| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| **97-98 Individual** | **1** | 2 | **2** | 40 | **3** | –12 | **4** | 3 | **5** | $-\frac{1}{2}$ |
| | **6** | 466 | **7** | 19 | **8** | 3 | **9** | 2 | **10** | 744 |

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| **97-98 Group** | **1** | 2 | **2** | 12 | **3** | 27 | **4** | 64 | **5** | 14 |
| | **6** | 14 | **7** | $-\frac{1}{2}$ | **8** | 1 | **9** | 20 | **10** | 19 |

**Individual Events**

**I1** Given that $x^3 - 5x^2 + 2x + 8$ is divisible by $(x - a)$ and $(x - 2a)$, where $a$ is an integer, find the value of $a$.

Let $f(x) = x^3 - 5x^2 + 2x + 8$

$f(-1) = -1 - 5 - 2 + 8 = 0 \Rightarrow x + 1$ is a factor

$f(2) = 8 - 20 + 4 + 8 = 0 \Rightarrow x - 2$ is a factor

$f(x) = (x + 1)(x - 2)(x - 4)$

$a = 2$

**I2** Given that 8, $a$, $b$ form an A.P. and $a$, $b$, 36 form a G.P. If $a$ and $b$ are both positive numbers, find the sum of $a$ and $b$.

$a = \frac{8+b}{2}$ … (1); $b^2 = 36a$ … (2)

Sub. (1) into (2): $b^2 = 18(8 + b)$

$b^2 - 18b - 144 = 0$

$(b + 6)(b - 24) = 0$

$b = -6$ (rejected) or $b = 24$

$a = \frac{8+24}{2} = 16$

$a + b = 40$

**I3** Find the smallest real root of the following equation: $\frac{x}{(x-4)(x+3)} = \frac{x}{(x+4)(x-6)}$.

**Reference: 1995 FI2.1**

$x(x + 4)(x - 6) = x(x - 4)(x + 3)$

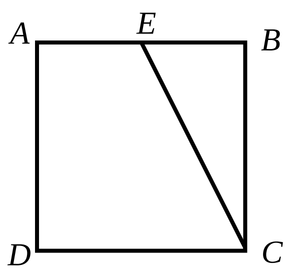
$x(x^2 - 2x - 24) = x(x^2 - x - 12)$

$0 = x(x + 12)$

$x = 0$ or $-12$

The smallest root = –12

**I4** In figure 1, $ABCD$ is a square. $E$ is a point on $AB$ such that $BE = 1$ and $CE = 2$. Find the area of the square $ABCD$.




$BC^2 = 2^2 - 1^2$ (Pythagoras' theorem on $\Delta BCE$)

Area of the square $= BC^2 = 3$

**I5** If $2x+3=\sqrt{2+\sqrt{2+\sqrt{2+\cdots}}}$, find the value of $x$.

$(2x+3)^2=2+\sqrt{2+\sqrt{2+\sqrt{2+\cdots}}}=2+(2x+3)$

$4x^2+12x+9=2x+5$

$4x^2+10x+4=0$

$2x^2+5x+2=0$

$(2x+1)(x+2)=0$

$x=-\frac{1}{2}$ or $-2$

$\because 2x+3=\sqrt{2+\sqrt{2+\sqrt{2+\cdots}}}>0$

$\therefore x\neq -2$, $x=-\frac{1}{2}$ only

**I6** Given that $n$ is a positive integer which is less than 1000. If $n$ is divisible by 3 or 5, find the number of possible values of $n$. **(Reference: 1993 FG8.3-4, 1994 FG8.1-2, 2015 FI3.1)**

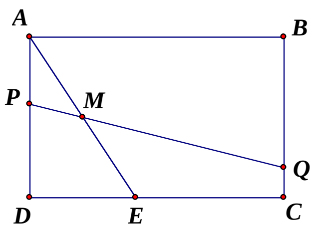
Number of multiples of 3 = 333

Number of multiples of 5 = 199

Number of multiples of 15 = 66

Number of possible $n$ = 333 + 199 – 66 = 466

**I7** In figure 2, $ABCD$ is a rectangle with $CD = 12$. $E$ is a point on $CD$ such that $DE = 5$. $M$ is the mid-point of $AE$ and $P$, $Q$ are points on $AD$ and $BC$ respectively such that $PMQ$ is a straight line. If $PM : MQ = 5 : k$, find the value of $k$.




