Individual Events

I1	***P***	4	**I2**	***a***	8	**I3**	***a***	6	**I4**	***a***	23	**I5**	***a***	2	**IS**	***a***	2
	Q	8		***b***	10		***b***	7		***b***	2		***b***	1	spare	***b***	770
	R	11		***c***	1		***c***	2		***c***	2		***c***	0		***c***	57
	S	10		***d***	2000		***d***	9902		***d***	8		***d***	6		***d***	58

Group Events

G1	***a***	1	**G2**	***a***	–1	**G3**	***a***	2	**G4**	***a***	4	**G5**	***P***	35	**GS**	***P***	4
	b	15		***b***	0		***b***	7		***b***	0		***Q***	6	spare	***Q***	6
	c	80		***c***	13		***c***	0		***c***	3		***R***	11		***R***	35
	d	1		***d***	5		***d***	*6 see the remark		***d***	3		***S***	150		***S***	8

Individual Event 1

I1.1 If the interior angles of a P-sided polygon form an Arithmetic Progression and the smallest and the largest angles are 20° and 160° respectively. Find the value of P.

Sum of all interior angles $= \frac{P}{2}(20° + 160°) = 180°(P - 2)$

$90P = 180P - 360$

$\Rightarrow P = 4$

I1.2 In ΔABC, $AB = 5$, $AC = 6$ and $BC = P$. If $\frac{1}{Q} = \cos 2A$, find the value of Q.

(<u>Hint</u>: $\cos 2A = 2\cos^2 A - 1$)

$\cos A = \frac{6^2+5^2-4^2}{2\times6\times5} = \frac{3}{4}$

$\cos 2A = 2\cos^2 A - 1$

$= 2\times\left(\frac{3}{4}\right)^2 - 1 = \frac{1}{8}$

$Q = 8$

I1.3 If $\log_2 Q + \log_4 Q + \log_8 Q = \frac{R}{2}$, find the value of R.

$\frac{R}{2} = \log_2 8 + \log_4 8 + \log_8 8$

$= 3 + \frac{3}{2} + 1 = \frac{11}{2}$

$R = 11$

I1.4 If the product of the numbers R and $\frac{11}{S}$ is the same as their sum, find the value of S.

$11\times\frac{11}{S} = 11 + \frac{11}{S}$

$\Rightarrow \frac{110}{S} = 11$

$S = 10$

Individual Event 2

I2.1 If x, y and z are positive real numbers such that $\frac{x+y-z}{z}=\frac{x-y+z}{y}=\frac{-x+y+z}{x}$ and $a=\frac{(x+y)\cdot(y+z)\cdot(z+x)}{xyz}$, find the value of a.

Reference: 1992 HG2

Let $\frac{x+y-z}{z}=k$, $\frac{x-y+z}{y}=k$, $\frac{-x+y+z}{x}=k$.

$$\begin{cases} x+y-z=kz\cdots\cdots(1) \\ x-y+z=ky\cdots\cdots(2) \\ -x+y+z=kx\cdots\cdots(3) \end{cases}$$

(1) + (2) + (3): $x+y+z=k(x+y+z)$

$\Rightarrow k=1$

From (1), $x+y=2z$, (2): $x+z=2y$, (3): $y+z=2x$

$\therefore a=\frac{(x+y)\cdot(y+z)\cdot(z+x)}{xyz}=\frac{8xyz}{xyz}=8$

I2.2 Let u and t be positive integers such that $u+t+ut=4a+2$. If $b=u+t$, find the value of b.

$u+t+ut=34\Rightarrow 1+u+t+ut=35$

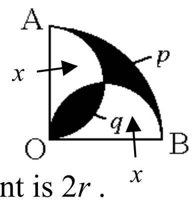
$\Rightarrow(1+u)(1+t)=35$

$\Rightarrow 1+u=5$, $1+t=7$

$u=4$, $t=6$

$\Rightarrow b=4+6=10$

I2.3 In Figure 1, OAB is a quadrant of a circle and semi-circles are drawn on OA and OB. If p, q denotes the areas of the shaded regions, where $p=(b-9)$ cm^2 and $q=c$ cm^2, find the value of c.



