Individual Events

I1	***a***	6	**I2**	***P***	2	**I3**	***a***	–2	**I4**	***a***	2	**IS**	***P***	84
	b	$2+4\sqrt{2}$		***Q***	6		***b***	9		***b***	11		***Q***	8
	c	7		***R***	56		***c***	$\frac{1}{24}$		***c***	462		***R***	$\frac{\sqrt{3}}{4}$
	d	11		***S***	2352		***d***	$-\frac{7}{18}$		***d***	*334 see the remark		***S***	$1+\sqrt{2}$

Group Events

G1	***a***	47	**G2**	***a***	*2 see the remark	**G3**	***a***	–10	**G4**	***P***	500	**GS**	***a***	16
	b	101		***b***	4.5		***b***	0		***Q***	15		***b***	1
	c	43		***c***	15		***c***	2005		***R***	$\frac{1}{2}$		***c***	12
	d	0		***d***	1		***d***	2005		***S***	1		***d***	9

Individual Event 1

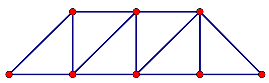
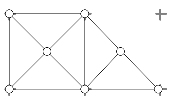
I1.1 Given that there are a positive integers less than 200 and each of them has exactly three positive factors, find the value of a.

If $x = rs$, where r and s are positive integers, then the positive factors of x may be 1, r, s and x.

In order to have exactly three positive factors, $r = s$ = a prime number.

Possible x = 4, 9, 25, 49, 121, 169. $a = 6$.

I1.2 If a copies of a right-angled isosceles triangle with hypotenuse $\sqrt{2}$ cm can be assembled to form a trapezium with perimeter equal to b cm, find the least possible value of b. (give the answer in surd form).

 or 

The perimeter = $6 + 2\sqrt{2} \approx 8.8$ or $2 + 4\sqrt{2} \approx 7.7$

The least possible value of $b = 2 + 4\sqrt{2}$

I1.3 If $\sin(c^2 - 3c + 17)° = \dfrac{4}{b-2}$, where $0 < c^2 - 3c + 17 < 90$ and $c > 0$, find the value of c.

$\sin(c^2 - 3c + 17)° = \dfrac{4}{2+4\sqrt{2}-2} = \dfrac{1}{\sqrt{2}}$

$c^2 - 3c + 17 = 45$

$c^2 - 3c - 28 = 0$

$(c - 7)(c + 4) = 0$

$c = 7$ or -4 (rejected)

I1.4 Given that the difference between two 3-digit numbers $\overline{xyz}$ and $\overline{zyx}$ is $700 - c$, where $x > z$. If d is the greatest value of $x + z$, find the value of d.

$\overline{xyz} - \overline{zyx} = 700 - c$

$100x + 10y + z - (100z + 10y + x) = 700 - 7$

$99x - 99z = 693$

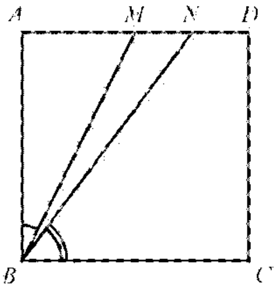
$x - z = 7$

Possible answers: $x = 8, z = 1$ or $x = 9, z = 2$

d is the greatest value of $x + z = 9 + 2 = 11$

Individual Event 2

I2.1 In Figure 1, $ABCD$ is a square, M is the mid-point of AD and N is the mid-point of MD. If $\angle CBN : \angle MBA = P : 1$, find the value of P.



Let $\angle ABM = \theta$, $\angle CBM = P\theta$. Let $AB = 4$, $AM = 2$, $MN = 1 = ND$.

$$\tan\theta = \frac{1}{2}$$

$$\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta} = \frac{2\times\frac{1}{2}}{1-\left(\frac{1}{2}\right)^2} = \frac{4}{3} = \tan P\theta,\ P = 2$$

I2.2 Given that $ABCD$ is a rhombus on a Cartesian plane, and the co-ordinates of its vertices are $A(0, 0)$, $B(P, 1)$, $C(u, v)$ and $D(1, P)$ respectively. If $u + v = Q$, find the value of Q.

