^{-2x})$

$= 3(7 - 1) = 18$

I1.3 The figure shows an open cylindrical tube (radius = 1 cm) with a hemispherical bottom of radius 1 cm. The height of the tube is b cm and the external surface area of the tube is $c\pi$ cm^2. Find the value of c.



$c\pi = 2\pi r\text{o} + 2\pi r^2 = 2\pi(1)(17) + 2\pi(1^2) = 36\pi$

$c = 36$

I1.4 Two fair dice are thrown. Let d be the probability of getting the sum of scores to be $\frac{c}{6}$. Find the value of d.

Sum = 6, d = P(6) = P((1,5), (2, 4), (3, 3), (4, 2), (5, 1)) $= \frac{5}{36}$.

Individual Event 2

I2.1 It is given that m, $n > 0$ and $m + n = 1$. If the minimum value of $\left(1+\frac{1}{m}\right)\left(1+\frac{1}{n}\right)$ is a, find the value of a.

$(m + n)^2 = 1$

$\Rightarrow m^2 + n^2 + 2mn = 1$

$\Rightarrow m^2 + n^2 = 1 - 2mn$

$(m - n)^2 \ge 0$

$\Rightarrow m^2 - 2mn + n^2 \ge 0$

$\Rightarrow 1 - 2mn - 2mn \ge 0$

$\Rightarrow mn \le \frac{1}{4} \Rightarrow \frac{1}{mn} \ge 4$

$\left(1+\frac{1}{m}\right)\left(1+\frac{1}{n}\right) = \frac{(1+m)(1+n)}{mn} = \frac{1+m+n+mn}{mn} = \frac{1+1+mn}{mn} = \frac{2}{mn} + 1 \ge 2\times 4 + 1 = 9$

I2.2 If the roots of the equation $x^2 - (10 + a)x + 25 = 0$ are the square of the roots of the equation $x^2 + bx = 5$, find the positive value of b. **(Reference: 2001 FI3.4)**

$x^2 - 19x + 25 = 0$, roots $= \alpha, \beta$; $\alpha + \beta = 19$, $\alpha\beta = 25$

$x^2 + bx = 5$, roots r, s; $r + s = -b$, $rs = -5$

Now $r^2 = \alpha$, $s^2 = \beta$

$19 = \alpha + \beta = r^2 + s^2 = (r + s)^2 - 2rs = b^2 - 2(-5)$

$b^2 = 9$

$\Rightarrow b = 3$ (positive root)

Method 2 Replace x by $\sqrt{x}$ in $x^2 + bx = 5$

$x + b\sqrt{x} = 5$

$b\sqrt{x} = 5 - x$

$b^2x = 25 - 10x + x^2$

$x^2 - (10 + b^2)x + 25 = 0$, which is identical to $x^2 - 19x + 25 = 0$

$\Rightarrow b = 3$ (positive root)

I2.3 If $(xy - 2)^{b-1} + (x - 2y)^{b-1} = 0$ and $c = x^2 + y^2 - 1$, find the value of c.

Reference: 2005FI4.1, 2006 FI4.2, 2009 FG1.4, 2013 FI1.4, 2015 HG4, 2015 FI1.1

$(xy - 2)^2 + (x - 2y)^2 = 0$

$\Rightarrow xy = 2$ and $x = 2y$

$\Rightarrow 2y^2 = 2$

$\Rightarrow y = \pm 1, x = \pm 2$

$c = x^2 + y^2 - 1 = 4 + 1 - 1 = 4$

I2.4 If $f(x)$ is a polynomial of degree two, $f(f(x)) = x^4 - 2x^2$ and $d = f(c)$, find the value of d.

Let $f(x) = px^2 + qx + r$

$f(f(x)) = p(px^2 + qx + r)^2 + q(px^2 + qx + r) + r \equiv x^4 - 2x^2$

Compare coefficient of x^4 : $p = 1$

Compare coefficient of x^3 : $2q = 0 \Rightarrow q = 0$

Compare the constant: $r^2 + qr + r = 0 \Rightarrow r = 0$ or $r + q + 1 = 0$ …… (1)

Compare coefficient of x: $2qr + q^2 = 0 \Rightarrow q = 0$ or $2r + q = 0$ …… (2)

Sub. $q = 0$ into (1): $r = 0$ or -1

But $q = 0$, $r = -1$ does not satisfy $2r + q = 0$ in (2)

$\therefore$ $(p, q, r) = (1, 0, 0)$ or $(1, 0, -1)$

Sub. $(p, q, r) = (1, 0, 0)$ into $f(f(x)) = x^4 - 2x^2 \equiv x^4$, which is a contradiction $\therefore$ rejected

Sub. $(p, q, r) = (1, 0, -1)$ into $f(f(x)) = x^4 - 2x^2 \equiv (x^2 - 1)^2 - 1$

R.H.S. $= x^4 - 2x^2 + 1 - 1 =$ L.H.S.

