Individual Events

I1	***P***	5	**I2**	***P***	23	**I3**	***P***	4	**I4**	***P***	12
	Q	4		***Q***	4		***Q***	33		***Q***	$\frac{2}{3}$
	R	1		***R***	8		***R***	3		***R***	4
	S	62		***S***	8		***S***	$3\sqrt{2}$		***S***	144

Group Events

G1	***a***	29	**G2**	***a***	12	**G3**	***a***	334501	**G4**	α	$\frac{180}{7}$
	b	7		***b***	6		***b***	$\frac{1}{3}$		***b***	$\frac{1}{5}$
	c	100		***c***	16		***c***	$1+\sqrt{2}$		***c***	10
	d	206		***d***	$\frac{44}{125}$		***d***	3		***d***	$\frac{1+\sqrt{5}}{2}$

Individual Event 1

I1.1 Let P be the units digit of $3^{2003} \times 5^{2002} \times 7^{2001}$. Find the value of P.

$3^{2003} \times 7^{2001}$ is an odd number, and the units digit of 5^{2002} is 5; $P = 5$

I1.2 If the equation $(x^2 - x - 1)^{x+P-1} = 1$ has Q integral solutions, find the value of Q.

The equation is $(x^2 - x - 1)^{x+4} = 1$

Either $x^2 - x - 1 = 1$ ....(1) or $x + 4 = 0$ ....(2) or ($x^2 - x - 1 = -1$ and $x + 4$ is even) ......(3)

(1): $x = 2$ or -1; (2): $x = -4$; (3): $x = 0$ or 1 and x is even $\Rightarrow x = 0$ only

Conclusion: $x = -4, -1, 0, 2$

$Q = 4$

I1.3 Let x, y be real numbers and $xy = 1$.

If the minimum value of $\frac{1}{x^4} + \frac{1}{Qy^4}$ is R, find the value of R.

$$\frac{1}{x^4} + \frac{1}{Qy^4} = \frac{1}{x^4} + \frac{1}{4y^4} \ge 2\sqrt{\frac{1}{x^4} \cdot \frac{1}{4y^4}} = 1 = R \text{ (A.M.} \ge \text{G.M.)}$$

I1.4 Let $x_R, x_{R+1}, \ldots, x_K$ ($K > R$) be $K - R + 1$ distinct positive integers and $x_R + x_{R+1} + \ldots + x_K = 2003$. If S is the maximum possible value of K, find the value of S. **(Reference: 2004 HI4)**

$x_1 + x_2 + \ldots + x_K = 2003$

For maximum possible value of K, $x_1 = 1$, $x_2 = 2$, … , $x_{K-1} = K - 1$

$1 + 2 + \ldots + K - 1 + x_K = 2003$

$\frac{(K-1)K}{2} + x_K = 2003$, $x_K \ge K$

$2003 \ge \frac{(K-1)K}{2} + K$

$4006 \ge K^2 + K$

$K^2 + K - 4006 \le 0$

$$\left(K - \frac{-1-\sqrt{1+4\times 4006}}{2}\right)\left(K - \frac{-1+\sqrt{1+4\times 4006}}{2}\right) \le 0$$

$$0 \le K \le \frac{-1+\sqrt{1+4\times 4006}}{2}$$

$$\frac{-1+\sqrt{1+4\times 4006}}{2} \approx \frac{-1+\sqrt{4\times 4006}}{2} = \sqrt{4006} - 0.5 \ge \sqrt{3969} - 0.5 = \sqrt{63^2} - 0.5 = 62.5$$

Maximum possible $K = 62 = S$

$1 + 2 + \ldots + 62 = 1953 = 2003 - 50$; $1 + 2 + \ldots + 61 + 112 = 2003$

Individual Event 2

I2.1 If the 50^{th} power of a two-digit number P is a 69-digit number, find the value of P.
(Given that log 2 = 0.3010, log 3 = 0.4771, log 11 = 1.0414.)

Reference: 1995 HG5 $\cdots 37^{100}$ $\cdots$ 157-digit number, $37^{15} \cdots$ n-digit $\cdots$.

$P^{50} = y$, $10 < P \le 99$, $10^{68} \le y < 10^{69}$

$P = y^{\frac{1}{50}}$; $10^{68 \div 50} < P < 10^{69 \div 50}$

$1.34 < \log P < 1.38$

$\log 22 = \log 2 + \log 11 = 1.3424$; $\log 24 = 3\log 2 + \log 3 = 1.3801$

$\log 22 < \log P < \log 24$, $P = 23$

I2.2 The roots of the equation $x^2 + ax - P + 7 = 0$ are α and β, whereas the roots of the equation $x^2 + bx - r = 0$ are $-\alpha$ and $-\beta$. If the positive root of the equation $(x^2+ax-P+7) + (x^2+bx-r) = 0$ is Q, find the value of Q.

