

02-03 Individual	1	$\begin{matrix} *8 \\ \text{see the remark} \end{matrix}$	2	$-\frac{1}{2}$	3	9	4	6	5	8
	6	$\frac{3}{4}$	7	8	8	55	9	9	10	4

02-03 Group	1	$\frac{63}{2525}$	2	2	3	$\sqrt{3}$	4	$\frac{3\sqrt{5}}{4}$	5	-1
	6	$\frac{9}{8}$	7	10	8	40	9	20.56	10	10

- I1** Let f be a function such that for all integers m and n , $f(m)$ is an integer and $f(mn) = f(m)f(n)$. It is given that $f(m) > f(n)$ when $9 > m > n$, $f(2) = 3$ and $f(6) > 22$, find the value of $f(3)$.

Reference: 2012 FI4.3

$$f(4) = f(2) \times f(2) = 9$$

$$f(6) = f(2) \times f(3) = 3f(3) > 22 \Rightarrow f(3) > \frac{22}{3}$$

$$\therefore 3 < 4 \therefore \frac{22}{3} < f(3) < f(4) \Rightarrow \frac{22}{3} < f(3) < 9$$

$$f(3) \text{ is an integer} \Rightarrow f(3) = 8$$

Remark: The old version of the question was:

Let f be a function such that for all integers m and n , $f(m)$ is an integer and $f(mn) = f(m)f(n)$. It is given that $f(m) > f(n)$ when $m > n$, $f(2) = 3$ and $f(6) > 22$, find the value of $f(3)$.

The old version of the question was wrong because it can be proved that $f(15) > f(16)$.

Proof: $f(4) = f(2) \times f(2) = 9$

$$f(6) = f(2) \times f(3) = 3f(3) > 22 \Rightarrow f(3) > \frac{22}{3}$$

$$\therefore 3 < 4 \therefore \frac{22}{3} < f(3) < f(4) \Rightarrow \frac{22}{3} < f(3) < 9$$

$$f(3) \text{ is an integer} \Rightarrow f(3) = 8$$

$$f(8) = f(4) \times f(2) = 27; f(16) = f(8) \times f(2) = 81$$

$$f(9) = f(3) \times f(3) = 64; f(12) = f(4) \times f(3) = 72$$

$$f(9) < f(10) < f(12)$$

$$\Rightarrow 64 < f(10) < 72$$

$$\Rightarrow 64 < f(2) \times f(5) < 72$$

$$\Rightarrow 64 < 3f(5) < 72$$

$$\Rightarrow 21\frac{1}{3} < f(5) < 24$$

$$f(5) = 22 \text{ or } 23$$

$$\Rightarrow f(15) = f(3) \times f(5) = 8f(5)$$

$$= 176 \text{ or } 184 > 81$$

$$= f(16), \text{ which is a contradiction}$$

- I2** If $P = \frac{1}{4}$, find the value of $P \log_2 P$.

$$P \log_2 P = \frac{1}{4} \log_2 \frac{1}{4} = \frac{1}{4} \log_2 2^{-2} = \frac{1}{4} \times (-2) = -\frac{1}{2}$$

- 13** If $0 \leq x \leq 1$, find the maximum value of $\left[\log_{10} \left(\frac{99999x+1}{1000} \right) \right]^2$.

$$0 \leq 99999x \leq 99999$$

$$\Rightarrow 1 \leq 99999x + 1 \leq 100000$$

$$\Rightarrow \frac{1}{1000} \leq \frac{99999x+1}{1000} \leq 100$$

$$-3 \leq \log_{10} \left(\frac{99999x+1}{1000} \right) \leq 2$$

$$\Rightarrow \left[\log_{10} \left(\frac{99999x+1}{1000} \right) \right]^2 \leq 9$$

- 14** Given that a quadratic equation $a(x+1)(x+2) + b(x+2)(x+3) + c(x+3)(x+1) = 0$ has roots 0 and 1, and $k = \frac{a}{b}$, find the value of k .

$$\text{Put } x = 0, 2a + 6b + 3c = 0 \dots\dots\dots(1)$$

$$\text{Put } x = 1, 6a + 12b + 8c = 0$$

$$\Rightarrow 3a + 6b + 4c = 0 \dots\dots\dots(2)$$

$$3(2) - 4(1): a - 6b = 0$$

$$k = \frac{a}{b} = 6$$

- 15** There are n persons in the classroom. If each person in the classroom shakes hands exactly once with each other person in the classroom and there are altogether 28 handshakes. Find the value of n .