$p=1$, let the area of each of two unshaded regions be x cm^2

Let the radius of each of the smaller semicircles be r. The radius of the quadrant is $2r$.

$x+q$ = area of one semi-circle $=\frac{\pi r^2}{2}$; $2x+p+q$ = area of the quadrant $=\frac{1}{4}\pi(2r)^2=\pi r^2$

$2\times(1)=(2)$, $2x+2q=2x+p+q\Rightarrow q=p$; $c=1$

I2.4 Let $f_0(x)=\frac{1}{c-x}$ and $f_n(x)=f_0(f_{n-1}(x))$, $n=1, 2, 3, \ldots$. If $f_{2000}(2000)=d$, find the value of d.

Reference: 2009 HI6

$f_0(x)=\frac{1}{1-x}$, $f_1(x)=f_0(\frac{1}{1-x})=\frac{1}{1-\frac{1}{1-x}}=\frac{1-x}{-x}=\frac{x-1}{x}=1-\frac{1}{x}$

$f_2(x)=f_0(1-\frac{1}{x})=\frac{1}{1-\left(1-\frac{1}{x}\right)}=x$, which is an identity function.

So $f_5(x)=f_2(x)=x, \cdots, f_{2000}(x)=x$;

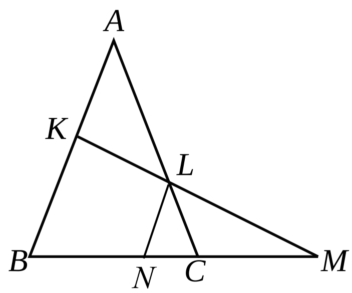
$f_{2000}(2000)=2000=d$

Individual Event 3 (2000 Sample Individual Event)

I3.1 For all integers m and n, $m \otimes n$ is defined as: $m \otimes n = m^n + n^m$.
If $2 \otimes a = 100$, find the value of a .
Reference: 1990 HI4
$2^a + a^2 = 100$
$64 + 36 = 2^6 + 6^2 = 100$
$a = 6$

I3.2 If $\sqrt[3]{13b+6a+1}-\sqrt[3]{13b-6a-1}=\sqrt[3]{2}$, where $b > 0$, find the value of b .
Reference: 2005 FI2.2, 2016 FG3.3, 2019 HI10
$\left(\sqrt[3]{13b+37}-\sqrt[3]{13b-37}\right)^3=2$
$13b + 37 - 3\sqrt[3]{(13b+37)^2}\sqrt[3]{13b-37}+3\sqrt[3]{(13b-37)^2}\sqrt[3]{13b+37} - (13b - 37) = 2$
$24 = \sqrt[3]{(13b)^2-37^2}\sqrt[3]{13b+37} - \sqrt[3]{(13b)^2-37^2}\sqrt[3]{13b-37}$
$24 = \sqrt[3]{(13b)^2-37^2}\ \sqrt[3]{2}$; ($\because \sqrt[3]{13b+37}-\sqrt[3]{13b-37}=\sqrt[3]{2}$)
$13824 = [(13b)^2 - 1369] \times 2$
$6912 + 1369 = 169\, b^2$
$b^2 = 49$
$\Rightarrow b = 7$
Method 2 $\sqrt[3]{13b+37}-\sqrt[3]{13b-37}=\sqrt[3]{2}$
We look for the difference of multiples of $\sqrt[3]{2}$
$\sqrt[3]{8\times2}-\sqrt[3]{2}=\sqrt[3]{2} \Rightarrow 13b + 37 = 16,\ 13b - 37 = 2$, no solution
$\sqrt[3]{27\times2}-\sqrt[3]{8\times2}=\sqrt[3]{2} \Rightarrow 13b + 37 = 54,\ 13b - 37 = 16$, no solution
$\sqrt[3]{64\times2}-\sqrt[3]{27\times2}=\sqrt[3]{2} \Rightarrow 13b + 37 = 128,\ 13b - 37 = 54$
$\Rightarrow b = 7$

I3.3 In figure 2, $AB = AC$ and $KL = LM$. If $LC = b - 6$ cm and $KB = c$ cm, find the value of c .