$\alpha + \beta = -a$, $\alpha\beta = -16$; $-\alpha - \beta = -b$, $(-\alpha)(-\beta) = -r$

$\therefore b = -a$, $r = 16$

$(x^2 + ax - P + 7) + (x^2 + bx - r) = 0$ is equivalent to $(x^2 + ax - 16) + (x^2 - ax - 16) = 0$

$2x^2 - 32 = 0$

$x = 4$ or -4

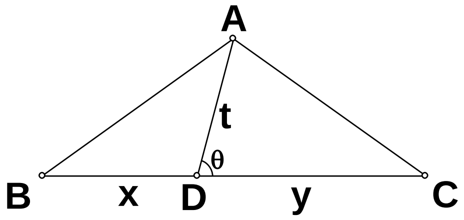
Q = positive root = 4

I2.3 Given that ΔABC is an isosceles triangle, $AB = AC = \sqrt{2}$, and $D_1, D_2, \ldots, D_Q$ are Q points on BC. Let $m_i = AD_i^2 + BD_i \times D_iC$. If $m_1 + m_2 + m_3 + \ldots + m_Q = R$, find the value of R.

Reference: 2010 HIS

As shown in the figure, $AB = AC = \sqrt{2}$

$BD = x$, $CD = y$, $AD = t$, $\angle ADC = \theta$



Apply cosine formula on ΔABD and ΔACD

$$\cos\theta = \frac{t^2 + y^2 - 2}{2ty}$$

$$\cos(180^\circ - \theta) = \frac{t^2 + x^2 - 2}{2tx}$$

since $\cos(180° – \theta) = –\cos\theta$

Add these equations and multiply by $2txy$:

$x(t^2 + y^2 - 2) + y(t^2 + x^2 - 2) = 0$

$(x + y)t^2 + (x + y)xy - 2(x + y) = 0$

$(x + y)(t^2 + xy - 2) = 0$

$t^2 + xy - 2 = 0$

$AD^2 + BD \cdot DC = 2$

$R = m_1 + m_2 + m_3 + m_4 = 2 + 2 + 2 + 2 = 8$

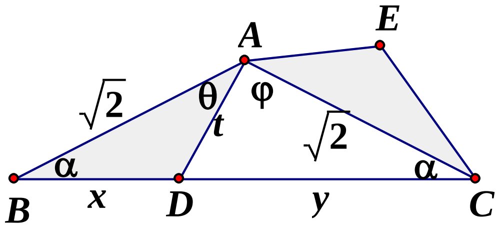
Method 2

Let $BD = x$, $CD = y$, $AD = t$, $\angle ABC = \alpha = \angle ACD$, $\angle BAD = \theta$, $\angle CAD = \varphi$.

Rotate AD anticlockwise about A to AE so that $\angle DAE = \angle BAC$.

$\angle CAE = \angle DAE - \varphi = \angle BAD = \theta$

By the property of rotation, $AE = AD = t$.



$\Delta CAE \cong \Delta BAD$ (S.A.S.)

$CE = BD = x$ (corr. sides, $\cong \Delta$s)

$\angle ACE = \angle ABD = \alpha$ (corr. $\angle$s, $\cong \Delta$s)

$\angle DAE + \angle DCE = \theta + \varphi + 2\alpha = 180°$ ($\angle$s sum of Δ)

$\Rightarrow 2\alpha = 180° - (\theta + \varphi)$ ……… (*)

The area of $ADCE = S_{\Delta ADE} + S_{\Delta CDE}$ = the area of ΔABC

$$\frac{1}{2}t^2\sin(\varphi+\theta) + \frac{1}{2}xy\sin 2\alpha = \frac{1}{2}\sqrt{2}^2\sin(\varphi+\theta)$$

$t^2 \sin(\theta + \varphi) + xy \sin[180° - (\theta + \varphi)] = 2 \sin(\theta + \varphi)$ by (*)

$\because \sin[180° - (\theta + \varphi)] = \sin(\theta + \varphi)$ $\therefore t^2 + xy = 2$

$AD^2 + BD \cdot DC = 2$

$R = m_1 + m_2 + m_3 + m_4 = 2 + 2 + 2 + 2 = 8$

I2.4 There are 2003 bags arranged from left to right. It is given that the leftmost bag contains R balls, and every 7 consecutive bags contains 19 balls altogether. If the rightmost bag contains S balls, find the value of S.

The leftmost bag contains 8 balls.

Starting from left to right, the total number of balls from 2^{nd} bag to the 7^{th} bag is 11.

The number of balls in the 8^{th} bag is therefore 8.

Similarly, the number of balls in the 15^{th} bag, 22^{th} bag, 29^{th} bag, … are all 8.

$2003 = 7 \times 286 + 1$, the rightmost bag should have the same number of balls as the leftmost bag.

$S = 8$

Individual Event 3

I3.1 Given that $\begin{cases} wxyz = 4 \\ w - xyz = 3 \end{cases}$ and $w > 0$. If the solution of w is P, find the value of P.

From (2), $xyz = w - 3$............(3), sub. into (1)

$w(w-3) = 4$

$w^2 - 3w - 4 = 0$

$w = 4$ or $w = -1$ (rejected)

$P = 4$

I3.2 Let $[y]$ represents the integral part of the decimal number y. For example, $[3.14] = 3$.