$$\text{There are altogether } {}_nC_2 \text{ hand-shaking: } \frac{n(n-1)}{2} = 28$$

$$\Rightarrow n = 8$$

- 16** If for any $0 < x < \frac{\pi}{2}$, $\cot \frac{1}{4}x - \cot x \equiv \frac{\sin kx}{\left(\sin \frac{1}{4}x \right)(\sin x)}$, where k is a constant, find the value of k

$$\frac{\cos \frac{1}{4}x}{\sin \frac{1}{4}x} - \frac{\cos x}{\sin x} \equiv \frac{\sin kx}{\left(\sin \frac{1}{4}x \right)(\sin x)}$$

$$\frac{\cos \frac{1}{4}x \sin x - \cos x \sin \frac{1}{4}x}{\left(\sin \frac{1}{4}x \right)(\sin x)} \equiv \frac{\sin kx}{\left(\sin \frac{1}{4}x \right)(\sin x)}$$

$$\frac{\sin \left(x - \frac{1}{4}x \right)}{\left(\sin \frac{1}{4}x \right)(\sin x)} \equiv \frac{\sin kx}{\left(\sin \frac{1}{4}x \right)(\sin x)}$$

$$k = \frac{3}{4}$$

- 17** In Figure 1, AB is a diagonal of the cube and $AB = \sqrt{12}$ cm. If the volume of the cube is M cm³, find the value of M .

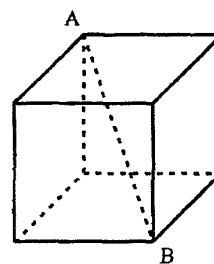
Reference: 1992 HI14, 1995 FI5.2

Let the side length of the square be x cm

$$x^2 + x^2 + x^2 = AB^2 = 12$$

$$x = 2$$

$$M = 2^3 = 8$$



- 18** In Figure 2, a square with area equal to 25 cm² is divided into 25 small squares with side length equal to 1 cm. If the total number of different squares in the figure is K , find the value of K .

The number of squares with side = 1 is 25

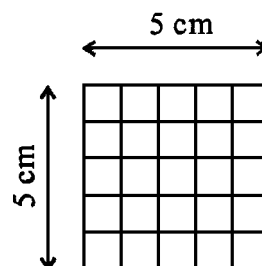
The number of squares with side = 2 is 16

The number of squares with side = 3 is 9

The number of squares with side = 4 is 4

The number of squares with side = 5 is 1

$$K = \text{total number of squares} = 1 + 4 + 9 + 16 + 25 = 55$$



- 19** It is given that the 6-digit number $N = \overline{x1527y}$ is a multiple of 4, and the remainder is 5 when N is divided by 11. Find the value of $x + y$.

N is a multiple of 4

\Rightarrow the last two digit of N must be divisible by 4

$\Rightarrow y = 2$ or 6

When N is divided by 11, the remainder is 5

$\Rightarrow (N - 5)$ is divisible by 11

When $y = 2$, $(N - 5) = \overline{x15267}$, it is divisible by 11

$$\Rightarrow x + 5 + 6 - (1 + 2 + 7) = 11m$$

$\Rightarrow x = 10$ rejected

When $y = 6$, $(N - 5) = \overline{x15271}$.

$$\text{It is divisible by 11} \Rightarrow x + 5 + 7 - (1 + 2 + 1) = 11m$$

$$\Rightarrow x = 3$$

$$x + y = 3 + 6 = 9$$

- 110** The sides of a triangle have lengths 7.5 cm, 11 cm and x cm respectively. If x is an integer, find the minimum value of x .

Triangle inequality gives:

$$7.5 + 11 > x \cdots (1)$$

$$7.5 + x > 11 \cdots (2)$$

$$11 + x > 7.5 \cdots (3)$$

Solve (1), (2), (3) gives

$$18.5 > x, x > 3.5 \text{ and } x > -3.5$$

$$\therefore 3.5 < x < 18.5$$

For integral value of x , the minimum value is 4.

Group Events

- G1** If $k = \frac{1}{4 \times 5 \times 6} + \frac{1}{5 \times 6 \times 7} + \frac{1}{6 \times 7 \times 8} + \cdots + \frac{1}{99 \times 100 \times 101}$, find the value of k .

Reference: 2014 FI3.2

$$\frac{1}{r(r+1)} - \frac{1}{(r+1)(r+2)} = \frac{(r+2) - r}{r(r+1)(r+2)} = 2 \cdot \frac{1}{r(r+1)(r+2)}$$

Put $r = 4$, $\frac{1}{4 \times 5} - \frac{1}{5 \times 6} = 2 \cdot \frac{1}{4 \times 5 \times 6}$

Put $r = 5$, $\frac{1}{5 \times 6} - \frac{1}{6 \times 7} = 2 \cdot \frac{1}{5 \times 6 \times 7}$

.....