Draw a straight line $HMG$ // $CD$ ($H$ lies on $AD$, $G$ lies on $BC$)

$AH = HD$ (Intercept theorem)

$\Delta PHM \sim \Delta QGM$ (equiangular)

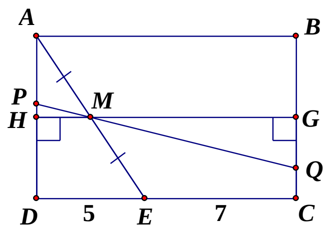
$\Delta AHM \sim \Delta ADE$ (equiangular)

$PM : MQ = HM : MG$ (ratio of sides, ~Δ's)

$\quad = \frac{1}{2}DE : (HG - HM)$ (ratio of sides, ~Δ's)

$\quad = 2.5 : (12 - 2.5) = 5 : 19$ (opp. sides, rectangle)

$k = 19$




**I8** Find the last digit of the value of $6^{20}-5^{12}-8$.

$6^1=6$, $6^2=36$, …, the last digit of $6^{20}$ is 6, the last digit of $5^{12}$ is 5.

The last digit of the number is $6-5-8 \pmod{10} = -7 = 3 \pmod{10}$

**I9** Let $a$ be the positive root of the equation $\sqrt{\frac{x+2}{x-1}}+\sqrt{\frac{x-1}{x+2}}=\frac{5}{2}$, find the value of $a$.

Cross multiplying: $2(x+2+x-1)=5\sqrt{(x-1)(x+2)}$

$4(4x^2+4x+1)=25(x^2+x-2)$

$\Rightarrow 9x^2+9x-54=0$

$\Rightarrow 9(x-2)(x+3)=0$

$\Rightarrow a=x=2$

**I10** Find the sum of all positive factors of 240.

**Reference 1993 HI8, 1994 FI3.2, 1997 HI3, 1998 FI1.4, 2002 FG4.1, 2005 FI4.4**

$240=2^4\times 3\times 5$

Positive factors are in the form $2^a3^b5^c$, $0\le a\le 4$, $0\le b, c\le 1$, $a, b, c$ are integers.

Sum of positive factors $=(1+2+2^2+2^3+2^4)(1+3)(1+5)=31\times 4\times 6=744$

**Group Events**

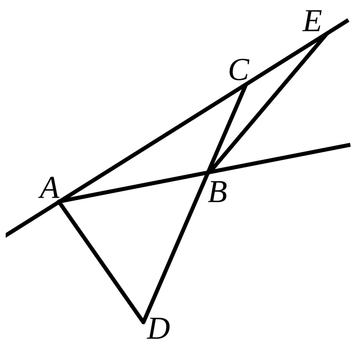
**G1** If $x+\frac{1}{x}=2$, find the value of $x^3+\frac{1}{x^3}$.

**Reference: 1984 FG10.2**

$x^2+\frac{1}{x^2}=\left(x+\frac{1}{x}\right)^2-2=2$

$x^3+\frac{1}{x^3}=\left(x+\frac{1}{x}\right)\left(x^2-1+\frac{1}{x^2}\right)=2\times(2-1)=2$

**G2** In Figure 1, $ABC$ is a triangle. $AD$ and $BE$ are the bisectors of the exterior angles $A$ and $B$ respectively meeting $CB$ and $AC$ produced at $D$ and $E$. Let $AD = BE = AB$ and $\angle BAC = a°$. Find the value of $a$. **Reference: 1986 上海市初中數學競賽**




$\angle BAD=\frac{180^\circ-a^\circ}{2}=90^\circ-\frac{a^\circ}{2}$ (adj. ∠s on st. line, ∠ bisector)