If $\left[\left(\sqrt{2}+1\right)^P\right] = Q$, find the value of Q. **(Reference: HKAL PM 1991 P1 Q11, 2005 HG5)**

Note that $0 < \sqrt{2} - 1 < 1$ and $0 < (\sqrt{2} - 1)^4 < 1$

$(\sqrt{2} + 1)^4 + (\sqrt{2} - 1)^4 = 2(\sqrt{2}^4 + 6\sqrt{2}^2 + 1) = 2(4 + 12 + 1) = 34$

$33 < (\sqrt{2} + 1)^4 < 34$

$Q = \left[\left(\sqrt{2}+1\right)^4\right] = 33$

I3.3 Given that $x_0y_0 \neq 0$ and $Qx_0^2 - 22\sqrt{3}x_0y_0 + 11y_0^2 = 0$. If $\frac{6x_0^2 + y_0^2}{6x_0^2 - y_0^2} = R$, find the value of R.

$33x_0^2 - 22\sqrt{3}x_0y_0 + 11y_0^2 = 0$

$3x_0^2 - 2\sqrt{3}x_0y_0 + y_0^2 = 0$

$(\sqrt{3}x_0 - y_0)^2 = 0$

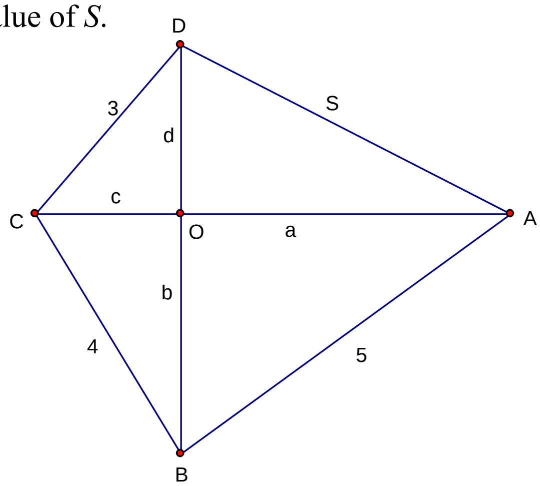
$y_0 = \sqrt{3}x_0$

$R = \frac{6x_0^2 + y_0^2}{6x_0^2 - y_0^2} = \frac{6x_0^2 + 3x_0^2}{6x_0^2 - 3x_0^2} = 3$

I3.4 The diagonals AC and BD of a quadrilateral $ABCD$ are perpendicular to each other.

Given that $AB = 5$, $BC = 4$, $CD = R$. If $DA = S$, find the value of S.

Reference 1994 FG10.1-2, 2001 FG2.2, 2018HI7



Suppose AC and BD intersect at O.

Let $OA = a$, $OB = b$, $OC = c$, $OD = d$.

$a^2 + b^2 = 5^2$ ....(1)

$b^2 + c^2 = 4^2$...........(2)

$c^2 + d^2 = 3^2$...........(3)

$d^2 + a^2 = S^2$ ....(4)

(1) + (3) – (2): $S^2 = d^2 + a^2 = 5^2 + 3^2 - 4^2 = 18$

$S = 3\sqrt{2}$

Individual Event 4

I4.1 Suppose the 9-digit number $\overline{32x35717y}$ is a multiple of 72, and $P = xy$, find the value of P.

$72 = 8\times9$, the number is divisible by 8 and 9. **(Reference: 2001 FG1.3, 2017 HI1)**

$\overline{17y}$ is divisible by 8, i.e. $y = 6$.

$3 + 2 + x + 3 + 5 + 7 + 1 + 7 + 6 = 9m$, where m is an integer.

$34 + x = 9m$, $x = 2$

$P = xy = 2\times6 = 12$

I4.2 Given that the lines $4x + y = \frac{P}{3}$, $mx + y = 0$ and $2x - 3my = 4$ cannot form a triangle. Suppose that $m > 0$ and Q is the minimum possible value of m, find Q.

Slope of $L_1 = -4$, slope of $L_2 = -m$, slope of $L_3 = \frac{2}{3m}$

If $L_1 // L_2$: $m = 4$; if $L_2 // L_3$: $m^2 = -\frac{2}{3}$ (no solution); if $L_1 // L_3$: $m = -\frac{1}{6}$ (rejected, $\because m > 0$)

If they are concurrent: $\begin{cases} 4x+y=4 & \cdots\cdots(1) \\ mx+y=0 & \cdots\cdots(2) \\ 2x-3my=4 & \cdots\cdots(3) \end{cases}$

Solve (1), (2) gives: $x = \frac{4}{4-m}; y = \frac{-4m}{4-m}$

Sub. into (3): $\frac{2\times4}{4-m} - \frac{3m(-4m)}{4-m} = 4$

$3m^2 + m - 2 = 0$

$(m + 1)(3m - 2) = 0$

$m = \frac{2}{3}$ (rejected -1, $\because m > 0$)

Minimum positive $m = \frac{2}{3}$

I4.3 Given that R, x, y, z are integers and $R > x > y > z$. If R, x, y, z satisfy the equation $2^R + 2^x + 2^y + 2^z = \frac{495Q}{16}$, find the value of R. **Reference: 2019 FI2.4**

$2^R + 2^x + 2^y + 2^z = \frac{495\cdot\frac{2}{3}}{16} = \frac{165}{8} = 20 + \frac{5}{8} = 2^4 + 2^2 + \frac{1}{2} + \frac{1}{2^3}$

$R = 4$

I4.4 In Figure 1, Q is the interior point of ΔABC. Three straight lines passing through Q are parallel to the sides of the triangle such that $FE // AB$, $GK // AC$ and $HJ // BC$. Given that the areas of ΔKQE, ΔJFQ and ΔQGH are R, 9 and 49 respectively. If the area of ΔABC is S, find the value of S. (**Reference: IMO (HK) Preliminary Contest 2001 Q13**)

It is easy to show that all triangles are similar.