Reference: 1992 HG6
Draw $LN \parallel AB$ on BM.
$BN = NM$ intercept theorem
$\angle LNC = \angle ABC = \angle LCN$ (corr. $\angle$s, $AB \parallel LN$, base $\angle$s, isos. Δ)
$LN = LC = b - 6$ cm = 1 cm (sides opp. eq. $\angle$s)
c cm = $KB = 2\ LN$ = 2 cm (mid point theorem)

I3.4 The sequence $\{a_n\}$ is defined as $a_1 = c$, $a_{n+1} = a_n + 2n$ ($n \ge 1$). If $a_{100} = d$, find the value of d .
$a_1 = 2$, $a_2 = 2 + 2$, $a_3 = 2 + 2 + 4$, $\cdots$,
$a_{100} = 2 + 2 + 4 + \cdots + 198$
$= 2 + \frac{1}{2}(2+198)\cdot 99 = 9902 = d$

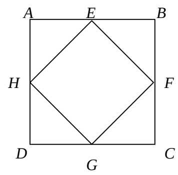
Individual Event 4

I4.1 Mr. Lee is a years old, $a < 100$.
If the product of a and his month of birth is 253, find the value of a.
$253 = 11\times 23$
11 = his month of birth and $a = 23$

I4.2 Mr. Lee has $a + b$ sweets. If he divides them equally among 10 persons, 5 sweets will be remained. If he divides them equally among 7 persons, 3 more sweets are needed. Find the minimum value of b.
$10m + 5 = 7n - 3 = 23 + b$
$7n - 10m = 8$
By trial and error $n = 4$, $m = 2$
$23 + b = 7\times 4 - 3 = 25$
$b = 2$

I4.3 Let c be a positive real number. If $x^2 + 2\sqrt{c}\,x + b = 0$ has one real root only, find the value of c.
$x^2 + 2\sqrt{c}\,x + 2 = 0$
$\Delta = 4(c - 2) = 0$
$\Rightarrow c = 2$

I4.4 In figure 3, the area of the square $ABCD$ is equal to d. If E, F, G, H are the mid-points of AB, BC, CD and DA respectively and $EF = c$, find the value of d.



Area of $EFGH = c^2 = 2^2 = 4$
Area of $ABCD = 2 \times$ area of $EFGH = 8$
$\Rightarrow d = 8$

Individual Event 5

I5.1 If $144^p = 10$, $1728^q = 5$ and $a = 12^{2p-3q}$, find the value of a.
$a = 12^{2p-3q} = 144^p \div 1728^q = 10 \div 5 = 2$

I5.2 If $1 - \frac{4}{x} + \frac{4}{x^2} = 0$, $b = \frac{a}{x}$, find b.
Reference: 1994 FI5.1
$\left(1 - \frac{2}{x}\right)^2 = 0$; $x = 2$, $b = \frac{2}{2} = 1$

I5.3 If the number of real roots of the equation $x^2 - bx + 1 = 0$ is c, find the value of c.
$x^2 - x + 1 = 0$
$\Delta = 1^2 - 4 < 0$
c = number of real roots = 0

I5.4 Let $f(1) = c + 1$ and $f(n) = (n - 1)\, f(n - 1)$, where $n > 1$. If $d = f(4)$, find the value of d.
Reference: 2009 FI1.4
$f(1) = 1$
$f(2) = f(1) = 1$
$f(3) = 2f(2) = 2$
$f(4) = 3f(3) = 3\times 2 = 6$

Individual Event (Spare)

IS.1 If a is the smallest prime number which can divide the sum $3^{11} + 5^{13}$, find the value of a .

Reference: 2010 FG3.1

3^{11} is an odd number

5^{13} is also an odd number

So $3^{11} + 5^{13}$ is an even number, which is divisible by 2 .

IS.2 For all real number x and y, $x \oplus y$ is defined as: $x \oplus y = \frac{1}{xy}$.