$\because$ $ABCD$ is a rhombus, $\therefore$ It is also a parallelogram

By the property of parallelogram, the diagonals bisect each other

Mid point of B, D = mid point of AC

$$\left(\frac{1+2}{2},\frac{2+1}{2}\right) = \left(\frac{0+u}{2},\frac{0+v}{2}\right)$$

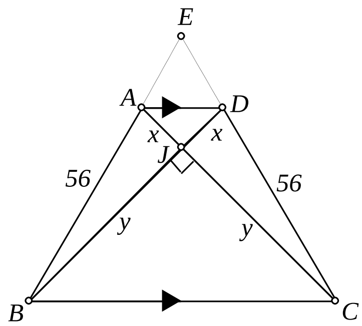
$u = 3$, $v = 3 \Rightarrow Q = u + v = 6$

I2.3 If $1 + (1 + 2) + (1 + 2 + 3) + \ldots + (1 + 2 + 3 + \ldots + Q) = R$, find the value of R.

$R = 1 + (1 + 2) + (1 + 2 + 3) + \ldots + (1 + 2 + 3 + \ldots + 6)$

$R = 1 + 3 + 6 + 10 + 15 + 21 = 56$

I2.4 In the figure, EBC is an equilateral triangle, and A, D lie on EB and EC respectively. Given that $AD // BC$, $AB = CD = R$ and $AC \perp BD$. If the area of the trapezium $ABCD$ is S, find the value of S.



$\angle ABC = \angle BCD = 60°$, AC intersects BD at J, $AC \perp BD$.

$\Delta ACD \cong \Delta DBA$ (S.A.S.)

$AC = BD$ (corr. sides, $\cong$ Δ's)

$\angle ABD = \angle DCA$ (corr. $\angle$s, $\cong$ Δ's)

$\therefore$ $\angle JBC = 60° - \angle ABD = \angle JCB$

ΔJBC is a right-angled isosceles triangle.

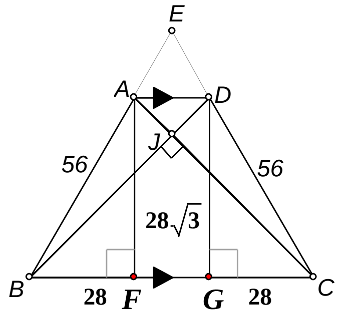
$\angle JBC = \angle JCB = 45°$

$BJ = CJ = y$, $AJ = AC - y = BD - y = DJ = x$

$\angle JAD = \angle JDA = 45°$, $\angle ADC = 120°$

Apply sine formula on ΔACD, $\dfrac{56}{\sin 45^\circ} = \dfrac{x+y}{\sin 120^\circ}$

$x + y = 28\sqrt{6}$, area $= \frac{1}{2}(x+y)^2\sin 90^\circ = \frac{1}{2}(784)(6) = 2352$



Method 2 $\angle EBC = 60° = \angle ECB$

Draw $AF \perp BC$ and $DG \perp BC$, cutting BC at F and G.

$BF = CG = 56\cos 60° = 28$

$AF = DG = 56\sin 60° = 28\sqrt{3}$

AC intersects BD at J, $AC \perp BD$.

$\Delta ACD \cong \Delta DBA$ (S.A.S.)

$\angle ABD = \angle DCA$ (corr. $\angle$s, $\cong$ Δ's)

$\therefore$ $\angle JBC = 60° - \angle ABD = \angle JCB$

ΔJBC is a right-angled isosceles triangle.

$\angle JBC = \angle JCB = 45°$

$CF = AF\cot 45° = 28\sqrt{3}$

$FG = CF - CG = 28\left(\sqrt{3}-1\right) = AD$

Area of the trapezium $ABCD = S = \dfrac{28\left(\sqrt{3}-1\right)\times 2 + 28\times 2}{2}\cdot\left(28\sqrt{3}\right) = 2352$

Individual Event 3

I3.1 Let $x \neq \pm 1$ and $x \neq -3$. If a is the real root of the equation $\frac{1}{x-1}+\frac{1}{x+3}=\frac{2}{x^2-1}$, find the value of a.

$$\frac{1}{x+3}=\frac{2}{x^2-1}-\frac{1}{x-1}$$

$$\frac{1}{x+3}=\frac{2-(x+1)}{(x-1)(x+1)}$$

$$\frac{1}{x+3}=\frac{1-x}{(x-1)(x+1)}$$

$$\frac{1}{x+3}=-\frac{1}{x+1}$$

$x+1=-x-3$

$\Rightarrow x=-2=a$

I3.2 If $b > 1$, $f(b)=\frac{-a}{\log_2 b}$ and $g(b)=1+\frac{1}{\log_3 b}$. If b satisfies the equation $|f(b)-g(b)|+f(b)+g(b)=3$, find the value of b.