$\therefore$ $f(x) = x^2 - 1$; $d = f(4) = 4^2 - 1 = 15$

Individual Event 3

I3.1 If a is a real number and $2a^3 + a^2 - 275 = 0$, find the value of a.

Let $f(a) = 2a^3 + a^2 - 275$; $275 = 5\times5\times11$

$f(5) = 2\times125 + 25 - 275 = 0$

$f(a) = (a - 5)(2a^2 + 11a + 55)$

Δ of $2a^2 + 11a + 55$ is $11^2 - 4(2)(55) < 0$

$\therefore a = 5$

$$\begin{array}{r} 2a^2 + 11a + 55 \\ a-5 \overline{\smash{)}\,2a^3 + a^2 \qquad - 275} \\ \underline{2a^3 - 10a^2 \qquad\qquad} \\ 11a^2 \qquad\quad \\ \underline{11a^2 - 55a \quad} \\ 55a - 275 \\ \underline{55a - 275} \end{array}$$

I3.2 Find the value of b if $3^2\cdot3^5\cdot3^8\cdots3^{3b-1} = 27^a$.

$3^{2+5+8+\ldots(3b-1)} = 3^{3\times5}$

$\because 2 + 5 + 8 = 15$

$\therefore 3b - 1 = 8$

$b = 3$

I3.3 Find the value of c if $\log_b\left(b^c - 8\right) = 2 - c$.

$\log_3\left(3^c - 8\right) = 2 - c$

$\Rightarrow 3^c - 8 = 3^{2-c}$

Let $y = 3^c$; then $3^{2-c} = 3^2\cdot3^{-c} = \frac{9}{y}$

$y - 8 = \frac{9}{y}$

$\Rightarrow y^2 - 8y - 9 = 0$

$\Rightarrow (y - 9)(y + 1) = 0$

$\Rightarrow y = 9$ or -1

$\Rightarrow 3^c = 9$ or -1 (rejected)

$c = 2$

I3.4 If $[(4^c)^c]^c = 2^d$, find the value of d.

$[(4^2)^2]^2 = 2^d$

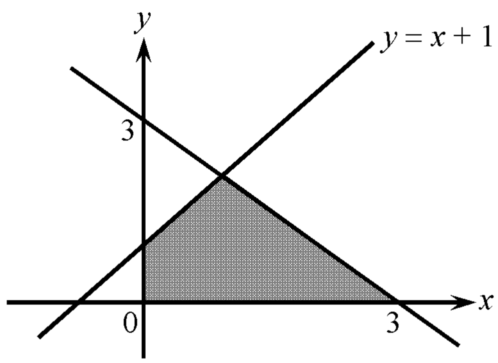
$\Rightarrow 4^8 = 2^d$

$\Rightarrow 2^{16} = 2^d$

$d = 16$

Individual Event 4

I4.1 In the figure, the area of the shaded region is a.



Find the value of a.

Equation of the other straight line is $x + y = 3$

The intersection point is (1, 2)

y-intercept of $y = x + 1$ is 1

Shaded area = area of big triangle – area of small Δ

$$=\frac{1}{2}\cdot 3\times 3-\frac{1}{2}(3-1)\cdot 1=\frac{7}{2}$$

I4.2 If $8^b = 4^a - 4^3$, find the value of b.

$8^b = 4^{3.5} - 4^3 = 4^3\cdot(2 - 1) = 64$

$b = 2$

I4.3 Given that c is the positive root of the equation $x^2 - 100b + \frac{10000}{x^2} = 0$, find the value of c.