If $b = 4 \oplus (a \oplus 1540)$, find the value of b .

$a \oplus 1540 = \frac{1}{2\times 1540} = \frac{1}{3080}$

$b = 4 \oplus (a \oplus 1540) = \frac{3080}{4} = 770$

IS.3 W and F are two integers which are greater than 20. If the product of W and F is b and the sum of W and F is c, find the value of c .

$$\begin{cases} WF = 770 \cdots\cdots(1) \\ W + F = c \quad \cdots\cdots(2) \end{cases}$$

$770 = 22\times 35$

$W = 22$, $F = 35$

$c = 22 + 35 = 57$

IS.4 If $\frac{d}{114} = \left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\cdots\left(1-\frac{1}{c^2}\right)$, find the value of d .

Reference: 1986 FG10.4, 2014 FG3.1

$$\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\cdots\left(1-\frac{1}{57^2}\right) = \left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\cdots\left(1-\frac{1}{57}\right)\left(1+\frac{1}{2}\right)\left(1+\frac{1}{3}\right)\cdots\left(1+\frac{1}{57}\right)$$

$$= \frac{1}{2}\cdot\frac{2}{3}\cdots\cdots\frac{56}{57}\times\frac{3}{2}\cdot\frac{4}{3}\cdots\cdots\frac{58}{57} = \frac{1}{57}\times\frac{58}{2} = \frac{58}{114}$$

$d = 58$

Group Event 1 (2000 Final Sample Group Event)

G1.1 Let $x * y = x + y - xy$, where x, y are real numbers. If $a = 1 * (0 * 1)$, find the value of a.

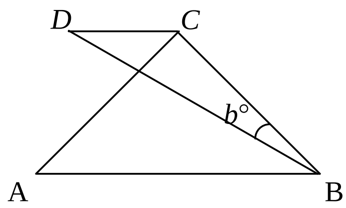
$0 * 1 = 0 + 1 - 0 = 1$

$a = 1 * (0 * 1)$

$= 1 * 1$

$= 1 + 1 - 1 = 1$

G1.2 In figure 1, AB is parallel to DC, $\angle ACB$ is a right angle, $AC = CB$ and $AB = BD$. If $\angle CBD = b°$, find the value of b.



ΔABC is a right angled isosceles triangle.

$\angle BAC = 45°$ ($\angle$s sum of Δ, base $\angle$s isos. Δ)

$\angle ACD = 45°$ (alt. $\angle$s, $AB \parallel DC$)

$\angle BCD = 135°$

Apply sine law on ΔBCD,

$$\frac{BD}{\sin 135°} = \frac{BC}{\sin D}$$

$$AB\sqrt{2} = \frac{AB \sin 45°}{\sin D}$$, given that $AB = BD$

$\sin D = \frac{1}{2}$; $D = 30°$

$\angle CBD = 180° - 135° - 30° = 15°$ ($\angle$s sum of ΔBCD)

$b = 15$

G1.3 Let x, y be non-zero real numbers. If x is 250% of y and $2y$ is c% of x, find the value of c.

$x = 2.5y$ ⋯⋯ (1)

$2y = \frac{c}{100} \cdot x$ ⋯⋯(2)

Sub. (1) into (2): $2y = \frac{c}{100} \cdot 2.5y$

$c = 80$

G1.4 If $\log_p x = 2$, $\log_q x = 3$, $\log_r x = 6$ and $\log_{pqr} x = d$, find the value of d.