Similar question: 2007 HI9

$$\left|\frac{2}{\log_2 b}-1-\frac{1}{\log_3 b}\right|+\frac{2}{\log_2 b}+1+\frac{1}{\log_3 b}=3$$

$$|2\log_b 2-\log_b b-\log_b 3|+2\log_b 2+\log_b 3=2$$

$$\left|\log_b \frac{4}{3b}\right|+\log_b 12=\log_b b^2$$

$$\log\frac{4}{3b}=\pm\log\frac{b^2}{12}$$

$$\log\frac{4}{3b}=\log\frac{b^2}{12} \text{ or } \log\frac{4}{3b}=\log\frac{12}{b^2}$$

$b^3=16$ or $b=9$

When $b^3=16$, $(b^3)^2=256<1728=12^3$

$\Rightarrow b^2<12 \Rightarrow \left|\log\frac{4}{3b}\right|=\log\frac{b^2}{12}<0$ rejected.

When $b=9$, $\left|\log\frac{4}{3b}\right|=\log\frac{b^2}{12}=\log\frac{81}{12}>0$ accepted.

I3.3 Given that x_0 satisfies the equation $x^2-5x+(b-8)=0$. If $c=\frac{x_0^2}{x_0^4+x_0^2+1}$, find the value of c.

$x^2-5x+1=0$, $x^2+1=5x$, $x^4+2x^2+1=25x^2$, $x^4+x^2+1=24x^2$

$$c=\frac{x_0^2}{x_0^4+x_0^2+1}=\frac{x_0^2}{24x_0^2}=\frac{1}{24}$$

I3.4 If -2 and $216c$ are the roots of the equation $px^2+dx=1$, find the value of d.

-2 and 9 are roots of $px^2+dx-1=0$

Product of roots $=-\frac{1}{p}=-2\times 9 \Rightarrow p=\frac{1}{18}$

Sum of roots $=-\frac{d}{p}=-2+9 \Rightarrow -18d=7 \Rightarrow d=-\frac{7}{18}$

Individual Event 4

I4.1 Let a be a real number.

If a satisfies the equation $\log_2(4^x + 4) = x + \log_2(2^{x+1} - 3)$, find the value of a.

$\log_2(4^x + 4) = \log_2 2^x + \log_2(2^{x+1} - 3)$

$4^x + 4 = 2^x \cdot (2^{x+1} - 3)$

$(2^x)^2 + 4 = 2\cdot(2^x)^2 - 3\cdot 2^x$

$0 = (2^x)^2 - 3\cdot 2^x - 4$

$(2^x - 4)(2^x + 1) = 0$

$2^x = 4$, $x = 2 = a$

I4.2 Given that n is a natural number.

If $b = n^3 - 4an^2 - 12n + 144$ is a prime number, find the value of b. **(Reference: 2011 FI3.3)**

Let $f(n) = n^3 - 8n^2 - 12n + 144$

$f(6) = 6^3 - 8\cdot 6^2 - 12\cdot 6 + 144 = 216 - 288 - 72 + 144 = 0$

$\therefore f(6)$ is a factor

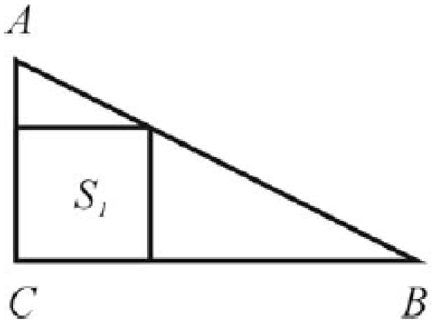
By division, $f(n) = (n - 6)(n - 6)(n + 4)$

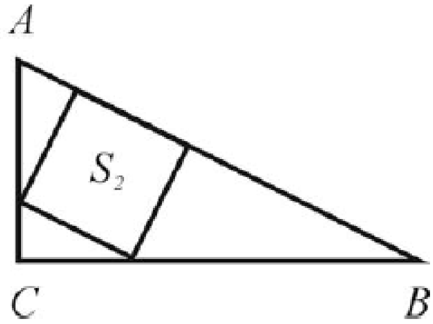
$b = n^3 - 8n^2 - 12n + 144$, it is a prime $\Rightarrow n - 6 = 1$, $n = 7$, $b = 11$

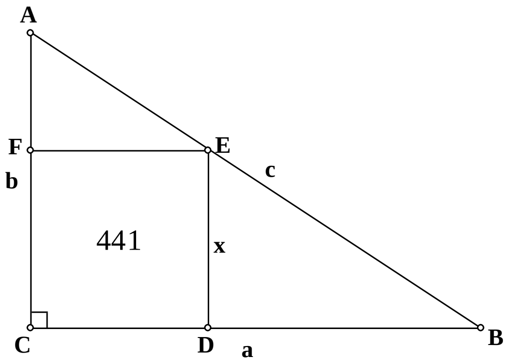
I4.3 In Figure 1, S_1 and S_2 are two different inscribed squares of the right-angled triangle ABC.

