99-00 Individual	1	$\dfrac{170}{891}$	2	3	3	10	4	35	5	540
	6	190	7	$\dfrac{1}{3}$	8	428571	9	24	10	0

99-00 Group	1	−3	2	5	3	6	4	10	5	10
	6	60	7	0.93	8	421	9	12	10	0

Individual Events

I1　Let $x = 0.1\dot{7} + 0.01\dot{7} + 0.001\dot{7} + \cdots$, find the value of x. (**Reference: 2009 HI1**)

$0.1\dot{7} = \dfrac{17}{99}; 0.01\dot{7} = \dfrac{17}{990}; 0.001\dot{7} = \dfrac{17}{9900}, \cdots$

$x = \dfrac{17}{99} + \dfrac{17}{990} + \dfrac{17}{9900} + \cdots$

$= \dfrac{17}{99}\left(1 + \dfrac{1}{10} + \dfrac{1}{100} + \cdots\right)$　(It is an infinite geometric series, $a = \dfrac{17}{99}, r = \dfrac{1}{10}$)

$= \dfrac{17}{99} \cdot \dfrac{1}{1 - \dfrac{1}{10}}$

$= \dfrac{17}{99} \cdot \dfrac{10}{9} = \dfrac{170}{891}$

I2　Solve the following equation:

$\dfrac{1}{x+12} + \dfrac{1}{(x+1)(x+2)} + \dfrac{1}{(x+2)(x+3)} + \dfrac{1}{(x+3)(x+4)} + \cdots + \dfrac{1}{(x+10)(x+11)} + \dfrac{1}{(x+11)(x+12)} = \dfrac{1}{4}$

$\dfrac{1}{x+12} + \left(\dfrac{1}{x+1} - \dfrac{1}{x+2}\right) + \left(\dfrac{1}{x+2} - \dfrac{1}{x+3}\right) + \left(\dfrac{1}{x+3} - \dfrac{1}{x+4}\right) \cdots + \left(\dfrac{1}{x+10} - \dfrac{1}{x+11}\right) + \left(\dfrac{1}{x+11} - \dfrac{1}{x+12}\right) = \dfrac{1}{4}$

$\dfrac{1}{x+1} = \dfrac{1}{4}$

$\Rightarrow x = 3$

I3　Using digits 0, 1, 2, and 5, how many 3-digit numbers can be formed, which are divisible by 5? (If no digit may be repeated.)

Possible numbers are: 105, 120, 125, 150, 205, 210, 215, 250, 510, 520.

Altogether 10 numbers.

I4　Figure 1 represents a 4×3 rectangular spiderweb. If a spider walks along the web from A to C and it always walks either due East or due North. Find the total number of possible paths.
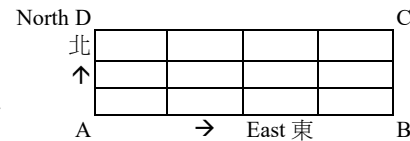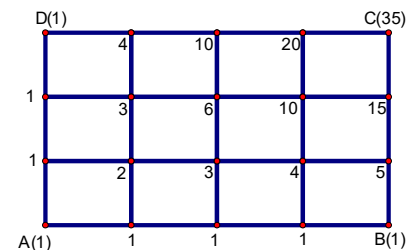

Figure 1 圖一

Reference: 1983 FI4.1, 1998 HG6, 2007 HG5

The numbers at each of the vertices of in the following figure show the number of possible ways.

So the total number of ways = 35

I5   In Figure 2, let $\angle A+\angle B+\angle C+\angle D+\angle E+\angle F+\angle G = x°$, find the value of x.

Reference: 1992 HI13, 2012 FG3.2

In the figure, let P, Q, R, S, T, U, V be as shown.