By the ratio of areas of similar triangles,

$S_{\Delta KQE} : S_{\Delta JFQ} : S_{\Delta QGH} = (QE)^2 : (FQ)^2 : (GH)^2$

$4 : 9 : 49 = (QE)^2 : (FQ)^2 : (GH)^2$

$QE : FQ : GH = 2 : 3 : 7$

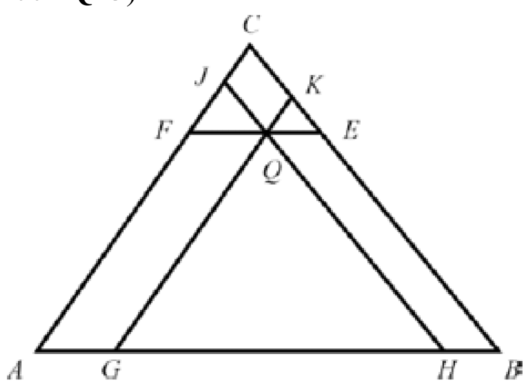
Let $QE = 2t$, $FQ = 3t$, $GH = 7t$

$AFQG$ and $BEQH$ are parallelograms.

$AG = 3t$, $BH = 2t$ (opp. sides of //-gram)

$AB = 3t + 7t + 2t = 12t$

$S_{\Delta ABC} = 4\times6^2 = 144 = S$



Group Event 1

G1.1 Given that n and k are natural numbers and $1 < k < n$. If $\frac{(1+2+3+\cdots+n)-k}{n-1}=10$ and $n + k = a$, find the value of a.

$\frac{n(n+1)}{2}-k=10n-10 \Rightarrow n^2 - 19n + 2(10 - k) = 0$

$\Delta = 281 + 8k$, n is an integer $\Rightarrow \Delta$ is a perfect square.

$281 + 8k = 289, 361, 441, \ldots \Rightarrow k = 1, 10, 20, \ldots$ Given $1 < k < n$, $\therefore k = 10, 20, \ldots$

when $k = 10$, $n = 19$; $a = n + k = 29$; when $k = 20$, $n = 20$ rejected.

G1.2 Given that $(x - 1)^2 + y^2 = 4$, where x and y are real numbers.

If the maximum value of $2x + y^2$ is b, find the value of b. **Reference: 2009 HI5, 2011 HI2**

$2x + y^2 = 2x + 4 - (x - 1)^2$

$= -x^2 + 2x - 1 + 2x + 4$

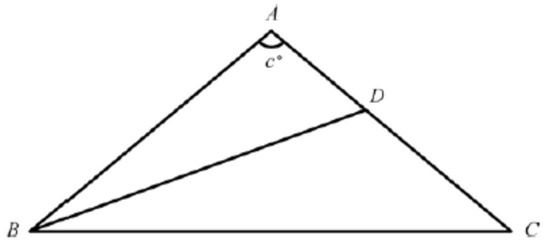
$= -x^2 + 4x + 3$

$= -(x^2 - 4x + 4) + 7$

$= -(x - 2)^2 + 7 \le 7 = b$

G1.3 In Figure 1, ΔABC is an isosceles triangle and $AB = AC$. Suppose the angle bisector of $\angle B$ meets AC at D and $BC = BD + AD$.

Let $\angle A = c°$, find the value of c.

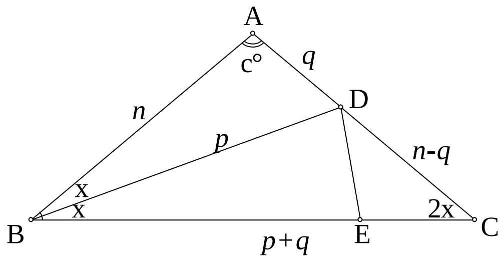


Let $AB = n = AC$; $AD = q$, $BD = p$, $CD = n - q$

$\angle ABD = x = \angle CBD$; $\angle ACB = 2x$.

Let E be a point on BC such that $BE = p$, $EC = q$

Apply sine formula on ΔABD and ΔBCD.