Reference: 2001 FG1.4, 2015 HI7

$\frac{\log x}{\log p} = 2$; $\frac{\log x}{\log q} = 3$; $\frac{\log x}{\log r} = 6$

$\frac{\log p}{\log x} = \frac{1}{2}$; $\frac{\log q}{\log x} = \frac{1}{3}$; $\frac{\log r}{\log x} = \frac{1}{6}$

$\frac{\log p}{\log x} + \frac{\log q}{\log x} + \frac{\log r}{\log x} = \frac{1}{2} + \frac{1}{3} + \frac{1}{6} = 1$

$\frac{\log pqr}{\log x} = 1$

$\Rightarrow \frac{\log x}{\log pqr} = 1$

$d = \log_{pqr} x = 1$

Group Event 2

G2.1 If $a = x^4 + x^{-4}$ and $x^2 + x + 1 = 0$, find the value of a .

$$\frac{x^2+x+1}{x}=0$$

$$\Rightarrow x+\frac{1}{x}=-1$$

$$\Rightarrow \left(x+\frac{1}{x}\right)^2=1$$

$$\Rightarrow \quad x^2+\frac{1}{x^2}=-1$$

$$\Rightarrow \left(x^2+\frac{1}{x^2}\right)^2=1$$

$$a=x^4+\frac{1}{x^4}=-1$$

G2.2 If $6^b + 6^{b+1} = 2^b + 2^{b+1} + 2^{b+2}$, find the value of b .

$6^b \cdot (1 + 6) = 2^b \cdot (1 + 2 + 4)$

$\Rightarrow b = 0$

G2.3 Let c be a prime number. If $11c + 1$ is the square of a positive integer, find the value of c .

$11c + 1 = m^2$

$\Rightarrow m^2 - 1 = 11c$

$\Rightarrow (m + 1)(m - 1) = 11c$

$\Rightarrow m - 1 = 11$ and $m + 1 = c$

$m = 13$

G2.4 Let d be an odd prime number. If $89 - (d + 3)^2$ is the square of an integer, find the value of d .

$\because$ d is odd, $d + 3$ must be even, $89 - (d + 3)^2$ must be odd.

$89 = (d + 3)^2 + m^2$

By trial and error, $m = 5$, $89 = 8^2 + 5^2$

$\Rightarrow d + 3 = 8$

$\Rightarrow d = 5$

Group Event 3

G3.1 Let a be the number of positive integers less than 100 such that they are both square and cubic numbers, find the value of a .

Reference 1998 HG4, 2021 P2Q4

The positive integers less than 100 such that they are both square and cubic numbers are:

1 and $2^6 = 64$ only, so there are only 2 numbers satisfying the condition.

G3.2 The sequence $\{a_k\}$ is defined as: $a_1 = 1$, $a_2 = 1$ and $a_k = a_{k-1} + a_{k-2}$ $(k > 2)$.

If $a_1 + a_2 + \ldots + a_{10} = 11\ a_b$, find the value of b .

$a_1 = 1, a_2 = 1, a_3 = 2, a_4 = 3, a_5 = 5, a_6 = 8, a_7 = 13, a_8 = 21, a_9 = 34, a_{10} = 55$

$a_1 + a_2 + \ldots + a_{10} = 1 + 1 + 2 + 3 + 5 + 8 + 13 + 21 + 34 + 55 = 143 = 11\times13 = 11a_7$

$b = 7$

G3.3 If c is the maximum value of $\log(\sin x)$, where $0 < x < \pi$, find the value of c.

$0 < \sin x \le 1$

$\log(\sin x) \le \log 1 = 0$

$\Rightarrow c = 0$

G3.4 Let $x \ge 0$ and $y \ge 0$. Given that $x + y = 18$. If the maximum value of $\sqrt{x}+\sqrt{y}$ is d,

find the value of d . **(Reference: 1999 FGS.2, 2019 FG1.1)**

$x+y=\left(\sqrt{x}+\sqrt{y}\right)^2-2\sqrt{xy}$

$\Rightarrow\left(\sqrt{x}+\sqrt{y}\right)^2=18+2\sqrt{xy}\le18+2\left(\frac{x+y}{2}\right)=36$ (GM ≤ AM)

$\sqrt{x}+\sqrt{y}\le 6 = d$ (It is easy to get the answer by letting $x = y$ in $x + y = 18$)

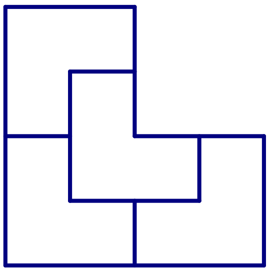
Remark The original question is Given that $x + y = 18$. If the maximum value of $\sqrt{x}+\sqrt{y}$ ⋯

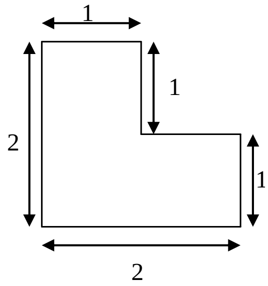
$\sqrt{x}+\sqrt{y}$ is undefined for $x < 0$ or $y < 0$.