$x^4 - 200x^2 + 10000 = 0$

$\Rightarrow (x^2 - 100)^2 = 0$

$\Rightarrow x = 10$

I4.4 If $d=\frac{1}{1\times 2}+\frac{1}{2\times 3}+\cdots+\frac{1}{(c-1)\times c}$, find the value of d.

$$d=\frac{1}{1\times 2}+\frac{1}{2\times 3}+\cdots+\frac{1}{9\times 10}$$

$$=\left(1-\frac{1}{2}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\cdots+\left(\frac{1}{9}-\frac{1}{10}\right)$$

$$=1-\frac{1}{10}=\frac{9}{10}$$

Individual Event 5

I5.1 Four fair dice are thrown. Let *a* be the probability of getting at least half of the outcome of the dice to be even. Find the value of *a*.

P(at least half of the outcome of the dice to be even)

= P(even, even, odd, odd) + P(even, even, even, odd) + P(even, even, even, even)

$=6\times\frac{1}{16}+4\times\frac{1}{16}+\frac{1}{16}=\frac{11}{16}$

I5.2 It is given that $f(x)=\frac{3}{8}x^2(81)^{-\frac{1}{x}}$ and $g(x) = 4\log_{10}(14x) - 2\log_{10}49$.

Find the value of $b = f\{g[16(1-a)]\}$.

$g[16(1-a)] = g[16(1-\frac{11}{16})] = g(5)$

$= 4\log 70 - 2\log 49$

$= 4\log 70 - 4\log 7 = 4\log 10 = 4$

$b = f(g(5)) = f(4) = \frac{3}{8}(4)^2(81)^{-\frac{1}{4}} = 6\times 9^{-\frac{1}{2}} = \frac{6}{3} = 2$

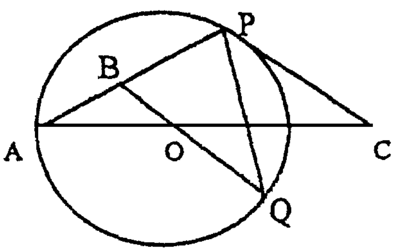
I5.3 Let $c=\frac{1}{b^2-1}+\frac{1}{(2b)^2-1}+\frac{1}{(3b)^2-1}+\cdots+\frac{1}{(10b)^2-1}$, find the value of *c*.

Hint: $\frac{1}{x^2-1}=\frac{1}{2}\left(\frac{1}{x-1}-\frac{1}{x+1}\right)$

$c=\frac{1}{2^2-1}+\frac{1}{4^2-1}+\frac{1}{6^2-1}+\cdots+\frac{1}{20^2-1}=\frac{1}{2}\left(1-\frac{1}{3}\right)+\frac{1}{2}\left(\frac{1}{3}-\frac{1}{5}\right)+\frac{1}{2}\left(\frac{1}{5}-\frac{1}{7}\right)+\cdots+\frac{1}{2}\left(\frac{1}{19}-\frac{1}{21}\right)$

$=\frac{1}{2}\left(1-\frac{1}{21}\right)=\frac{10}{21}$

I5.4 In the following diagram, *PC* is a tangent to the circle (centre *O*) at the point *P*, and ΔABO is an isosceles triangle, $AB = OB$, $\angle PCO = c$ $(=\frac{10}{21})$ and $d = \angle QPC$, where *c*, *d* are radian measures. Find the value of *d*. (Take $\pi=\frac{22}{7}$)



$\because AB = OB$, let $\angle BAO = \angle AOB = \theta$ (base $\angle$s, isos. Δ)

Join *OP*. $\angle OPC = \frac{\pi}{2}$ (tangent $\perp$ radius)

$\because OA = OP$ (radii), $\angle OPA = \theta$ (base $\angle$s, isos. Δ)

In ΔAPC, $\theta + \theta + \frac{\pi}{2} + c = \pi$ ($\angle$s sum of Δ)

$2\theta = \frac{\pi}{2}-\frac{10}{21}=\frac{11}{7}-\frac{10}{21}=\frac{23}{21}\Rightarrow\theta=\frac{23}{42}$

$\angle COQ = \theta = \frac{23}{42}$ (vert. opp. $\angle$s)

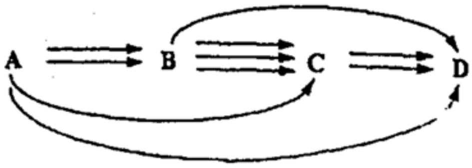
Join *AQ*. $\angle QAO = \frac{\theta}{2} = \frac{23}{84}$ ($\angle$ at centre twice $\angle$ at $\odot^{ce}$)

$d = \angle BAO + \angle QAO$ ($\angle$ in alt. segment)

$=\theta+\frac{\theta}{2}=\frac{23}{42}+\frac{23}{84}=\frac{23}{28}$

Spare Event (Individual)

IS.1 From the following figure, determine the number of routes a from A to D.