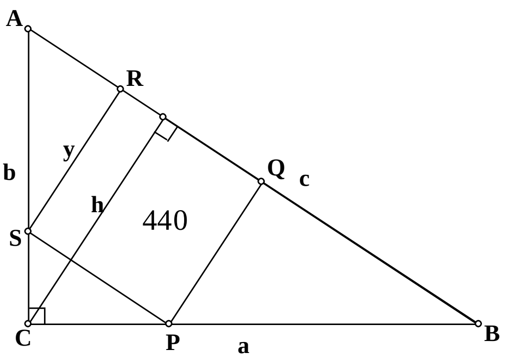
If the area of S_1 is $40b + 1$, the area of S_2 is $40b$ and $AC + CB = c$, find the value of c.

Reference: American Invitation Mathematics Examination 1987 Q15









Add the label D, E, F, P, Q, R, S as shown. $CDEF$, $PQRS$ are squares.

Let $DE = x$, $SR = y$, then $x = \sqrt{441} = 21$, $y = \sqrt{440}$. Let $BC = a$, $AC = b$, $AB = c = \sqrt{a^2 + b^2}$.

Let the height of the triangle drawn from C onto AB be h, then $ab = ch = 2$ area of Δ .....(*)

$\Delta AFE \sim \Delta ACB$: $\dfrac{b-x}{b} = \dfrac{x}{a} \Rightarrow x = \dfrac{ab}{a+b} = 21$ ....(1)

$\Delta CSP \sim \Delta CAB$: $\dfrac{\text{height of } \Delta \text{CSP from C}}{SP} = \dfrac{h}{c} \Rightarrow \dfrac{h-y}{y} = \dfrac{h}{c} \Rightarrow y = \dfrac{ch}{c+h} = \sqrt{440}$

By (*), $\dfrac{ab}{c+\dfrac{ab}{c}} = \sqrt{440} \Rightarrow \sqrt{440} = \dfrac{ab\sqrt{a^2+b^2}}{a^2+ab+b^2}$ ......(2)

From (1) $ab = 21(a + b)$ .... (3), sub. (3) into (2):

$$\sqrt{440} = \frac{21(a+b)\sqrt{(a+b)^2-2ab}}{(a+b)^2-ab} = \frac{21(a+b)\sqrt{(a+b)^2-42(a+b)}}{(a+b)^2-21(a+b)} = \frac{21\sqrt{(a+b)^2-42(a+b)}}{(a+b)-21}$$

Cross multiplying and squaring both sides:

$440[(a+b)^2 - 42(a+b) + 441] = 441[(a+b)^2 - 42(a+b)]$

$(a+b)^2 - 42(a+b) - 440\times441 = 0$

$(a+b-462)(a+b+420) = 0$

$AC + CB = a + b = 462$

I4.4 Given that $241c + 214 = d^2$, find the positive value of d.

$d^2 = 241\times462 + 214 = 111556$

$d = \sqrt{111556}$

Reference: 昌爸工作坊圖解直式開平方

Divide 111556 into 3 groups of numbers 11, 15, 56.

Find the maximum integer p such that $p^2 \le 11 \Rightarrow p = 3$

$11 - p^2 = 2$

$3 + 3 = 6$

Find the maximum integer q such that $(60 + q)q \le 215$

$\Rightarrow q = 3$

$215 - 63\times3 = 26$

$60 + q + q = 66$

Find the maximum integer r such that $(660 + r)r \le 2656$

$\Rightarrow r = 4$

$d = 334$

Method 2: $d^2 = 241\times462 + 214 = 111556$

$300^2 = 90000 < 111556 < 160000 = 400^2 \Rightarrow 300 < d < 400$

$330^2 = 108900 < 111556 < 115600 = 340^2 \Rightarrow 330 < d < 340$

The unit digit of d^2 is 6 $\Rightarrow$ the unit digit of d is 4 or 6

$335^2 = 112225 \Rightarrow d = 334$

or 111556 is not divisible by 3, but 336 is divisible by 3

$\therefore d = 334$

Remark: Original question: ……, find the value of d. $\Rightarrow d = \pm334$

Method 3

Observe the number patterns:

$34^2 = 1156$

$334^2 = 111556$

$3334^2 = 11115556$

……………………

$\therefore d = 334$

Also, $33^2 = 1089$

$333^2 = 110889$

$3333^2 = 11108889$

………………………

$3\times4 = 12$

$33\times34 = 1122$

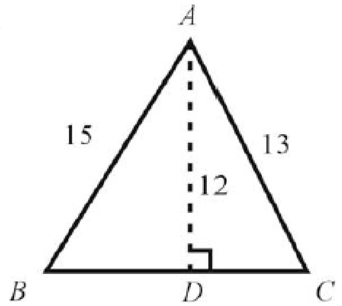
$333\times334 = 111222$

$3333\times3334 = 11112222$

…………………………

Individual Event (Spare)

IS.1 In figure 1, ΔABC is an acute triangle, $AB = 15$, $AC = 13$, and its altitude $AD = 12$. If the area of the ΔABC is P, find the value of P.



$BD = \sqrt{15^2 - 12^2} = 9$

$CD = \sqrt{13^2 - 12^2} = 5$

$P = \text{area of } \Delta = \frac{1}{2}(9+5)\times 12 = 84$

IS.2 Given that x and y are positive integers. If $x^4 + y^4$ is divided by $x + y$, the quotient is $P + 13$ and the remainder is Q, find the value of Q.

$P + 13 = 84 + 13 = 97$

$x^4 + y^4 = 97(x + y) + Q$, $0 \le Q < x + y$

Without loss of generality, assume $x \ge y$,

$x^4 < x^4 + y^4 = 97(x + y) + Q < 98(x + y) \le 98(2x) = 196x$

$x^3 < 196$

$x \le 5$

On the other hand, $x^4 + y^4 = 97(x + y) + Q = x^3(x + y) - y(x^3 - y^3)$

$\Rightarrow (x^3 - 97)(x + y) = y(x^3 - y^3) + Q$

RHS $\ge 0 \Rightarrow$ LHS $\ge 0 \Rightarrow x^3 \ge 97$

$x \ge 5$

$\therefore x = 5$

$5^4 + y^4 = 625 + y^4$ is divided by $5 + y$, the quotient is 97 and the remainder is Q, $1 \le y \le 5$

$625 + y^4 = 97(5 + y) + Q$, $0 \le Q < 5 + y \le 10$

$y^4 + 140 = 97y + Q$

$97y \le y^4 + 140 \le 97y + 9$

$131 \le 97y - y^4 \le 140$

By putting $y = 1, 2, 3, 4$ and 5 into the above inequalities, only $y = 4$ satisfies it.

$5^4 + 4^4 = 881 = (5 + 4)\times 97 + 8$, $Q = 8$

IS.3 Given that the perimeter of an equilateral triangle equals to that of a circle with radius $\frac{12}{Q}$ cm. If the area of the triangle is $R\pi^2$ cm^2, find the value of R.

Radius of circle $= \frac{12}{8}$ cm $= 1.5$ cm $\Rightarrow$ circumference $= 2\times\pi\times 1.5$ cm $= 3\pi$ cm

Side of the equilateral triangle $= \pi$ cm

Area of the triangle $= \frac{1}{2}\pi^2 \sin 60^\circ \text{ cm}^2 = \frac{\sqrt{3}}{4}\pi^2 \text{ cm}^2 \Rightarrow R = \frac{\sqrt{3}}{4}$

IS.4 Let $W = \frac{\sqrt{3}}{2R}$, $S = W + \cfrac{1}{W + \cfrac{1}{W + \cfrac{1}{W + \cdots}}}$, find the value of S.

$W = 2$, $S = 2 + \frac{1}{S} \Rightarrow S^2 - 2S - 1 = 0$, $S = \frac{1 \pm \sqrt{2}}{2}$, $S > 0$, $\therefore S = 1 + \sqrt{2}$ only

Group Event 1

G1.1 Given that a is an integer. If 50! is divisible by 2^a, find the largest possible value of a.

2, 4, 6, 8, … , 50 are divisible by 2, there are 25 even integers.

Reference: 1990 HG6, 1994 FG7.1, 1996 HI3, 2004 FG1.1, 2011 HG7, 2012 FI1.4, 2012 FG1.3

4, 8, … , 48 are divisible by 4, there are 12 multiples of 4.

8, …, 48 are divisible by 8, there are 6 multiples of 8.