$\angle AVP + \angle BPQ + \angle CQR + \angle DRS + \angle EST + \angle FTU+\angle GUV=360°$

(sum of ext. $\angle$ of polygon)

$\angle A = 180° – (\angle AVP + \angle BPQ)$ ($\angle$s sum of Δ)
$\angle B = 180° – (\angle BPQ + \angle CQR)$ ($\angle$s sum of Δ)
$\angle C = 180° – (\angle CQR + \angle DRS)$ ($\angle$s sum of Δ)
$\angle D = 180° – (\angle DRS + \angle EST)$ ($\angle$s sum of Δ)
$\angle E = 180° – (\angle EST + \angle FTU)$ ($\angle$s sum of Δ)
$\angle F = 180° – (\angle FTU + \angle GUV)$ ($\angle$s sum of Δ)
$\angle G = 180° – (\angle GUV + \angle AVP)$ ($\angle$s sum of Δ)
$\angle A+\angle B+\angle C+\angle D+\angle E+\angle F+\angle G =180°×7 – 2×360°$
$x = 540$
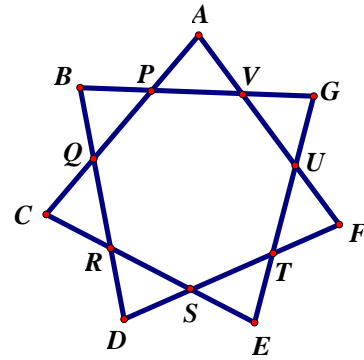
I6   Twenty straight lines were drawn on a white paper. Among them, no two or more straight lines are parallel; also no three or more than three straight lines are concurrent. What is the maximum number of intersections that these 20 lines can form?

2 lines give at most 1 intersection.
3 lines give at most 3 intersections.
4 lines give at most 6 intersections. ($6 = 1 + 2 + 3$)

....

20 lines give at most $1 + 2 + 3 + \cdots + 19$ intersections $=\dfrac{1+19}{2}\cdot 19= 190$ intersections

I7   In a family of 2 children, given that one of them is a girl, what is the probability of having another girl? (Assuming equal probabilities of boys and girls.)

Sample space = {(girl, boy), (girl, girl), (boy, girl)} and each outcome is equal probable.

$\therefore$ P(another child is also a girl) $=\dfrac{1}{3}$

I8   A particular 6-digit number has a unit-digit "1". Suppose this unit-digit "1" is moved to the place of hundred thousands, while the original ten thousand-digit, thousand-digit, hundred-digit, … are moved one digit place to the right. The value of the new 6-digit number is one-third of the value of the original 6-digit number. Find the original 6-digit number.

(**Reference: 1986 FG8**) Let the original number be: $\overline{abcde1}$, and the new number be: $\overline{1abcde}$.

$3\times \overline{1abcde}=\overline{abcde1}$
$3(100000 + 10000a + 1000b + 100c + 10d + e) = 100000a+10000b + 1000c + 100d + 10e + 1$
Compare the unit digit: $e = 7$ with carry digit 2 to the tens digit
Compare the tens digit: $d = 5$ with carry digit 1 to the hundreds digit
Compare the hundreds digit: $c = 8$ with carry digit 2 to the thousands digit
Compare the thousands digit: $b = 2$ with no carry digit to the ten-thousands digit
Compare the ten-thousands digit: $a = 4$ with carry digit 1 to the hundred-thousands digit
The original number is 428571

I9   Find the value of $\dfrac{12\sin^2 48° +12\sin^2 42°}{\sin 330° \tan 135° – \sin^2 48° \sin^2 42° \tan 180°}$.

$\dfrac{12\sin^2 48° +12\sin^2 42°}{\sin 330° \tan 135° – \sin^2 48° \sin^2 42° \tan 180°}=\dfrac{12 \sin^2 48° + 12\cos^2 48°}{\left(-\dfrac{1}{2}\right)(-1) – \sin^2 48° \sin^2 42° \times 0}=\dfrac{12}{\frac{1}{2}}=24$

I10   Find the shortest distance between the line $3x – y – 4 = 0$ and the point (2, 2).

$d =\left|\dfrac{Ax_0 + By_0 + C}{\sqrt{A^2 + B^2}}\right|=\left|\dfrac{3\times 2 – 2 – 4}{\sqrt{3^2 +(-1)^2}}\right|= 0$

Method 2 Sub. (2, 2) into $3x – y – 4 = 0$, LHS = $3×2 – 2 – 4 = 0$ = RHS
$\therefore$ (2, 2) lies on the line, the shortest distance = 0

Group Events

G1   If a is a root of $x^2 + 2x + 3 = 0$, find the value of $\dfrac{a^5 + 3a^4 + 3a^3 - a^2}{a^2 + 3}$.