$a = 2\times3\times2 + 1\times2 + 1 + 2\times1 = 17$

IS.2 If $\sin(2b° + 2a°) = \cos(6b° - 16°)$, where $0 < b < 90$, find the value of b.

$\cos(90° - 2b° - 34°) = \cos(6b° - 16°)$

$56 - 2b = 6b - 16$

$72 = 8b$

$b = 9$

IS.3 The lines $(bx - 6y + 3) + k(x - y + 1) = 0$, where k is any real constant, pass through a fixed point $P(c, m)$, find the value of c.

The fixed point is the intersection of $9x - 6y + 3 = 0$ … (1) and $x - y + 1 = 0$ … (2)

$(1)\div3 - 2(2)$: $x - 1 = 0$

$x = 1 \Rightarrow c = 1$

IS.4 It is known that $d^2 - c = 257\times259$. Find the positive value of d.

$d^2 - 1 = 257\times259$

$= (258 - 1)(258 + 1)$

$= 258^2 - 1$

$d = 258$

Group Event 6

G6.1 The number of eggs in a basket was a. Eggs were given out in three rounds. In the first round half of egg plus half an egg were given out. In the second round, half of the remaining eggs plus half an egg were given out. In the third round, again, half of the remaining eggs plus half an egg were given out. The basket then became empty. Find a.

$$\frac{a}{2}+\frac{1}{2}+\frac{1}{2}\left(\frac{a}{2}-\frac{1}{2}\right)+\frac{1}{2}+\frac{1}{2}\left\{\frac{a}{2}-\frac{1}{2}-\left[\frac{1}{2}\left(\frac{a}{2}-\frac{1}{2}\right)+\frac{1}{2}\right]\right\}+\frac{1}{2}=a$$

$$\frac{3}{2}+\frac{1}{2}\left(\frac{a}{2}-\frac{1}{2}\right)+\frac{1}{2}\left\{\frac{a}{2}-\frac{1}{2}-\left[\frac{1}{2}\left(\frac{a}{2}-\frac{1}{2}\right)+\frac{1}{2}\right]\right\}=\frac{a}{2}$$

$$3+\left(\frac{a}{2}-\frac{1}{2}\right)+\left\{\frac{a}{2}-\frac{1}{2}-\left[\frac{1}{2}\left(\frac{a}{2}-\frac{1}{2}\right)+\frac{1}{2}\right]\right\}=a$$

$$6+a-1+a-1-\left[\left(\frac{a}{2}-\frac{1}{2}\right)+1\right]=2a$$

$$3-\left(\frac{a}{2}-\frac{1}{2}\right)=0$$

$a = 7$

G6.2 If $p - q = 2$; $p - r = 1$ and $b = (r - q)[(p - q)^2 + (p - q)(p - r) + (p - r)^2]$. Find the value of b.

$b = [p - q - (p - r)][(p - q)^2 + (p - q)(p - r) + (p - r)^2]$

$= (2 - 1)[2^2 + 2\cdot 1 + 1^2]$

$= 2^3 - 1^3 = 7$

G6.3 If n is a positive integer, $m^{2n} = 2$ and $c = 2m^{6n} - 4$, find the value of c.

$c = 2m^{6n} - 4$

$= 2(m^{2n})^3 - 4$

$= 2\times 2^3 - 4 = 12$

G6.4 If r, s, t, u are positive integers and $r^5 = s^4$, $t^3 = u^2$, $t - r = 19$ and $d = u - s$, find the value of d.

Reference: 1998 HG4

Let $w=u^{\frac{1}{15}}$, $v=s^{\frac{1}{15}}$, then $t=u^{\frac{2}{3}}=u^{\frac{10}{15}}=w^{10}$, $r=s^{\frac{4}{5}}=s^{\frac{12}{15}}=v^{12}$

$t - r = 19 \Rightarrow w^{10} - v^{12} = 19$

$\Rightarrow (w^5 + v^6)(w^5 - v^6) = 19\times 1$

$\because$ 19 is a prime number, $w^5 + v^6 = 19$, $w^5 - v^6 = 1$

Solving these equations give $w^5 = 10$, $v^6 = 9 \Rightarrow w^5 = 10$, $v^3 = 3$

$u = w^{15} = 1000$, $s = v^{15} = 3^5 = 729$

$d = u - s = 1000 - 243 = 757$

Group Event 7

G7.1 If the two distinct roots of the equation $ax^2 - mx + 1996 = 0$ are primes, find the value of a.