16, …, 48 are divisible by 16, there are 3 multiples of 16.

32 is the only multiple of 32.

$a = 25 + 12 + 6 + 3 + 1 = 47$

G1.2 Let $[x]$ be the largest integer not greater than x. For example, $[2.5] = 2$.

If $b = \left[100\times\frac{11\times77+12\times78+13\times79+14\times80}{11\times76+12\times77+13\times78+14\times79}\right]$, find the value of b.

$$100\times\frac{11\times77+12\times78+13\times79+14\times80}{11\times76+12\times77+13\times78+14\times79}$$

$$=100\times\frac{11\times76+12\times77+13\times78+14\times79+11+12+13+14}{11\times76+12\times77+13\times78+14\times79}$$

$$=100\times\left(1+\frac{11+12+13+14}{11\times76+12\times77+13\times78+14\times79}\right)$$

$$= 100 + \frac{11\times100+12\times100+13\times100+14\times100}{11\times76+12\times77+13\times78+14\times79}$$

$11\times76+12\times77+13\times78+14\times79 < 11\times100+12\times100+13\times100+14\times100 < 2(11\times76+12\times77+13\times78+14\times79)$

$$1<\frac{11\times77+12\times78+13\times79+14\times80}{11\times76+12\times77+13\times78+14\times79}<2$$

$$101<100\times\frac{11\times77+12\times78+13\times79+14\times80}{11\times76+12\times77+13\times78+14\times79}<102,\ b = 101$$

G1.3 If there are c multiples of 7 between 200 and 500, find the value of c.

$\frac{200}{7} = 28.6$, the least multiple of 7 is $7\times29 = 203$

$\frac{500}{7} = 71.4$, the greatest multiple of 7 is $7\times71 = 497$

$497 = a + (c - 1)d = 203 + (n - 1)\cdot7$; $c = 43$

G1.4 Given that $0 \le x_0 \le \frac{\pi}{2}$ and x_0 satisfies the equation $\sqrt{\sin x+1}-\sqrt{1-\sin x}=\sin\frac{x}{2}$. If $d = \tan x_0$, find the value of d.

$$\left(\sqrt{\sin x+1}-\sqrt{1-\sin x}\right)^2=\left(\sin\frac{x}{2}\right)^2$$

$$1 + \sin x + 1 - \sin x - 2\sqrt{1-\sin^2 x}=\frac{1-\cos x}{2}$$

$2(2 - 2\cos x) = 1 - \cos x$

$\cos x = 1$

$x_0 = 0$

$d = \tan x_0 = 0$

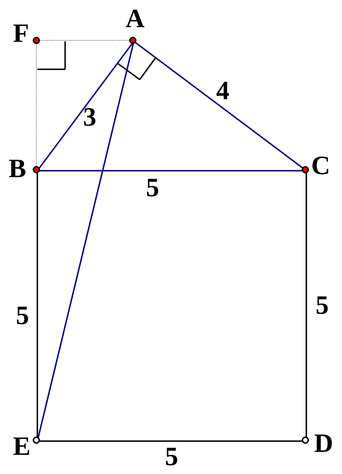
Group Event 2

G2.1 If the tens digit of 5^{5^5} is *a*, find the value of *a*.

$5^{5^5} = \cdots\cdots 125$, the tens digit = $a = 2$

Remark Original question: If the tenth-place digit ……, this is the position of the first digit to the right of the decimal point.

G2.2 In Figure 1, ΔABC is a right-angled triangle, $AB = 3$ cm, $AC = 4$ cm and $BC = 5$ cm. If $BCDE$ is a square and the area of ΔABE is b cm^2, find the value of b.



$\cos B = \frac{AB}{BC} = \frac{3}{5}$

Height of ΔABE from A

$= AF = AB \sin(90° - B) = 3 \times \frac{3}{5} = \frac{9}{5}$

$b = \frac{1}{2} \cdot AF \times BE$

$= \frac{1}{2} \cdot 5 \times \frac{9}{5} = \frac{9}{2}$

G2.3 Given that there are *c* prime numbers less than 100 such that their unit digits are not square numbers, find the values of *c*.

The prime are: {2, 3, 5, 7, 13, 17, 23, 37, 43, 47, 53, 67, 73, 83, 97}

$c = 15$

G2.4 If the lines $y = x + d$ and $x = -y + d$ intersect at the point $(d - 1, d)$, find the value of *d*.

$x = -(x + d) + d$

$x = 0 = d - 1$

$d = 1$

Group Event 3

G3.1 If a is the smallest real root of the equation $\sqrt{x(x+1)(x+2)(x+3)+1}=71$, find the value of a.

