00-01 Individual	1	2	2	33	3	17	4	–47	5	4
	6	147	7	3.5	8	75	9	9	10	36

00-01 Group	1	$\frac{60}{11}$	2	36	3	10	4	8	5	$\frac{7}{9}$
	6	7	7	2	8	120	9	3	10	$\frac{3}{11}$

Individual Events

I1 If $4^a = 25^b = 10$, find the value of $\frac{1}{a}+\frac{1}{b}$.

Reference: 2003 FG2.2, 2004 FG4.3, 2005 HI9, 2006 FG4.3

$\log 4^a = \log 25^b = \log 10$

$a \log 4 = b \log 25 = 1$

$\frac{1}{a}+\frac{1}{b} = \log 4 + \log 25 = \log (4\times 25) = \log 100 = 2$

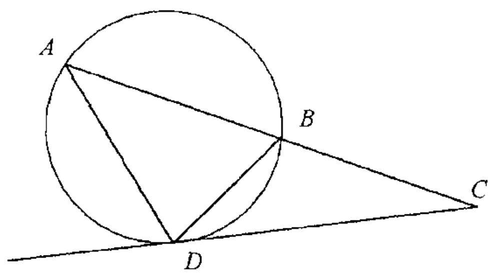
Method 2 $4 = 10^{\frac{1}{a}}$, $25 = 10^{\frac{1}{b}}$

$4\times 25 = 10^{\frac{1}{a}} \times 10^{\frac{1}{b}}$

$10^2 = 100 = 10^{\frac{1}{a}+\frac{1}{b}}$

$\frac{1}{a}+\frac{1}{b} = 2$

I2 In figure 1, ABC is a straight line, $AB = AD$, $\angle BDC = 38°$, CD is a tangent to the circle ABD. Let $\angle BCD = x°$, find the value of x.



圖一 Figure 1

$\angle BAD = \angle BDC = 38°$ ($\angle$ in alt. segment)

$\angle ADB = \angle ABD$ (base $\angle$s isosceles Δ)

$= (180° – 38°) \div 2$ ($\angle$ sum of Δ)

$= 71°$

$x° = \angle ABD – \angle BDC = 71° – 38° = 33°$ (ext. $\angle$ of Δ)

I3 If $p = 10x – 4xy – 5x^2 – y^2 – 8$, where x and y are real numbers, find the largest value of p.

Reference: 1999 HG7, 2012 HG5, 2018 HI1

$p = –x^2 + 10x – 8 – (4x^2 + 4xy + y^2)$

$= 17 – (x – 5)^2 – (2x + y)^2$

≤ 17 = the largest value

I4 If the following three straight lines intersect at one point, find the value of c.

L_1: $6x + 6y – 19 = 0$ …… (1)

L_2: $18x + 12y + c = 0$ …… (2)

L_3: $2x + 3y – 8 = 0$ …… (3)

(1) – 2(3): $2x – 3 = 0 \Rightarrow x = 1.5$ …… (4)

Sub. $x = 1.5$ into (3): $3 + 3y – 8 = 0 \Rightarrow y = \frac{5}{3}$

Sub. $x = 1.5$, $y = \frac{5}{3}$ into (2): $27 + 20 + c = 0$

$\Rightarrow c = –47$

I5 It is known that $2 - 6\cos^2\theta = 7\sin\theta\cos\theta$, find the largest value of $\tan\theta$.

$2(\sin^2\theta + \cos^2\theta) - 6\cos^2\theta = 7\sin\theta\cos\theta$

$2\sin^2\theta - 7\sin\theta\cos\theta - 4\cos^2\theta = 0$

Divide the equation by $\cos^2\theta$: $2\tan^2\theta - 7\tan\theta - 4 = 0$

$(2\tan\theta + 1)(\tan\theta - 4) = 0$

$\tan\theta = -\frac{1}{2}$ or 4

The largest value of $\tan\theta = 4$

I6 The total cost for 88 tickets was \$□293□. Because the printing machine was not functioning well, the first and the last digits of the 5-digit number were missing. If the cost for each ticket is \$$P$, where P is an integer, find the value of P.

Let the total cost of 88 tickets be $10000a + 2930 + b$, where a, b are integers between 0 and 9.

$88P = 10000a + 2930 + b$

$= (88\times113 + 56)a + (88\times33 + 26) + b$

$= 88\times(113a + 33) + 56a + 26 + b$

$\therefore$ $56a + 26 + b$ = multiple of 88

One possible guess is $a = 1$, $b = 6$

$P = 113 + 33 + 1 = 147$

I7 If p is the positive real root of $2x^3 + 7x^2 - 29x - 70 = 0$, find the value of p.