Group Event 4

G4.1 If a tiles of L-shape are used to form a larger similar figure (figure 2) without overlapping, find the least possible value of a.

From the figure, $a = 4$.





G4.2 Let α, β be the roots of $x^2 + bx - 2 = 0$.

If $\alpha > 1$ and $\beta < -1$, and b is an integer, find the value of b.

$\alpha - 1 > 0$ and $\beta + 1 < 0$

$\Rightarrow (\alpha - 1)(\beta + 1) < 0$

$\Rightarrow \alpha\beta + \alpha - \beta - 1 < 0$

$\Rightarrow \alpha - \beta < 3$

$\Rightarrow (\alpha - \beta)^2 < 9$

$\Rightarrow (\alpha + \beta)^2 - 4\alpha\beta < 9$

$\Rightarrow b^2 + 8 < 9$

$\Rightarrow -1 < b < 1$

$\because$ b is an integer

$\therefore$ $b = 0$

G4.3 Given that m, c are positive integers less than 10.

If $m = 2c$ and $0.\dot{m}\dot{c} = \frac{c+4}{m+5}$, find the value of c.

$0.\dot{m}\dot{c} = \frac{10m+c}{99} = \frac{c+4}{m+5}$

$\Rightarrow \frac{20c+c}{99} = \frac{c+4}{2c+5}$

$\Rightarrow \frac{7c}{33} = \frac{c+4}{2c+5}$

$\Rightarrow 14c^2 + 35c = 33c + 132$

$14c^2 + 2c - 132 = 0$

$\Rightarrow 7c^2 + c - 66 = 0$

$\Rightarrow (7c + 22)(c - 3) = 0$

$\Rightarrow c = 3$

G4.4 A bag contains d balls of which x are black, $x + 1$ are red and $x + 2$ are white. If the probability of drawing a black ball randomly from the bag is less than $\frac{1}{6}$, find the value of d.

$\frac{x}{3x+3} < \frac{1}{6}$

$\Rightarrow \frac{x}{x+1} < \frac{1}{2}$

$\Rightarrow 2x < x + 1$

$\Rightarrow x < 1$

$\Rightarrow x = 0$

$\Rightarrow d = 3x + 3 = 3$

Group Event 5

G5.1 If the roots of $x^2 - 2x - P = 0$ differ by 12, find the value of P .

Reference: 1999 FGS.3

$\alpha + \beta = 2$, $\alpha\beta = -P$

$\alpha - \beta = 12$

$\Rightarrow (\alpha - \beta)^2 = 144$

$\Rightarrow (\alpha + \beta)^2 - 4\alpha\,\beta = 144$

$\Rightarrow 4 + 4P = 144$

$\Rightarrow P = 35$

G5.2 Given that the roots of $x^2 + ax + 2b = 0$ and $x^2 + 2bx + a = 0$ are both real and $a, b > 0$.
If the minimum value of $a + b$ is Q, find the value of Q. **(Reference: 2013 HG6)**

$a^2 - 8b \ge 0$ and $4b^2 - 4a \ge 0$

$a^2 \ge 8b$ and $b^2 \ge a$

$\Rightarrow a^4 \ge 64b^2 \ge 64a$

$\Rightarrow a^4 - 64a \ge 0$

$\Rightarrow a(a^3 - 64) \ge 0$

$\Rightarrow a^3 \ge 64$

$\Rightarrow a \ge 4$

Minimum $a = 4$, $b^2 \ge a$

$\Rightarrow b^2 \ge 4 \Rightarrow$ minimum $b = 2$

Q = minimum value of $a + b = 4 + 2 = 6$

G5.3 If $R^{2000} < 5^{3000}$, where R is a positive integer, find the largest value of R .