Reference: 1993 HG6, 1995 FI4.4, 1996 FG10.1, 2000 FG3.1, 2012 FI2.3

Let $t = x + 1.5$, then the equation becomes $\sqrt{(t-1.5)(t-0.5)(t+0.5)(t+1.5)+1}=71$

$\sqrt{\left(t^2-\frac{9}{4}\right)\left(t^2-\frac{1}{4}\right)+1}=71$

$\sqrt{t^4-\frac{5}{2}t^2+\frac{9}{16}+1}=71$

$\Rightarrow\sqrt{t^4-\frac{5}{2}t^2+\frac{25}{16}}=71$

$\Rightarrow\sqrt{\left(t^2-\frac{5}{4}\right)^2}=71$

$t^2-\frac{5}{4}=71\Rightarrow t^2=\frac{289}{4}\Rightarrow t=\frac{17}{2}$ or $t=-\frac{17}{2}$

$x = t - 1.5 = \pm\frac{17}{2}-\frac{3}{2} = 7$ or -10

a = the smallest root = -10

G3.2 Given that p and q are prime numbers satisfying the equation $18p + 30q = 186$.

If $\log_8\frac{p}{3q+1} = b \geq 0$, find the value of b.

$18p + 30q = 186 \Rightarrow 3p + 5q = 31$

Note that the number "2" is the only prime number which is even.

$3p + 5q = 31$ = odd number $\Rightarrow$ either $p = 2$ or $q = 2$

If $p = 2$, then $q = 5$; $b=\log_8\frac{p}{3q+1}=\log_8\frac{2}{3\times5+1}=\log_8\frac{1}{16}<0$ (rejected)

If $q = 2$, then $p = 7$; $b=\log_8\frac{p}{3q+1}=\log_8\frac{7}{3\times2+1}=0$

G3.3 Given that for any real numbers x, y and z, $\oplus$ is an operation satisfying

(i) $x \oplus 0 = 1$, and

(ii) $(x \oplus y) \oplus z = (z \oplus xy) + z$.

If $1\oplus2004 = c$, find the value of c.

$c = 1 \oplus 2004$

$= (1\oplus0) \oplus2004$

$= (2004\oplus0)+ 2004$

$= 1 + 2004 = 2005$

G3.4 Given that $f(x) = (x^4 + 2x^3 + 4x - 5)^{2004} + 2004$. If $f\left(\sqrt{3}-1\right)=d$, find the value of d.

$x=\sqrt{3}-1$

$(x + 1)^2 = \left(\sqrt{3}\right)^2$

$x^2 + 2x - 2 = 0$

By division, $x^4 + 2x^3 + 4x - 5 = (x^2 + 2x - 2)(x^2 + 2) - 1 = -1$

$d=f\left(\sqrt{3}-1\right)$

$= f(x)$

$= (x^4 + 2x^3 + 4x - 5)^{2004} + 2004$

$= (-1)^{2004} + 2004 = 2005$

Group Event 4

G4.1 If $f(x)=\dfrac{4^x}{4^x+2}$ and $P=f\left(\dfrac{1}{1001}\right)+f\left(\dfrac{2}{1001}\right)+\cdots+f\left(\dfrac{1000}{1001}\right)$, find the value of P.

Reference: 2011 HG5, 2012 FI2.2

$f(x)+f(1-x)=\dfrac{4^x}{4^x+2}+\dfrac{4^{1-x}}{4^{1-x}+2}=\dfrac{4+4+2\cdot 4^x+2\cdot 4^{1-x}}{4+4+2\cdot 4^x+2\cdot 4^{1-x}}=1$

$P=f\left(\dfrac{1}{1001}\right)+f\left(\dfrac{2}{1001}\right)+\cdots+f\left(\dfrac{1000}{1001}\right)$

$=f\left(\dfrac{1}{1001}\right)+f\left(\dfrac{1000}{1001}\right)+f\left(\dfrac{2}{1001}\right)+f\left(\dfrac{999}{1001}\right)+\cdots+f\left(\dfrac{500}{1001}\right)+f\left(\dfrac{501}{1001}\right)=500$

G4.2 Let $f(x)=|x-a|+|x-15|+|x-a-15|$, where $a\le x\le 15$ and $0<a<15$.
If Q is the smallest value of $f(x)$, find the value of Q.