$\frac{n}{\sin\angle ADB}=\frac{q}{\sin x}$; $\frac{p+q}{\sin\angle BDC}=\frac{n-q}{\sin x}$

$\because \sin\angle ADB = \sin\angle BDC$

Dividing the above two equations

$\frac{n}{p+q}=\frac{q}{n-q}$

$\frac{AB}{BC}=\frac{EC}{CD}$ and $\angle ABC = \angle ECD = 2x$

$\Delta ABC \sim \Delta ECD$ (2 sides proportional, included angle)

$\therefore \angle CDE = 2x$ (corr. $\angle$s, $\sim \Delta$'s)

$\angle BED = 4x$ (ext. $\angle$ of ΔCDE)

$\angle BDE = 4x$ ($BD = BE = p$, base $\angle$s, isos. Δ)

$\angle ADB = 3x$ (ext. $\angle$ of ΔBCD)

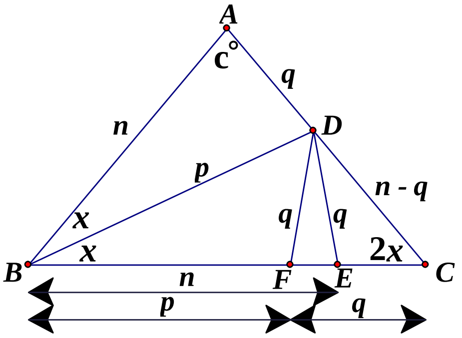
$2x + 3x + 4x = 180°$ (adj. $\angle$s on st. line ADC)

$x = 20°$

$c° = 180° - 4x = 100°$ ($\angle$ sum of ΔABC)

$c = 100$

Method 2



Claim $c° > 90°$

Proof: Otherwise, either $c° < 90°$ or $c° = 90°$

If $c° < 90°$, then locate a point E on BC so that $BE = n$

$\Delta ABD \cong \Delta EBD$ (S.A.S.)

$DE = q$ (corr. sides $\cong \Delta$s)

$\angle DEB = c°$ (corr. $\angle$s $\cong \Delta$s)

Locate a point F on BE so that $DF = q$

ΔDEF is isosceles

$\angle DFE = c°$ (base $\angle$s isos. Δ) …… (1)

$\angle ABD = x = \angle CBD$, $\angle ACB = 2x$ …… (2)

Consider ΔABC and ΔFCD

$\angle BAC = c° = \angle CFD$ (by (1))

$\angle ABC = 2x = \angle FCD$ (by (2))

$\therefore \Delta ABC \sim \Delta FCD$ (equiangular)

$CF : FD = BA : AC$ (corr. sides, $\sim\Delta$s)

$CF : FD = 1 : 1$ ($\because \Delta ABC$ is isosceles)

$\therefore CF = FD = q$

$BF = BC - CF = (p + q) - q = p$

$\because BF = p = BD$

$\therefore \Delta BDF$ is isosceles

$\angle BFD = \angle BDF$ (base $\angle$s isos. Δ)

$= \dfrac{180° - x}{2}$ ($\angle$ sum of Δ)

$< 90°$

$180° = \angle BFD + \angle EFD < 90° + 90° = 180°$, which is a contradiction

If $c° = 90°$, we use the same working steps as above, with $E = F$.

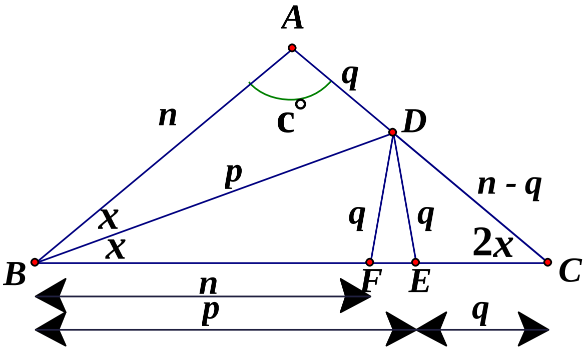
$\Delta ABC \sim \Delta FCD$ (equiangular)

$BE = n = BF = p$

$\therefore \Delta BDF$ is isosceles

$c° = 90° = \angle BFD = \dfrac{180° - x}{2} < 90°$, which is a contradiction

Conclusion: $c° > 90°$



Locate a point F on BC so that $BF = n$

$\Delta ABD \cong \Delta FBD$ (S.A.S.)

$DF = q$ (corr. sides $\cong \Delta$s)

$\angle DFB = c°$ (corr. $\angle$s $\cong \Delta$s)

$\angle DFC = 180° - c° < 90°$ (adj. $\angle$s on st. line)

Locate a point E on FC so that $DE = q$

ΔDEF is isosceles

$\angle DEF = 180° - c°$ (base $\angle$s isos. Δs)

$\angle DEC = c°$ (adj. $\angle$s on st. line) …… (3)

$\angle ABD = x = \angle CBD$; $\angle ACB = 2x$ …… (4)

Consider ΔABC and ΔECD

$\angle BAC = c° = \angle CED$ (by (3))

$\angle ABC = 2x = \angle ECD$ (by (4))

$\therefore \Delta ABC \sim \Delta ECD$ (equiangular)

$CE : ED = BA : AC$ (corr. sides, $\sim\Delta$s)

$CE : ED = 1 : 1$ ($\because \Delta ABC$ is isosceles)

$\therefore CE = ED = q$

$BE = BC - CE = (p + q) - q = p$

$\because BE = p = BD$

$\therefore \Delta BDE$ is isosceles

$\angle BED = \angle BDE = 180° - c°$ (adj. $\angle$s on st. line, base $\angle$s isos. Δ)

In ΔBDE, $x + 2(180° - c°) = 180°$ ($\angle$ sum of Δ)

$\Rightarrow x = 2c° - 180°$ $\cdots\cdots$ (5)

In ΔABC, $c° + 4x = 180°$ ($\angle$ sum of Δ) $\cdots\cdots$ (6)

Sub. (5) into (6), $c° + 4(2c° - 180°) = 180°$

$c = 100$

G1.4 Given that the sum of two prime numbers is 105. If the product of these prime numbers is d, find the value of d.