Let $f(x) = 2x^3 + 7x^2 - 29x - 70$.

$f(-2) = -16 + 28 + 58 - 70 = 0$

$\therefore$ $(x + 2)$ is a factor

$f(-5) = -250 + 175 + 145 - 70 = 0$

$\therefore$ $(x + 5)$ is a factor

By comparing coefficients, $f(x) = (x + 2)(x + 5)(2x - 7) = 0$

The positive root is 3.5, $p = 3.5$

I8 Two persons A, B can complete a task in 30 days when they work together. If they work together for 6 days and then A quits, B needs 40 days more in order to complete the task. If the proportion of the task A can finish each day is $\frac{1}{q}$, find the value of q.

Suppose B can finish the task alone in p days.

Then $\frac{1}{p} + \frac{1}{q} = \frac{1}{30}$ ....(1)

$6\left(\frac{1}{p} + \frac{1}{q}\right) + \frac{40}{p} = 1$ ....(2)

From (1): $\frac{1}{p} = \frac{1}{30} - \frac{1}{q}$ ....(3)

Sub. (3) into (2): $\frac{6}{q} + 46\left(\frac{1}{30} - \frac{1}{q}\right) = 1$

$\frac{23}{15} - 1 = \frac{40}{q}$

$\Rightarrow \frac{8}{15} = \frac{40}{q}$

$\Rightarrow q = 75$

I9 Let a, b, c be three distinct constants. It is given that

$$\frac{a^2}{(a-b)(a-c)(a+x)}+\frac{b^2}{(b-c)(b-a)(b+x)}+\frac{c^2}{(c-a)(c-b)(c+x)}\equiv\frac{p+qx+rx^2}{(a+x)(b+x)(c+x)}$$

where p, q r are constants, and $s = 7p + 8q + 9r$, find the value of s.

Theorem If α, β, γ are three distinct roots of $dx^2 + ex + f = 0$, then $d = e = f = 0$.

Proof: The quadratic equation has at most two distinct roots. If α, β, γ are three distinct roots, then it is identically equal to 0. i.e. $d = e = f = 0$.

Now multiply the given equation by $(a + x)(b + x)(c + x)$:

$$\frac{a^2(b+x)(c+x)}{(a-b)(a-c)}+\frac{b^2(a+x)(c+x)}{(b-c)(b-a)}+\frac{c^2(a+x)(b+x)}{(c-a)(c-b)}\equiv p+qx+rx^2$$

Put $x = -a, -b, -c$ respectively.

$$\begin{cases}a^2=p-qa+ra^2\\ b^2=p-qb+rb^2\\ c^2=p-qc+rc^2\end{cases}\Rightarrow\begin{cases}a^2(r-1)-qa-p=0\\ b^2(r-1)-qb-p=0\\ c^2(r-1)-qc-p=0\end{cases}$$

$\therefore$ a, b and c are three distinct roots of $(r-1)x^2 - qx - p = 0$

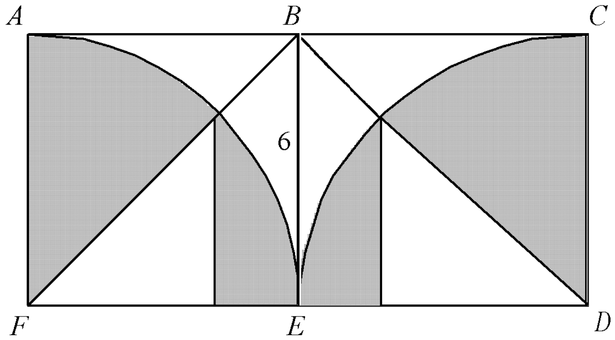
By the above theorem, $p = 0$, $q = 0$, $r = 1$.

$s = 7p + 8q + 9r = 9$

I10 In figure 2, $ABEF$, $BCDE$ are two squares, $BE = 6$ cm, and $\overset{\frown}{AE}$ and $\overset{\frown}{CE}$ are the arcs drawn with centres F and D respectively. If the total area of the shaded parts is S cm^2, find the value of S. (Assume $\pi = 3$.)

Radii of the two quadrants = 6

The two unshaded triangles are identical, base = height = $6\sin 45° = 3\sqrt{2}$



圖二 Figure 2

Shaded area = area of 2 quadrants – area of 2 unshaded Δs

$$=2\left[\frac{\pi\cdot 6^2}{4}-\frac{(3\sqrt{2})^2}{2}\right]=36$$

Group Events

G1 The time on the clock face is now one o'clock. After p minutes, the minute hand overlaps with the hour hand, find the minimum value of p.