Put $r = 99$, $\frac{1}{99 \times 100} - \frac{1}{100 \times 101} = 2 \cdot \frac{1}{99 \times 100 \times 101}$

Add up these equations, $k = \frac{1}{4 \times 5 \times 6} + \frac{1}{5 \times 6 \times 7} + \frac{1}{6 \times 7 \times 8} + \cdots + \frac{1}{99 \times 100 \times 101}$
 $= \frac{1}{2} \left(\frac{1}{4 \times 5} - \frac{1}{100 \times 101} \right) = \frac{1}{2} \cdot \frac{504}{10100} = \frac{63}{2525}$

- G2** Suppose $x^y + x^{-y} = 2\sqrt{2}$ and $x^y - x^{-y} = k$, where $x > 1$ and $y > 0$, find the value of k .

$$x^y + \frac{1}{x^y} = 2\sqrt{2}$$

$$\Rightarrow \left(x^y + \frac{1}{x^y} \right)^2 = 8$$

$$\Rightarrow x^{2y} + \frac{1}{x^{2y}} + 2 = 8$$

$$\Rightarrow x^{2y} + \frac{1}{x^{2y}} - 2 = 4$$

$$\Rightarrow \left(x^y - \frac{1}{x^y} \right)^2 = 4$$

$$k = x^y - x^{-y} = x^y - \frac{1}{x^y} = 2$$

- G3** In Figure 1, $\angle A : \angle B : \angle C = 3 : 2 : 1$,
 $a : b : c = 2 : k : 1$, find the value of k .

Let $\angle A = 3t$, $\angle B = 2t$, $\angle C = t$

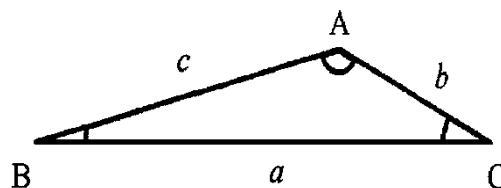
$3t + 2t + t = 180^\circ$ (\angle s sum of Δ)

$t = 30^\circ$, $\angle A = 90^\circ$, $\angle B = 60^\circ$, $\angle C = 30^\circ$

By sine formula, $a : b : c = \sin 90^\circ : \sin 60^\circ : \sin 30^\circ$

$$= 1 : \frac{\sqrt{3}}{2} : \frac{1}{2} = 2 : \sqrt{3} : 1$$

$$k = \sqrt{3}$$



- G4** In Figure 1, AMC and ANB are straight lines, $\angle NMC = \angle NBC = 90^\circ$, $AB = 4$, $BC = 3$, areas of $\triangle AMN$ and $\triangle ABC$ are in the ratio 1 : 4. Find the radius of the circle $BNMC$.

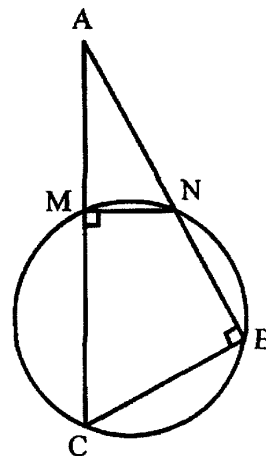
It is easy to show that $\triangle AMN \sim \triangle ABC$ (equiangular)

$$\frac{\text{Area of } \triangle AMN}{\text{Area of } \triangle ABC} = \left(\frac{MN}{BC}\right)^2 = \frac{1}{4}$$

$$\frac{MN}{BC} = \frac{1}{2} = \frac{AN}{AC} \Rightarrow \frac{AN}{5} = \frac{1}{2}, AN = 2.5$$

$$NB = AB - AN = 4 - 2.5 = 1.5$$

$$NC = \text{diameter} = \sqrt{1.5^2 + 3^2} = \frac{3}{2}\sqrt{5}, \text{ radius} = \frac{3\sqrt{5}}{4}$$



- G5** If the equation $x^2 + ax + 3b - a + 2 = 0$ has real root(s) for any real number a , find the maximum value of b .

$$\Delta = a^2 - 4(3b - a + 2)$$

$$= a^2 + 4a - 8 - 12b \geq 0$$

$$(a + 2)^2 - 12(1 + b) \geq 0$$

$$-12(1 + b) \geq 0$$

$$b \leq -1$$

- G6** Suppose the parabola $y = 4x^2 - 5x + c$ intersects the x -axis at $(\cos \theta, 0)$ and $(\cos \phi, 0)$ respectively. If θ and ϕ are two acute angles of a right-angled triangle, find the value of c .