"2" is the only prime number which is an even integer.

The sum of two prime number is 105, which is odd

$\Rightarrow$ One prime is odd and the other prime is even

$\Rightarrow$ One prime is odd and the other prime is 2

$\Rightarrow$ One prime is 103 and the other prime is 2

$d = 2\times103 = 206$

Group Event 2

G2.1 Given that the equation $ax(x + 1) + bx(x + 2) + c(x + 1)(x + 2) = 0$ has roots 1 and 2. If $a + b + c = 2$, find the value of a.

Expand and rearrange the terms in descending orders of x:

$(a + b + c)x^2 + (a + 2b + 3c)x + 2c = 0$

$2x^2 + (a + b + c + b + 2c)x + 2c = 0$

$2x^2 + (b + 2c + 2)x + 2c = 0$

It is identical to $2(x - 1)(x - 2) = 0$

$\therefore\ b + 2c + 2 = -6;\ 2c = 4$

Solving these equations give $c = 2$, $b = -12$, $a = 12$

G2.2 Given that $48^x = 2$ and $48^y = 3$. If $8^{\frac{x+y}{1-x-y}} = b$, find the value of b.

Reference: 2001 HI1, 2004 FG4.3, 2005 HI9, 2006 FG4.3

Take logarithms on the two given equations: $x \log 48 = \log 2$, $y \log 48 = \log 3$

$\therefore\ x = \frac{\log 2}{\log 48};\ y = \frac{\log 3}{\log 48}$

$$\frac{x+y}{1-x-y} = \frac{\frac{\log 2}{\log 48} + \frac{\log 3}{\log 48}}{1 - \frac{\log 2}{\log 48} - \frac{\log 3}{\log 48}}$$

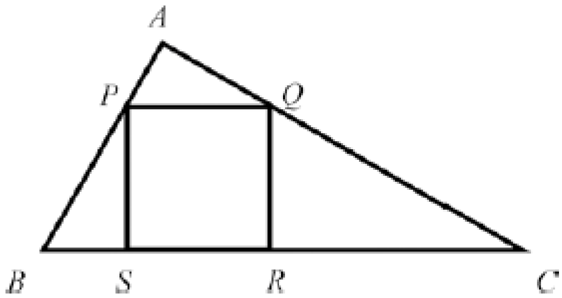
$$= \frac{\log 2 + \log 3}{\log 48 - \log 2 - \log 3}$$

$$= \frac{\log 6}{\log 8} \Rightarrow b = 8^{\frac{x+y}{1-x-y}}$$

$$\log b = \log\left(8^{\frac{x+y}{1-x-y}}\right) = \frac{x+y}{1-x-y}\log 8$$

$$= \frac{\log 6}{\log 8} \cdot \log 8 = \log 6 \Rightarrow b = 6$$

G2.3 In Figure 1, the square $PQRS$ is inscribed in ΔABC. The areas of ΔAPQ, ΔPBS and ΔQRC are 4, 4 and 12 respectively. If the area of the square is c, find the value of c.



Let $BC = a$, $PS = x$,

Let the altitude from A onto $BC = h$.

Area of $\Delta BPS = \frac{1}{2}x \cdot BS = 4 \Rightarrow BS = \frac{8}{x}$

Area of $\Delta CQR = \frac{1}{2}x \cdot CR = 12 \Rightarrow CR = \frac{24}{x}$

$BC = BS + SR + RC = \frac{8}{x} + x + \frac{24}{x} = x + \frac{32}{x}$ …… (1)

Area of $\Delta APQ = \frac{1}{2}x(h - x) = 4 \Rightarrow h = \frac{8}{x} + x$ …… (2)

Area of $\Delta ABC = \frac{1}{2}h \cdot BC = 4 + 4 + 12 + x^2 = 20 + x^2$ …… (3)

Sub. (1) and (2) into (3): $\frac{1}{2}\left(\frac{8}{x} + x\right) \cdot \left(x + \frac{32}{x}\right) = 20 + x^2$

$(8 + x^2)(x^2 + 32) = 40x^2 + 2x^4$

$x^4 + 40x^2 + 256 = 40x^2 + 2x^4$

$x^4 = 256$

c = area of the square $= x^2 = 16$

G2.4 In ΔABC, $\cos A = \frac{4}{5}$ and $\cos B = \frac{7}{25}$. If $\cos C = d$, find the value of d.