In one hour, the minute hand rotated 360°, the hour hand rotated 30°.

So the minute hand is 330° faster than the hour hand for every hour (= every 60 minutes).

At one o'clock, the minute hand is 30° behind of the hour hand.

After p minutes, the minute hand will catch up the hour hand.

By ratio, $\frac{p}{30^\circ}=\frac{60}{330^\circ}$

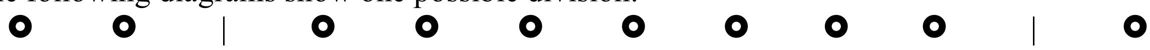
$p=\frac{60}{11}$

G2 In how many ways can 10 identical balls be distributed into 3 different boxes such that no box is to be empty?

Reference: 2006 HI6, 2010 HI1, 2012 HI2

Align the 10 balls in a row. There are 9 gaps between the 10 balls. Put 2 sticks into two of these gaps, so as to divide the balls into 3 groups.

The following diagrams show one possible division.



The three boxes contain 2 balls, 7 balls and 1 ball.

The number of ways is equivalent to the number of choosing 2 gaps as sticks from 9 gaps.

The number of ways is $C_2^9=\frac{9\times 8}{2}=36$

G3 Let $x=\sqrt{3-\sqrt{5}}+\sqrt{3+\sqrt{5}}$ and $y = x^2$, find the value of y.

$y=\left(\sqrt{3-\sqrt{5}}+\sqrt{3+\sqrt{5}}\right)^2$

$=3-\sqrt{5}+3+\sqrt{5}+2\sqrt{9-5}$

$= 6 + 2\times 2 = 10$

G4 If $\frac{4a}{1-x^{16}}\equiv\frac{2}{1-x}+\frac{2}{1+x}+\frac{4}{1+x^2}+\frac{8}{1+x^4}+\frac{16}{1+x^8}$, find the value of a.

$\frac{4a}{1-x^{16}}\equiv\frac{2}{1-x}+\frac{2}{1+x}+\frac{4}{1+x^2}+\frac{8}{1+x^4}+\frac{16}{1+x^8}$

$\equiv\frac{4}{1-x^2}+\frac{4}{1+x^2}+\frac{8}{1+x^4}+\frac{16}{1+x^8}$

$\equiv\frac{8}{1-x^4}+\frac{8}{1+x^4}+\frac{16}{1+x^8}$

$\equiv\frac{16}{1-x^8}+\frac{16}{1+x^8}\equiv\frac{32}{1-x^{16}}$

$a = 8$

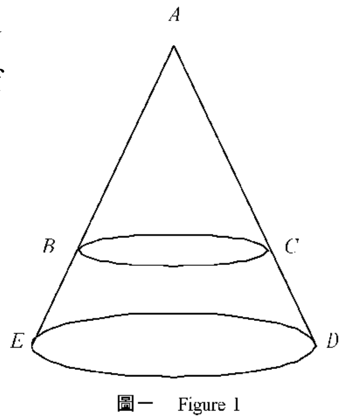
G5 In figure 1, ADE is a right circular cone. Suppose the cone is divided into two parts by a cut running parallel to the base and made $\frac{1}{4}$ of the way up, the ratio of the slant surface of the small cone ABC to that of the truncated base $BCDE$ is $1 : k$, find the value of k. **(Reference: 2000 FG5.3)**

curved surface area of small cone : that of large cone

$= 3^2 : 4^2 = 9 : 16$

curved surface area of small cone : that of the frustum

$= 9 : (16 - 9) = 9 : 7 = 1 : \frac{7}{9}\Rightarrow k=\frac{7}{9}$



圖一 Figure 1

G6 If a ten-digit number $2468m2468m$ is divisible by 3, find the maximum value of m.

$2 + 4 + 6 + 8 + m + 2 + 4 + 6 + 8 + m = 3k$

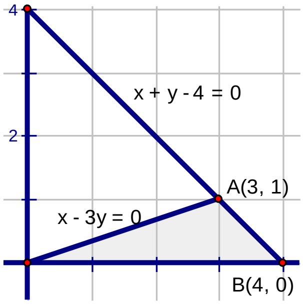
$40 + 2m = 3k$

$2m = 2, 8$ or 14

$m = 1, 4,$ or 7

Maximum = 7

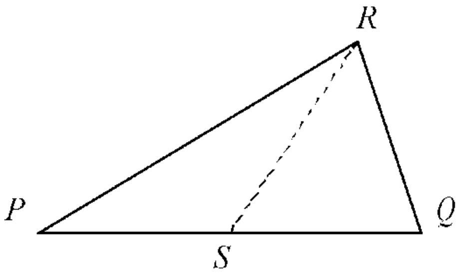
G7 Find the area enclosed by the x-axis and the straight lines $x - 3y = 0$, $x + y - 4 = 0$.