Reference: 1993 HI9, 2001 FG2.1, 2007 HG3, 2009 HG2

Divide $(a^5 + 3a^4 + 3a^3 - a^2)$ by $(a^2 + 2a + 3)$, quotient $= a^3 + a^2 - 2a$, remainder $= 6a$

$$\frac{a^5 + 3a^4 + 3a^3 - a^2}{a^2 + 3} = \frac{\left(a^2 + 2a + 3\right)\left(a^3 + a^2 - 2a\right) + 6a}{\left(a^2 + 2a + 3\right) - 2a}$$

$$= \frac{6a}{-2a} = -3$$

G2   There are exactly n roots in the equation $(\cos^2 \theta - 1)(2 \cos^2 \theta - 1) = 0$, where $0° < \theta < 360°$. Find the value of n.

$\cos \theta = 1, -1, \dfrac{1}{\sqrt{2}}$ or $-\dfrac{1}{\sqrt{2}}$.

$\theta = 180°, 45°, 315°, 135°, 225°$

$n = 5$

G3   Find the units digit of 2004^{2006}.

$4^1 = 4$, $4^2 = 16$, $4^3 = 64$, $4^4 = 256$, …

So the units digit of 2004^{2006} is 6.

G4   Let $x = |y - m| + |y - 10| + |y - m - 10|$, where $0 < m < 10$ and $m \le y \le 10$. Find the minimum value of x.

$x = y - m + 10 - y + 10 - y + m = 20 - y \ge 20 - 10 = 10$

The minimum $= 10$

G5   There are 5 balls with labels A, B, C, D, E respectively and there are 5 pockets with labels A, B, C, D, E respectively. A ball is put into each pocket. Find the number of ways in which exactly 3 balls have labels that match the labels on the pockets.

First choose any 3 bags out of five bags. Put the balls according to their numbers. The remaining 2 balls must be put in the wrong order.

The number of ways is $_5C_3 = 10$.

G6   In Figure 1, ΔPQR is an equilateral triangle, $PT = RS$; PS, QT meet at M; and QN is perpendicular to PS at N. Let $\angle QMN = x°$, find the value of x.

Reference: 2019 HI1

$PT = RS$ (given)

$\angle QPT = 60° = \angle PRS$ ($\angle$ of an equilateral Δ)

$PQ = PR$ (side of an equilateral Δ)

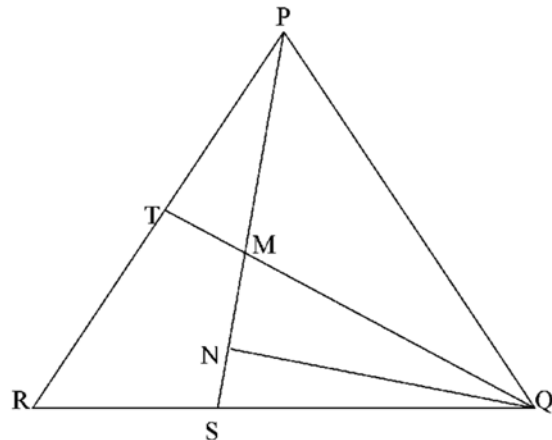$\Delta PQT \cong \Delta RPS$ (SAS)

$\therefore \angle PTQ = \angle PSR$ (corr. $\angle$s $\cong \Delta$)

R, S, M, T are concyclic (ext. $\angle$ = int. opp. $\angle$)

$\angle QMN = x° = \angle TRS = 60°$ (ext. $\angle$, cyclic quad.)