Reference: 1996 HG8, 2001 FG4.4, 2005 FG1.2, 2012 HI6

Let the roots be α, β. $\alpha + \beta = \frac{m}{a}$, $\alpha\beta = \frac{1996}{a}$

$1996 = 4\times499$ and 499 is a prime

$\therefore$ $a = 1, 2, 4, 499, 998$ or 1996

When $a = 1$, $\alpha\beta = 1996$, which cannot be expressed as a product of two primes $\therefore$ rejected

When $a = 2$, $\alpha\beta = 998$; $\alpha = 2$, $\beta = 499$ (accepted)

When $a = 4$, $\alpha\beta = 499$, which cannot be expressed as a product of two primes $\therefore$ rejected

When $a = 499$, $\alpha\beta = 4$, $\alpha = 2$, $\beta = 2$ (not distinct roots, rejected)

When $a = 998$, $\alpha\beta = 2$, which cannot be expressed as a product of two primes $\therefore$ rejected

When $a = 1996$, $\alpha\beta = 1$, which cannot be expressed as a product of two primes $\therefore$ rejected

Remark: the original question is:

If the two roots of the equation $ax^2 - mx + 1996 = 0$ are primes, find the value of a.

$a = 2$ or 499 (Not unique solution)

G7.2 A six-digit figure 111aaa is the product of two consecutive positive integers b and $b + 1$, find the value of b.

Reference: 2001 FG2.3

$111222 = 111000 + 222 = 111\times1000 + 2\times111 = 111\times1002 = 111\times3\times334 = 333\times334$; $b = 333$

G7.3 If p, q, r are non-zero real numbers;

$p^2 + q^2 + r^2 = 1$, $p\left(\frac{1}{q}+\frac{1}{r}\right)+q\left(\frac{1}{r}+\frac{1}{p}\right)+r\left(\frac{1}{p}+\frac{1}{q}\right)+3=0$ and $c = p + q + r$,

find the largest value of c.

The second equation becomes: $\frac{p^2(r+q)+q^2(p+r)+r^2(q+p)+3pqr}{pqr}=0$

$p^2(c - p) + q^2(c - q) + r^2(c - r) + 3pqr = 0$

$c(p^2 + q^2 + r^2) - (p^3 + q^3 + r^3 - 3pqr) = 0$

$c - (p + q + r)(p^2 + q^2 + r^2 - pq - qr - pr) = 0$

$c - c[1 - (pq + qr + pr)] = 0$

$c(pq + qr + pr) = 0$

$\frac{c}{2}[(p + q + r)^2 - (p^2 + q^2 + r^2)] = 0$

$c(c^2 - 1) = 0$

$c = 0, 1$ or -1

Maximum $c = 1$

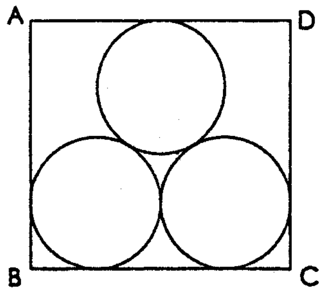
G7.4 If the units digit of 7^{14} is d, find the value of d.

$7^1 = 7$, $7^2 = 49$, $7^3 = 343$, $7^4 = 2401$; the units digit repeat in the pattern 7, 9, 3, 1, …

$7^{14} = (7^4)^3 \times 7^2$ $\therefore$ $d = 9$

Group Event 8 In this question, all unnamed circles are unit circles.

G8.1 If the area of the rectangle *ABCD* is $a + 4\sqrt{3}$, find the value of *a*.



The lines joining the centres form an equilateral triangle, side = 2.