From the figure, the vertices of the enclosed area are (0,0), (4, 0), (3, 1)

The area $= \frac{1}{2}\cdot 4\cdot 1 = 2$

G8 In figure 2, PQR is a triangle, S is the mid-point of PQ, $RQ = PS = SQ$, and $\angle RQS = 2\angle RPS$. Let $\angle PSR = x°$, find the value of x.



圖二 Figure 2

Reference: 1998 FG3.1

Let $\angle RPS = y°$, $\angle RQS = 2y°$ (given)

$\angle QRS = \angle QSR = 90° - y°$ ($\angle$s sum of isos. Δ)

$\angle PRS = \angle QSR - \angle SPR = 90° - y° - y° = 90° - 2y°$

$\angle PRQ = \angle PRS + \angle QRS = 90° - 2y° + 90° - y° = 180° - 3y°$

Apply sine formula on ΔPQR

$$\frac{PQ}{\sin\angle PRQ} = \frac{RQ}{\sin\angle QPR}$$

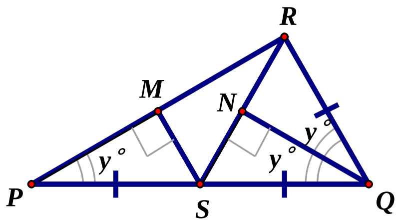
$$\Rightarrow \frac{2}{\sin(180° - 3y°)} = \frac{1}{\sin y°}$$

$2\sin y° = 3\sin y° - 4\sin^3 y°$

$4\sin^2 y° = 1$

$\sin y° = 0.5$, $y° = 30°$

$x° = 180° - (90° - y°) = 120°$ (adj. $\angle$ on st. line)



Method 2 Let $\angle RPS = y°$, $\angle RQS = 2y°$ (given)

Let M and N be the feet of perpendiculars drawn from S on PR and Q from RS respectively.

$\Delta QSN \cong \Delta QRN$ (R.H.S.)

$\Rightarrow \angle RQN = y° = \angle SQN$ (corr. $\angle$s $\cong$ Δ's)

$\Delta PSM \cong \Delta QSN \cong \Delta QRN$ (A.A.S.)

$MS = NS = NR$ (corr. sides $\cong$ Δ's)

$\sin\angle MRS = \frac{MS}{RS} = \frac{1}{2}$; $\angle MRS = 30°$, $\angle MSR = 60°$ ($\angle$s sum of ΔMRS)

$90° - y° + 60° + 90° - y° = 180°$ (adj. $\angle$ on st. line PSQ)

$\Rightarrow y° = 30°$

$x° = \angle PSR = 90° - y° + 60° = 120°$

G9 If x satisfies the equation $|x-3|+|x-5|=2$, find the minimum value of x.

Reference: 1994 HG1, 2004 FG4.2, 2008 HI8, 2008 FI1.3, 2010 HG6, 2011 FGS.1, 2012 FG2.3

Method 1

Case 1: $x \le 3$, $3-x+5-x=2$, $x=3$

Case 2: $3 < x \le 5$, $x-3+5-x=2$, always true, $3 < x \le 5$

Case 3: $5 < x$, $x-3+x-5=2$, $x=5$, no solution

Combined solution: $3 \le x \le 5$

The minimum value of $x = 3$

Method 2

Using the triangle inequality: $|a|+|b| \ge |a+b|$

$2=|x-3|+|5-x| \ge |x-3+5-x| = 2$

Equality holds when $x \ge 3$ and $5 \ge x$

$\Rightarrow$ the minimum value of $x = 3$

G10 3 shoes are chosen randomly from 6 pairs of shoes with different models, find the probability that exactly two out of the three shoes are of the same model.

In order that exactly two out of the three shoes are of the same model, either

Case 1 the first two chosen shoes are of the same model. (Probability = $1\times\frac{1}{11}$) or

Case 2 the last two chosen shoes are of the same model. (Probability = $1\times\frac{10}{11}\times\frac{1}{10}=\frac{1}{11}$) or

Case 3 the first and the 3rd chosen shoes are of the same model. (Probability = $1\times\frac{10}{11}\times\frac{1}{10}=\frac{1}{11}$)

$\therefore$ Required probability $=\frac{1}{11}+\frac{1}{11}+\frac{1}{11}=\frac{3}{11}$