$$\theta + \phi = 90^\circ$$

$$\Rightarrow \phi = 90^\circ - \theta$$

$$\cos \theta + \cos \phi = \frac{5}{4}$$

$$\Rightarrow \cos \theta + \cos(90^\circ - \theta) = \frac{5}{4}$$

$$\Rightarrow \cos \theta + \sin \theta = \frac{5}{4}$$

$$(\cos \theta + \sin \theta)^2 = \frac{25}{16}$$

$$\Rightarrow \cos^2 \theta + 2 \sin \theta \cos \theta + \sin^2 \theta = \frac{25}{16}$$

$$\Rightarrow \sin \theta \cos \theta = \frac{9}{32} = \frac{c}{4}$$

$$c = \frac{9}{8}$$

- G7** Suppose the straight line $y + 3x - 4 = 0$ intersects the parabola $y = x^2$ at points A and B respectively. If O is the origin, find the area of $\triangle OAB$.

$$\text{Sub. } y = 4 - 3x \text{ into } y = x^2$$

$$x^2 = 4 - 3x$$

$$\Rightarrow x^2 + 3x - 4 = 0$$

$$\Rightarrow (x - 1)(x + 4) = 0$$

$$\Rightarrow x = 1 \text{ or } -4$$

$$\text{When } x = 1, y = 1; \text{ when } x = -4, y = 16$$

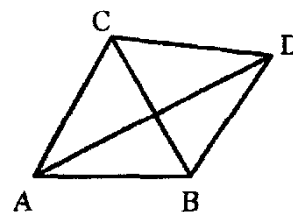
$$\text{Area of } \triangle OAB = \frac{1}{2} \begin{vmatrix} 1 & 1 \\ -4 & 16 \end{vmatrix} = 10$$

G8 In Figure 3, $AC = BC = CD$, $\angle ACB = 80^\circ$.

If $\angle ADB = x^\circ$, find the value of x . **Reference 2011 HG9, 2014 FG2.4**

We can use C as the centre, $AC = BC = CD$ as the radius to draw a circle to pass through A, B, D .

$$x = \frac{1}{2} \angle ACB = 40 \quad (\angle \text{ at centre twice } \angle \text{ at } \odot^{\text{ce}})$$



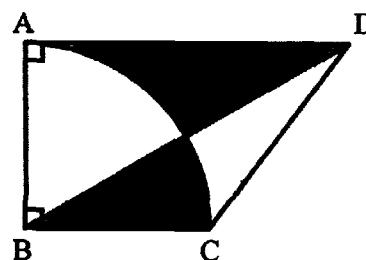
G9 In Figure 4, the sector ABC is one quarter of a circle with radius 4 cm. Suppose the areas of the two shaded parts are equal. Let the area of the trapezium $ABCD$ be $A \text{ cm}^2$, find the area of A . (Take $\pi = 3.14$)

Let $AD = x \text{ cm}$

Area of $\triangle ABD$ = area of sector ABC

$$\frac{1}{2} \cdot 4 \cdot x = \frac{1}{4} \cdot \pi (4)^2 \Rightarrow x = 2\pi$$

$$A = \frac{1}{2} \cdot (4 + x) \cdot 4 = 2(4 + 2 \times 3.14) = 20.56$$



G10 In Figure 5, the area of $\triangle DEF$ is 30 cm^2 . EIF , DJF and DKE are straight lines. P is the intersection point of DI and FK . Let $EI : IF = 1 : 2$, $FJ : JD = 3 : 4$, $DK : KE = 2 : 3$. Let the area of $\triangle DFP$ be $B \text{ cm}^2$, find the value of B .

Let $EI = t$, $IF = 2t$, $DK = 2x$, $KE = 3x$

Draw a line IM parallel to KF cutting DE at M .

By the theorem of equal ratio, $\frac{EM}{MK} = \frac{EI}{IF} = \frac{1}{2}$

$$\therefore EM = x, MK = 2x$$

$DP = PI$ (Intercept theorem)

$$\text{Area of } \triangle DIF = \frac{2}{3} \text{ Area of } \triangle DFE = 20 \text{ cm}^2$$

$$\text{Area of } \triangle DFP = \frac{1}{2} \text{ Area of } \triangle DIF = 10 \text{ cm}^2$$

$$\Rightarrow B = 10$$