$\angle ABD=\frac{180^\circ-\angle BAD}{2}=45^\circ+\frac{a^\circ}{4}$ (∠ sum of Δ$ABD$, base ∠s isos. Δ)

$\angle CBE=\frac{\angle ABD}{2}=22.5^\circ+\frac{a^\circ}{8}$ (vert. opp. ∠s, bisector)

$\angle ABE = \angle ABC + \angle CBE = 180° - \angle ABD + \angle CBE$

$=135^\circ-\frac{a^\circ}{4}+22.5^\circ+\frac{a^\circ}{8}=157.5^\circ-\frac{a^\circ}{8}$

$\angle AEB = a°$ (base ∠s isosceles Δ)

$a°+a°+157.5^\circ-\frac{a^\circ}{8}=180°$ (∠ sum of Δ$ABE$)

$a = 12$

**G3** If $-6 \le a \le 4$ and $3 \le b \le 6$, find the greatest value of $a^2 - b^2$.

$0 \le a^2 \le 36$ and $9 \le b^2 \le 36$

$-36 \le a^2 - b^2 \le 27$

⇒ The greatest value = 27.

**G4** Let $a$, $b$, $c$ be integers such that $a^2 = b^3 = c$. If $c > 1$, find the smallest value of $c$.

**Reference: 1999 FG3.1, 2021 P2Q4**

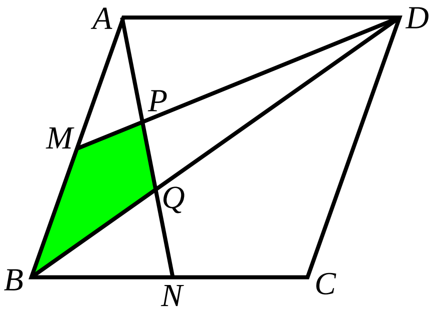
Let $a = k^3$, $b = k^2$, $c = k^6$

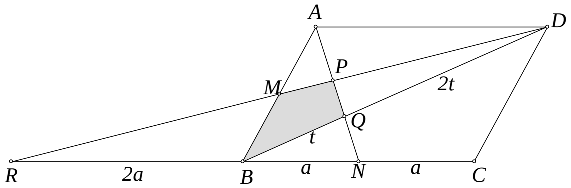
$c > 1 \Rightarrow k > 1$

The smallest $k = 2$

⇒ The smallest $c = 2^6 = 64$

**G5** In figure 2, the area of the parallelogram *ABCD* is 120. *M* and *N* are the mid-points of *AB* and *BC* respectively. *AN* intersects *MD* and *BD* at points *P* and *Q* respectively. Find the area of *BQPM*. (**Reference: 2016HI14, 2019 HI11**)




Produce *DM* and *CB* to meet at *R*.

Let $BC = 2a$. Then $BN = NC = a$ (mid-point)

$\Delta AQD \sim \Delta BQN$ (equiangular)

$\frac{BQ}{QD} = \frac{BN}{AD}$ (ratio of sides, ~Δ's)

$= \frac{1}{2}$ ($N$ = mid-point, opp. sides of //-gram)

Area of $\Delta ABD = \frac{1}{2} \times 120 = 60$

Area of $\Delta AQD = \frac{2}{3} \times \Delta \text{ABD} = 40$

$\frac{\text{Area of } \Delta\text{BQN}}{\text{Area of } \Delta\text{AQD}} = \left(\frac{BN}{AD}\right)^2 = \frac{1}{4}$

$\therefore$ Area of $\Delta BQN = \frac{1}{4} \times 40 = 10$ ……..(1)

As $M$ is the mid-point, $\Delta AMD \cong \Delta BMR$ (ASA)

$\Rightarrow RM = MD$ (corr. sides ≅ Δ's) ……………(2)

Also $\Delta APD \sim \Delta NPR$ (equiangular)

$\frac{DP}{PR} = \frac{AD}{NR}$ (ratio of sides, ~Δ's)