$x = 60$

G7 In Figure 2, three equal circles are tangent to each other, and inscribed in rectangle $PQRS$, find the value of $\dfrac{QR}{SR}$. (Use $\sqrt{3}=1.7$ and give the answer correct to 2 decimal places)

Let the radii of the circles be r.
Suppose the 3 circles touch the rectangle at A, B and C. Join O_1O_2, O_2O_3, O_1O_3, O_1C, O_2A, O_3B as shown. Then $O_1O_2=O_2O_3=O_1O_3=2r$
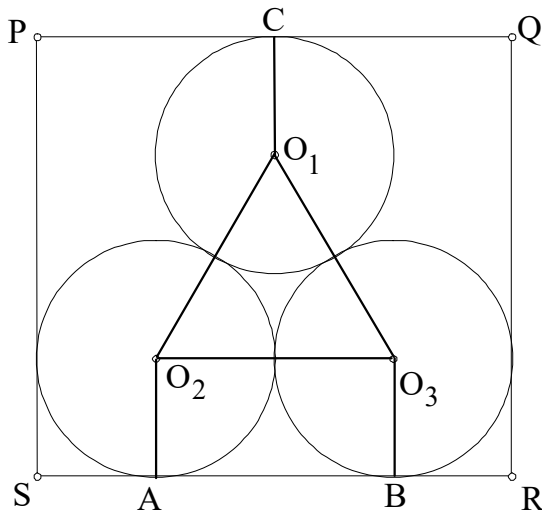$O_1C=O_2A=O_3B=r$
$O_1O_2O_3$ is an equilateral Δ
$QR=O_1C+O_1O_2\sin 60°+O_2A$

$$=r+2r\cdot\frac{\sqrt{3}}{2}+r=r(2+\sqrt{3})$$

$SR=4r$, $\dfrac{QR}{SR}=\dfrac{r(2+\sqrt{3})}{4r}=\dfrac{2+1.7}{4}=\dfrac{37}{40}\approx 0.93$

G8 The sum of two positive integers is 29, find the minimum value of the sum of their squares.

Let the two numbers be a and b.

$a^2+b^2=a^2+(29-a)^2=2a^2-58a+841=2(a-14.5)^2+420.5$

$\because$ a and b are integers, the minimum is attained when $a=15$, $b=14$

The minimum value of $a^2+b^2=15^2+14^2=225+196=421$

G9 Let $x=\sqrt{3+\sqrt{3}}$ and $y=\sqrt{3-\sqrt{3}}$, find the value of $x^2(1+y^2)+y^2$.

$$\begin{aligned}x^2(1+y^2)+y^2&=(3+\sqrt{3})(1+3-\sqrt{3})+3-\sqrt{3}\\&=(3+\sqrt{3})(4-\sqrt{3})+3-\sqrt{3}\\&=12+4\sqrt{3}-3\sqrt{3}-3+3-\sqrt{3}=12\end{aligned}$$

Method 2

$$\begin{aligned}x^2(1+y^2)+y^2&=(x^2+1)(y^2+1)-1\\&=(3+\sqrt{3}+1)(3-\sqrt{3}+1)-1\\&=16-3-1=12\end{aligned}$$

G10 There are nine balls in a pocket, each one having an integer label from 1 to 9. A draws a ball randomly from the pocket and puts it back, then B draws a ball randomly from the same pocket. Let n be the unit digit of the sum of numbers on the two balls drawn by A and B, and P(n) be the probability of the occurrence of n. Find the value of n such that P(n) is the maximum.

P(1) = P((2,9), (3,8), (4,7), (5,6), (6,5), (7,4), (8,3), (9,2))
P(2) = P((1,1), (3,9), (4,8), (5,7), (6,6), (7,5), (8,4), (9,3))
P(3) = P((1,2), (2,1), (4,9), (5,8), (6,7), (7,6), (8,5), (9,4))
P(4) = P((1,3), (2,2), (3,1), (5,9), (6,8), (7,7), (8,6), (9,5))
P(5) = P((1,4), (2,3), (3,2), (4,1), (6,9), (7,8), (8,7), (9,6))
P(6) = P((1,5), (2,4), (3,3), (4,2), (5,1), (7,9), (8,8), (9,7))
P(7) = P((1,6), (2,5), (3,4), (4,3), (5,2), (6,1), (8,9), (9,8))
P(8) = P((1,7), (2,6), (3,5), (4,4), (5,3), (6,2), (7,1), (9,9))
P(9) = P((1,8), (2,7), (3,6), (4,5), (5,4), (6,3), (7,2), (8,1))
P(0) = P((1,9), (2,8), (3,7), (4,6), (5,5), (6,4), (7,3), (8,2), (9,1))
$\therefore$ When $n=0$, P(n) is a maximum.