$AB = 2 + 2\sin 60° = 2 + \sqrt{3}$

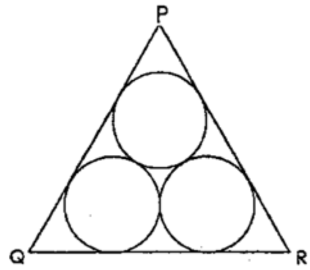
$BC = 4$

Area of $ABCD = (2 + \sqrt{3})\times 4 = 8 + 4\sqrt{3}$

$a = 8$

G8.2 If the area of the equilateral triangle *PQR* is $6 + b\sqrt{3}$,

find the value of *b*.



Reference: 1997 HG9, 2021 P1Q5

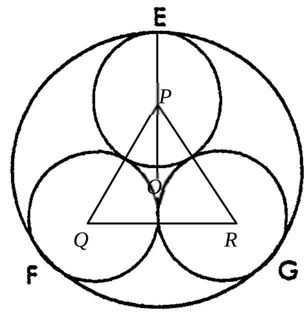
$PQ = 2 + 2\tan 60° = 2 + 2\sqrt{3}$

Area of $PQR = \frac{1}{2}\left(2+2\sqrt{3}\right)^2 \sin 60^\circ = 2\left(1+2\sqrt{3}+3\right)\cdot\frac{\sqrt{3}}{2}$

$6 + b\sqrt{3} = 6 + 4\sqrt{3}$

$b = 4$

G8.3 If the area of the circle *EFG* is $\frac{\left(c+4\sqrt{3}\right)\pi}{3}$, find the value of *c*.



Let the centre be *O*, the equilateral triangle formed by the lines joining the centres be *PQR*, the radius be *r*.

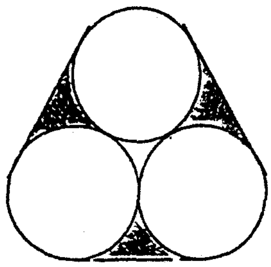
$r = OE = OP + PE = 1\sec 30° + 1 = \frac{2}{\sqrt{3}} + 1 = \frac{2+\sqrt{3}}{\sqrt{3}}$

Area of circle $= \pi\cdot\frac{\left(2+\sqrt{3}\right)^2}{3} = \frac{\left(7+4\sqrt{3}\right)\pi}{3}$

$c = 7$

G8.4 If all the straight lines in the diagram below are common tangents to the two circles, and the area of the shaded part is $6 + d\pi$,

find the value of *d*.



There are three identical shaded regions.

One shaded part = area of rectangle – area of semi-circle

$$= 2\cdot 1 - \frac{1}{2}\pi(1)^2 = 2 - \frac{\pi}{2}$$

Total shaded area $= 3\times\left(2-\frac{\pi}{2}\right) = 6 - \frac{3}{2}\pi$

$d = -\frac{3}{2}$

Group Event 9

G9.1 If $(1995)^a + (1996)^a + (1997)^a$ is divisible by 10, find the least possible integral value of a.

Reference: 2019 FG3.4

The unit digit of $(1995)^a + (1996)^a + (1997)^a$ is 0.

For any positive integral value of a, the units digit of 1995^a is 5, the units digit of 1996^a is 6.

The units digit of 1997^a repeats in the pattern 7, 9, 3, 1, …

$5 + 6 + 9 = 20$

So the least possible integral value of a is 2.

G9.2 If the expression $(x^2 + y^2)^2 \le b(x^4 + y^4)$ holds for all real values of x and y,

find the least possible integral value of b.

$b(x^4 + y^4) - (x^2 + y^2)^2 = b(x^4 + y^4) - (x^4 + y^4 + 2x^2y^2) = (b - 1)x^4 - 2x^2y^2 + (b - 1)y^4$

If $b = 1$, the expression $= -2x^2y^2$ which cannot be positive for all values of x and y.

If $b \ne 1$, discriminant $= (-2)^2 - 4(b - 1)^2 = 4(1 - b^2 + 2b - 1) = -4b(b - 2)$

In order that the expression is always non-negative, $(b - 1) > 0$ and discriminant ≤ 0

$b > 1$ and $-4b(b - 2) \le 0$

$b > 1$ and ($b \le 0$ or $b \ge 2$)

$\therefore\ b \ge 2$, the least possible integral value of b is 2.

G9.3 If $c = 1996\times19971997 - 1995\times19961996$, find the value of c.