Reference: 1996 HI4, 2008 FI4.3, 2018 FG2.4

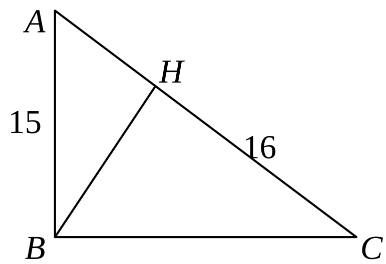
$(R^2)^{1000} < (5^3)^{1000}$

$\Rightarrow R^2 < 5^3 = 125$

$\Rightarrow R < \sqrt{125} < 12$

The largest integral value of $R = 11$

G5.4 In figure 3, ΔABC is a right-angled triangle and $BH \perp AC$.
If $AB = 15$, $HC = 16$ and the area of ΔABC is S, find the value of S.



Reference: 1998 FG1.3, 2022 P1Q3

It is easy to show that $\Delta ABH \sim \Delta BCH \sim \Delta ACB$.

Let $\angle ABH = \theta = \angle BCH$

In ΔABH, $BH = 15 \cos \theta$

In ΔBCH, $CH = BH \div \tan \theta \Rightarrow 16 \tan \theta = 15 \cos \theta$

$16 \sin \theta = 15 \cos^2 \theta \Rightarrow 16 \sin \theta = 15 - 15 \sin^2 \theta$

$15 \sin^2 \theta + 16 \sin \theta - 15 = 0$

$(3 \sin \theta + 5)(5 \sin \theta - 3) = 0$

$\sin \theta = \frac{3}{5}$; $\tan \theta = \frac{3}{4}$

$BC = AB \div \tan \theta = 15 \times \frac{4}{3} = 20$

Area of $\Delta ABC = \frac{1}{2} \cdot 15 \times 20 = 150 = S$

Group Event (Spare)

GS.1 If a number N is chosen randomly from the set of positive integers, the probability of the unit digit of N^4 being unity is $\frac{P}{10}$, find the value of P.

If the unit digit of N^4 is 1, then the unit digit of N may be 1, 3, 7, 9. So the probability $=\frac{4}{10}$

$P = 4$

GS.2 Let $x \ge 0$ and $y \ge 0$. Given that $x + y = 18$.

If the maximum value of $\sqrt{x}+\sqrt{y}$ is d, find the value of d.

Reference: 1999 FG3.4

$x+y=\left(\sqrt{x}+\sqrt{y}\right)^2-2\sqrt{xy}$

$\Rightarrow\left(\sqrt{x}+\sqrt{y}\right)^2=18+2\sqrt{xy}\le 18+2\left(\frac{x+y}{2}\right)=36$ (G.M. ≤ A.M.)

$\sqrt{x}+\sqrt{y} \le 6 = d$

GS.3 If the roots of $x^2 - 2x - R = 0$ differs by 12, find the value of R.

Reference: 1999 FG5.1

$\alpha + \beta = 2$, $\alpha\beta = -R$

$\alpha - \beta = 12$

$\Rightarrow (\alpha - \beta)^2 = 144$

$\Rightarrow (\alpha + \beta)^2 - 4\alpha\beta = 144$

$\Rightarrow 4 + 4R = 144$

$\Rightarrow R = 35$

GS.4 If the product of a 4-digit number ab$S$d and 9 is equal to another 4-digit number dSba, find the value of S.

Reference: 1987 FG9, 1994HI6

$a = 1$, $d = 9$,

Let the carry digit in the hundred digit be x.

Then $9S + 8 = 10x + b$ ⋯⋯ (1)

$9b + x = S$ ⋯⋯ (2)

$x = S - 9b$ ⋯⋯ (3)

Sub. (3) into (1): $9s + 8 = 10(S - 9b) + b$

$\Rightarrow 8 = S - 89b$

$\Rightarrow S = 8$, $b = 0$