Reference: 2012 FI3.2

$\sin A = \frac{3}{5}$, $\sin B = \frac{24}{25}$

$\cos C = \cos(180° - A - B) = -\cos(A + B) = -\cos A \cos B + \sin A \sin B$

$= -\frac{4}{5} \cdot \frac{7}{25} + \frac{3}{5} \cdot \frac{24}{25} = \frac{44}{125}$

Group Event 3

G3.1 Let f be a function such that $f(1) = 1$ and for any integers m and n, $f(m + n) = f(m) + f(n) + mn$.

If $a = \frac{f(2003)}{6}$, find the value of a.

$f(n+1) = f(n) + n + 1 = f(n-1) + n + n + 1 = f(n-2) + n-1 + n + n+1 = \ldots = 1 + 2 + \ldots + n + n+1$

$\therefore f(n) = 1 + 2 + \ldots + n = \frac{n(n+1)}{2}$

$\frac{f(2003)}{6} = \frac{2004\times 2003}{12} = 334501$

G3.2 Suppose $x^{\frac{1}{2}} + x^{-\frac{1}{2}} = 3$, $b = \frac{x^{\frac{3}{2}} + x^{-\frac{3}{2}} - 3}{x^2 + x^{-2} - 2}$, find the value of b.

$\left(x^{\frac{1}{2}} + x^{-\frac{1}{2}}\right)^2 = 9 \Rightarrow x + x^{-1} = 7 \Rightarrow \left(x + x^{-1}\right)^2 = 49 \Rightarrow x^2 + x^{-2} = 47$

$\left(x^{\frac{1}{2}} + x^{-\frac{1}{2}}\right)\left(x + x^{-1}\right) = 3\times 7 \Rightarrow x^{\frac{3}{2}} + x^{-\frac{3}{2}} + x^{\frac{1}{2}} + x^{-\frac{1}{2}} = 21 \Rightarrow x^{\frac{3}{2}} + x^{-\frac{3}{2}} = 18$

$b = \frac{x^{\frac{3}{2}} + x^{-\frac{3}{2}} - 3}{x^2 + x^{-2} - 2} = \frac{18-3}{47-2} = \frac{1}{3}$

G3.3 Given that $f(n) = \sin\frac{n\pi}{4}$, where n is an integer.

If $c = f(1) + f(2) + \cdots + f(2003)$, find the value of c.

$f(1) + f(2) + f(3) + f(4) + f(5) + f(6) + f(7) + f(8)$

$= \frac{1}{\sqrt{2}} + 1 + \frac{1}{\sqrt{2}} + 0 - \frac{1}{\sqrt{2}} - 1 - \frac{1}{\sqrt{2}} + 0 = 0$

and the function repeats for every multiples of 8.

$c = f(2001) + f(2002) + f(2003) = \frac{1}{\sqrt{2}} + 1 + \frac{1}{\sqrt{2}} = 1 + \sqrt{2}$

G3.4 Given that $f(x) = \begin{cases} -2x+1, \text{ when } x<1 \\ x^2-2x, \text{ when } x\geq 1 \end{cases}$.

If d is the maximum integral solution of $f(x) = 3$, find the value of d.

When $x \geq 1$, $f(x) = 3$

$\Rightarrow x^2 - 2x = 3$

$\Rightarrow x^2 - 2x - 3 = 0$

$\Rightarrow x = 3$ or -1 (rejected)

When $x < 1$,

$-2x + 1 = 3$

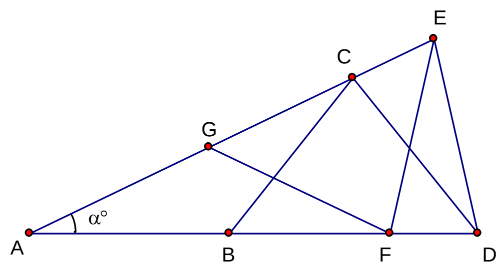
$\Rightarrow 2x = -2$

$\Rightarrow x = -1$

$\therefore d = 3$

Group Event 4

G4.1 In Figure 1, AE and AD are two straight lines and $AB = BC = CD = DE = EF = FG = GA$.
If $\angle DAE = \alpha°$, find the value of α.



$\angle AFG = \alpha° = \angle ACB$ (base $\angle$s isos. Δ)
$\angle CBD = 2\alpha° = \angle FGE$ (ext. $\angle$ of Δ)
$\angle FEG = 2\alpha° = \angle BDC$ (base $\angle$s isos. Δ)
$\angle DFE = 3\alpha° = \angle DCE$ (ext. $\angle$ of Δ)
$\angle ADE = 3\alpha° = \angle AED$ (base $\angle$s isos. Δ)
$\alpha° + 3\alpha° + 3\alpha° = 180°$ ($\angle$s sum of Δ)
$\alpha = \frac{180}{7}$

G4.2 Suppose $P(x) = a_0 + a_1x + a_2x^2 + \cdots + a_8x^8$ is a polynomial of degree 8 with real coefficients a_0, $a_1, \cdots, a_8$. If $P(k) = \frac{1}{k}$ when $k = 1, 2, \cdots, 9$, and $b = P(10)$, find the value of b.

Reference: 2018 HG1

$P(k) = \frac{1}{k}$, for $k = 1, 2, \cdots, 9$.