$= \frac{2a}{3a} = \frac{2}{3}$ (opp. sides of //-gram, corr. sides ≅ Δ's) ……….(3)

Combine (2) and (3)

$PD = \frac{2}{5} RD$; $MD = \frac{1}{2} RD$

$MP = MD - PD = \frac{1}{2} RD - \frac{2}{5} RD = \frac{1}{10} RD$

$\Rightarrow \frac{MP}{PD} = \frac{\frac{1}{10} RD}{\frac{2}{5} RD} = \frac{1}{4}$ ……………..(4)

Area of $\Delta AMD = \frac{1}{4} \times 120 = 30$

By (4): Area of $\Delta AMP = \frac{1}{5} \times$ Area of $\Delta\text{AMD} = \frac{1}{5} \times 30 = 6$ …………(5)

Area of $\Delta ABN = \frac{1}{4} \times 120 = 30$

$\therefore$ Area of *BQPM* = Area of $\Delta ABN$ – Area of $\Delta AMP$ – Area of $\Delta BQN$
= 30 – 6 – 10 = 14 (by (1) and (5))

**G6** In figure 3, find the number of possible paths from point $A$ to point $B$ following the direction of arrow heads.

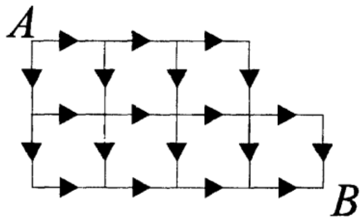
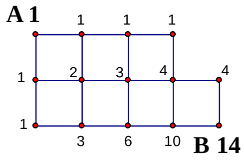



Figure 3

**Reference 1983 FI4.1, 2000 HI4, 2007 HG5**

The numbers at each of the vertices of in the following figure show the number of possible ways.




So the total number of ways = 14

**G7** Find the smallest real root of the equation $(x-2)(2x-1)=5$.

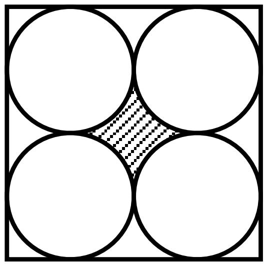
$2x^2-5x-3=0$

$(2x+1)(x-3)=0$

$x=-\frac{1}{2}$ or 3

The smallest real root is $-\frac{1}{2}$.

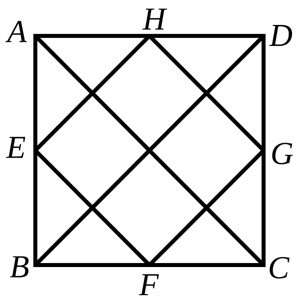
**G8** In figure 4, four circles with radius 1 touch each other inside a square. Find the shaded area. (Correct your answer to the nearest integer.)



The line segments joining the four centres form a square of sides = 2

Shaded area = $2^2-\pi\cdot 1^2\approx 1$

**G9** In figure 5, $ABCD$ is a square and points $E$, $F$, $G$, $H$ are the mid-points of sides $AB$, $BC$, $CD$, $DA$ respectively, find the number of right-angled triangles in the figure. **(Reference: 1995 HG9)**




Let the shortest side of the smallest right-angled triangle be 1.

Then $AE=\sqrt{2}$, $EH=2$, $AB=2\sqrt{2}$, $AC=4$

We count the number of right-angled triangles with different hypotenuses.

| Hypotenuse | Number of triangles |
|---|---|
| $\sqrt{2}$ | 8 |
| 2 | 4 |
| $2\sqrt{2}$ | 4 |
| 4 | 4 |

Total number of triangles = 20

**G10** A test is composed of 25 multiple-choice questions. 4 marks will be awarded for each correct answer and 1 mark will be deducted for each incorrect answer. A pupil answered all questions and got 70 marks. How many questions did the pupil answer correctly?

**Reference: 1994 FI1.2**

Suppose he answer $x$ questions correctly and $25-x$ question wrongly.

$4\cdot x-(25-x)=70$

$x=19$