Reference: 1994 HG1, 2001 HG9, 2008 HI8, 2008 FI1.3, 2010 HG6, 2011 FGS.1, 2012 Final G2.3

$f(x)=x-a+15-x+15-x+a=30-x\ge 30-15=15=Q$

G4.3 If $2^m=3^n=36$ and $R=\dfrac{1}{m}+\dfrac{1}{n}$, find the value of R.

Reference: 2001 HI1, 2003 FG2.2, 2004 FG4.3, 2005 HI9, 2006 FG4.3

$\log 2^m=\log 3^n=\log 36$

$m\log 2=n\log 3=\log 36$

$m=\dfrac{\log 36}{\log 2}$; $n=\dfrac{\log 36}{\log 3}$

$\dfrac{1}{m}+\dfrac{1}{n}=\dfrac{\log 2}{\log 36}+\dfrac{\log 3}{\log 36}=\dfrac{\log 6}{\log 36}=\dfrac{\log 6}{2\log 6}=\dfrac{1}{2}$

G4.4 Let $[x]$ be the largest integer not greater than x, for example, $[2.5]=2$.
If $a=1+\dfrac{1}{2^2}+\dfrac{1}{3^2}+\cdots+\dfrac{1}{2004^2}$ and $S=[a]$, find the value of a.

$1<a=1+\dfrac{1}{2^2}+\dfrac{1}{3^2}+\cdots+\dfrac{1}{2004^2}<1+\dfrac{1}{1\times 2}+\dfrac{1}{2\times 3}+\cdots+\dfrac{1}{2003\times 2004}$

$=1+1-\dfrac{1}{2}+\dfrac{1}{2}-\dfrac{1}{3}+\cdots+\dfrac{1}{2003}-\dfrac{1}{2004}$

$=2-\dfrac{1}{2004}<2$

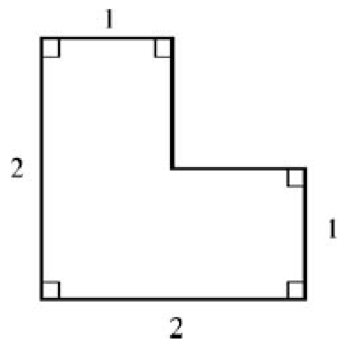
$S=[a]=1$

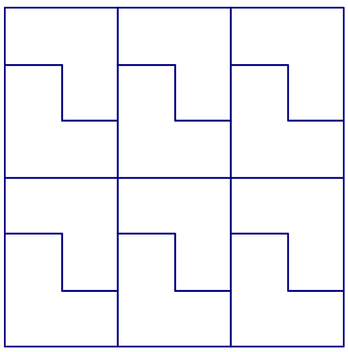
Group Event (Spare)

GS.1 For all integers n, F_n is defined by $F_n = F_{n-1} + F_{n-2}$, $F_0 = 0$ and $F_1 = 1$.
If $a = F_{-5} + F_{-4} + \dots + F_4 + F_5$, find the value of a.
$F_2 = 0 + 1 = 1$, $F_3 = 1 + 1 = 2$, $F_4 = 1 + 2 = 3$, $F_5 = 2 + 3 = 5$
$F_{-1} + 0 = 1 \Rightarrow F_{-1} = 1$, $F_{-2} + 1 = 0 \Rightarrow F_{-2} = -1$, $F_{-3} + (-1) = 1 \Rightarrow F_{-3} = 2$,
$F_{-4} + 2 = -1 \Rightarrow F_{-4} = -3$, $F_{-5} + (-3) = 2 \Rightarrow F_{-5} = 5$
$a = F_{-5} + F_{-4} + \dots + F_4 + F_5 = 5 - 3 + 2 - 1 + 1 + 0 + 1 + 1 + 2 + 3 + 5 = 16$

GS.2 Given that x_0 satisfies the equation $x^2 + x + 2 = 0$. If $b = x_0^4 + 2x_0^3 + 3x_0^2 + 2x_0 + 1$, find the value of b.
By division, $b = x_0^4 + 2x_0^3 + 3x_0^2 + 2x_0 + 1$
$= (x_0^2 + x_0 + 2)(x_0^2 + x_0) + 1 = 1$

GS.3 Figure 1 shows a tile. If C is the minimum number of tiles required to tile a square, find the value of C.





$C = 12$

GS.4 If the line $5x + 2y - 100 = 0$ has d points whose x and y coordinates are both positive integers, find the value of d.
$(x, y) = (18, 5), (16, 10), (14, 15), (12, 20), (10, 25), (8, 30), (6, 35), (4, 40), (2, 45)$
$d = 9$