Let $F(x) = x\,P(x) - 1$, then $F(k) = k\,P(k) - 1 = 0$, for $k = 1, 2, \cdots, 9$.

$F(x)$ is a polynomial of degree 9 and the roots are $1, 2, \cdots, 9$.

$F(x) = x\,P(x) - 1 = c(x-1)(x-2)\cdots(x-9)$

$P(x) = \frac{c(x-1)(x-2)\cdots(x-9)+1}{x}$, which is a polynomial of degree 8.

Compare the constant term of $c(x-1)(x-2)\cdots(x-9) + 1 = 0$:

$-c\cdot 9! + 1 = 0$

$c = \frac{1}{9!} \Rightarrow P(x) = \frac{(x-1)(x-2)\cdots(x-9)+9!}{9!x}$

$P(10) = \frac{9!+9!}{9!\cdot 10} = \frac{1}{5}$

G4.3 Given two positive integers x and y, $xy - (x + y) = \text{HCF}(x, y) + \text{LCM}(x, y)$, where HCF($x$, y) and LCM(x, y) are respectively the greatest common divisor and the least common multiple of x and y. If c is the maximum possible value of $x + y$, find c.
Without loss of generality assume $x \ge y$.
Let the H.C.F. of x and y be m and $x = ma$, $y = mb$ where the H.C.F. of a, b is 1.
L.C.M. of x and $y = mab$. $a \ge b$.
$xy - (x + y) = \text{HCF} + \text{LCM} \Rightarrow m^2ab - m(a + b) = m + mab$
$ab(m - 1) = a + b + 1$
$m - 1 = \frac{1}{a} + \frac{1}{b} + \frac{1}{ab}$
$1 \le m - 1 = \frac{1}{a} + \frac{1}{b} + \frac{1}{ab} \le 3$
$m = 2, 3$ or 4
when $m = 2$, $\frac{1}{a} + \frac{1}{b} + \frac{1}{ab} = 1 \Rightarrow a + b + 1 = ab \Rightarrow ab - a - b - 1 = 0$
$ab - a - b + 1 = 2$
$(a - 1)(b - 1) = 2$
$a = 3$, $b = 2$, $m = 2$, $x = 6$, $y = 4$, $c = x + y = 10$

When $m = 3$, $\frac{1}{a}+\frac{1}{b}+\frac{1}{ab}=2 \Rightarrow a + b + 1 = 2ab \Rightarrow 2ab - a - b - 1 = 0$

$4ab - 2a - 2b + 1 = 3$

$(2a - 1)(2b - 1) = 3$

$a = 2, b = 1, m = 3, x = 6, y = 3, c = x + y = 9$

When $m = 4$, $\frac{1}{a}+\frac{1}{b}+\frac{1}{ab}=3 \Rightarrow a + b + 1 = 3ab \Rightarrow 3ab - a - b - 1 = 0$

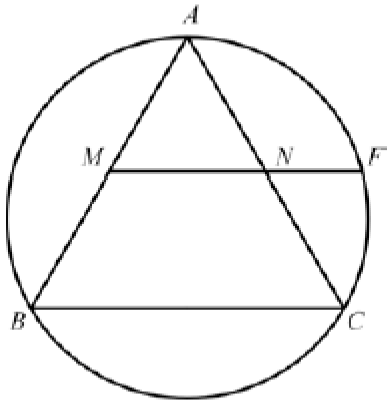
$9ab - 3a - 3b + 1 = 4$

$(3a - 1)(3b - 1) = 4$

$a = 1, b = 1, m = 4, x = 4, y = 4, c = x + y = 8$

Maximum $c = 10$

G4.4 In Figure 2, ΔABC is an equilateral triangle, points M and N are the midpoints of sides AB and AC respectively, and F is the intersection of the line MN with the circle ABC.



If $d = \frac{MF}{MN}$, find the value of d.

Let O be the centre, join AO.

Suppose MN intersects AO at H.

Produce FNM to meet the circle at E.

Then it is easy to show that:

$MN \parallel BC$ (mid-point theorem)

$\Delta AMO \cong \Delta ANO$ (SSS)

$\Delta AMH \cong \Delta ANH$ (SAS)

$AO \perp MN$ and $MH = HN$ (corr. sides, $\cong \Delta$s)

$EH = HF$ ($\perp$ from centre bisect chords)

Let $EM = t$, $MN = a$, $NF = p$.

$t = EH - MH = HF - HN = p$

By intersecting chords theorem,

$AN \times NC = FN \times NE$

$a^2 = p(p + a)$

$p^2 + ap - a^2 = 0$

$$\left(\frac{p}{a}\right)^2 + \frac{p}{a} - 1 = 0$$

$$\frac{p}{a} = \frac{-1+\sqrt{5}}{2} \quad \text{or} \quad \frac{-1-\sqrt{5}}{2} \quad \text{(rejected)}$$

$$d = \frac{MF}{MN} = \frac{a+p}{a}$$

$$= 1 + \frac{-1+\sqrt{5}}{2}$$

$$= \frac{1+\sqrt{5}}{2}$$