Reference: 1998 FG2.2

$c = 1996\times1997\times1001 - 1995\times1996\times1001 = 1001\times1996\times(1997 - 1995)$

$= 3992\times1001 = 39923992$

G9.4 Find the sum d where

$$d=\left(\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots+\frac{1}{60}\right)+\left(\frac{2}{3}+\frac{2}{4}+\frac{2}{5}+\cdots+\frac{2}{60}\right)+\left(\frac{3}{4}+\frac{3}{5}+\cdots+\frac{3}{60}\right)+\cdots+\left(\frac{58}{59}+\frac{58}{60}\right)+\frac{59}{60}$$

Reference: 1995 HG3, 2004 HG1, 2018 HG9

$$d=\frac{1}{2}+\left(\frac{1}{3}+\frac{2}{3}\right)+\left(\frac{1}{4}+\frac{2}{4}+\frac{3}{4}\right)+\left(\frac{1}{5}+\frac{2}{5}+\frac{3}{5}+\frac{4}{5}\right)+\cdots+\left(\frac{1}{60}+\frac{2}{60}+\cdots+\frac{59}{60}\right)$$

$$=\frac{1}{2}+\frac{\frac{3\times2}{2}}{3}+\frac{\frac{4\times3}{2}}{4}+\frac{\frac{5\times4}{2}}{5}+\cdots+\frac{\frac{60\times59}{2}}{60}$$

$$=\frac{1}{2}+\frac{2}{2}+\frac{3}{2}+\frac{4}{2}+\cdots+\frac{59}{2}$$

$$=\frac{1}{2}(1+2+3+4+\cdots+59)$$

$$=\frac{1}{2}\times\frac{1}{2}\cdot60\cdot59=885$$

Group Event 10

G10.1 It is given that $3\times4\times5\times6 = 19^2 - 1$

$4\times5\times6\times7 = 29^2 - 1$

$5\times6\times7\times8 = 41^2 - 1$

$6\times7\times8\times9 = 55^2 - 1$

If $a^2 = 1000\times1001\times1002\times1003 + 1$, find the value of a.

Reference: 1993 HG6, 1995 FI4.4, 2000 FG3.1, 2004 FG3.1, 2012 FI2.3

$19 = 4\times5 - 1$; $29 = 5\times6 - 1$; $41 = 6\times7 - 1$; $55 = 5\times6 - 1$

$a^2 = (1001\times1002 - 1)^2 - 1 + 1 = (1001\times1002 - 1)^2$

$a = 1003002 - 1 = 1003001$

G10.2 Let $f(x) = x^9 + x^8 + x^7 + x^6 + x^5 + x^4 + x^3 + x^2 + x + 1$.

When $f(x^{10})$ is divided by $f(x)$, the remainder is b. Find the value of b.

Reference: 2016 FI3.1

Consider the roots of $f(x) = x^9 + x^8 + x^7 + x^6 + x^5 + x^4 + x^3 + x^2 + x + 1 = 0$

f(x) can be rewritten as $\dfrac{x^{10}-1}{x-1} = 0$, where $x \neq 1$

There are 9 roots $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7, \alpha_8, \alpha_9$: where $\alpha_i^{10} = 1$and $\alpha_i \neq 1$ for $1 \leq i \leq 9$

Let $f(x^{10}) = f(x)\, Q(x) + a_8 x^8 + a_7x^7 + a_6x^6 + a_5x^5 + a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0$.

$f(\alpha_i^{10}) = f(\alpha_i)\, Q(\alpha_i) + a_8\, \alpha_i^8 + a_7\alpha_i^7 + a_6\alpha_i^6 + a_5\alpha_i^5 + a_4\alpha_i^4 + a_3\alpha_i^3 + a_2\alpha_i^2 + a_1\alpha_i + a_0$, $1 \leq i \leq 9$

$f(1) = 0\cdot Q(\alpha_i) + a_8\, \alpha_i^8 + a_7\alpha_i^7 + a_6\alpha_i^6 + a_5\alpha_i^5 + a_4\alpha_i^4 + a_3\alpha_i^3 + a_2\alpha_i^2 + a_1\alpha_i + a_0$ for $1 \leq i \leq 9$

$a_8\, \alpha_i^8 + a_7\alpha_i^7 + a_6\alpha_i^6 + a_5\alpha_i^5 + a_4\alpha_i^4 + a_3\alpha_i^3 + a_2\alpha_i^2 + a_1\alpha_i + a_0 = 10$ for $1 \leq i \leq 9$

$\therefore$ α_i ($1\leq i\leq 9$) are the roots of $a_8 x^8 + a_7x^7 + a_6x^6 + a_5x^5 + a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0 - 10 = 0$

Since a polynomial of degree 8 has at most 8 roots and it is satisfied by α_i for $1 \leq i \leq 9$.

$\therefore$ $a_8 x^8 + a_7x^7 + a_6x^6 + a_5x^5 + a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0 - 10$ must be a zero polynomial.

$a_8 = 0, a_7 = 0, a_6 = 0, a_5 = 0, a_4 = 0, a_3 = 0, a_2 = 0, a_1 = 0, a_0 = 10$

The remainder when $f(x^{10})$ is divided by $f(x)$ is $a_0 = 10$.

Method 2 (Provided by Pui Ching Middle School 李國柱老師)

$f(x^{10}) = x^{90} + x^{80} + x^{70} + x^{60} + x^{50} + x^{40} + x^{30} + x^{20} + x^{10} + 1$

$= x^{90} - 1 + x^{80} - 1 + x^{70} - 1+ x^{60} - 1 + x^{50} - 1 + x^{40} - 1 + x^{30} - 1 + x^{20} - 1 + x^{10} - 1 + 10$

$= (x^{10} - 1)g_1(x) + (x^{10} - 1)g_2(x) + (x^{10} - 1)g_3(x) + \cdots + (x^{10} - 1)g_9(x) + 10$

where $g_1(x) = x^{80} + x^{70} + \cdots + x^{10} + 1$, $g_2(x) = x^{70} + x^{60} + \cdots + x^{10} + 1$, $\cdots$, $g_9(x) = 1$

$f(x^{10}) = (x^{10} - 1)[g_1(x) + g_2(x) + \cdots + g_9(x)] + 10$

$= f(x)(x - 1)[g_1(x) + g_2(x) + \cdots + g_9(x)] + 10$

The remainder is 10 .

Method 3 (Provided by Pui Ching Middle School 李國柱老師)

Clearly $f(1) = 10$.

By division algorithm, $f(x) = (x - 1)Q(x) + 10$, where $Q(x)$ is a polynomial

$f(x^{10}) = (x^{10} - 1)Q(x^{10}) + 10$

$= (x - 1)(x^9 + x^8 + x^7 + x^6 + x^5 + x^4 + x^3 + x^2 + x + 1)Q(x^{10}) + 10$

$= f(x)(x - 1)Q(x^{10}) + 10$

The remainder is 10 .

G10.3 The fraction $\frac{p}{q}$ is in its simplest form. If $\frac{7}{10}<\frac{p}{q}<\frac{11}{15}$ where q is the smallest possible positive integer and $c = pq$. Find the value of c.

Reference 2005 HI1, 2010 HG7

$\frac{7}{10}<\frac{p}{q}<\frac{11}{15} \Rightarrow 1-\frac{7}{10}>1-\frac{p}{q}>1-\frac{11}{15}$

$\frac{3}{10}>\frac{q-p}{q}>\frac{4}{15} \Rightarrow \frac{10}{3}<\frac{q}{q-p}<\frac{15}{4}$

$\frac{1}{3}<\frac{q}{q-p}-3<\frac{3}{4} \Rightarrow \frac{1}{3}<\frac{3p-2q}{q-p}<\frac{3}{4}$

$\frac{3p-2q}{q-p}=\frac{1}{2} \Rightarrow 3p-2q=1,\ q-p=2$

Solving the equations gives $p = 5$, $q = 7$

$\frac{7}{10}<\frac{5}{7}<\frac{11}{15}$, $c = 35$

G10.4 A positive integer d when divided by 7 will have 1 as its remainder; when divided by 5 will have 2 as its remainder and when divided by 3 will have 2 as its remainder. Find the least possible value of d.

$d = 7m + 1 = 5n + 2 = 3r + 2$

$7m - 5n = 1$ …… (1)

$5n = 3r$ …… (2)

From (2), $n = 3k$, $r = 5k$

Sub. $n = 3k$ into (1), $7m - 5(3k) = 1$

$\Rightarrow 7m - 15k = 1$

$-14 + 15 = 1 \Rightarrow$ A possible solution is $m = -2$, $k = -1$

$m = -2 + 15t$, $k = -1 + 7t$

When $t = 1$, $m = 13$, $k = 6$, $n = 18$, $r = 30$.

The least possible value of $d = 3(30) + 2 = 92$