

03-04 Individual	1	-2009010	2	7	3	45	4	700	5	6
	6	12.5	7	6	8	$-\frac{3}{16}$	9	12	10	$\frac{19}{4}$

03-04 Group	1	2475	2	1	3	6	4	32	5	5
	6	500	7	34.56	8	$\frac{1}{6}$	9	10	10	$\frac{5}{3}$

Individual Events

- I1** Let $A = 1^2 - 2^2 + 3^2 - 4^2 + \dots + 2003^2 - 2004^2$, find the value of A .

Reference: 1997 HI5, 2002 FG2.3, 2015 FI3.2, 2015 FG4.1

$$A = (1^2 - 2^2) + (3^2 - 4^2) + \dots + (2003^2 - 2004^2)$$

$$= -3 - 7 - 11 - \dots - 4007, \text{ this is an arithmetic series, } a = -3, \ell = -4007 = a + (n-1)(-4), n = 1002$$

$$= -\frac{3 + 4007}{2} \times 1002 = -2009010$$

- I2** If $\sqrt[2003]{B} = 2003$, C is the units digit of B , find the value of C .
 $B = 2003^{2003}$; $3^1 = 3, 3^2 = 9, 3^3 = 27, 3^4 = 81, 3^5 = 243$; the units digit repeats for every multiples of 4. $2003^{2003} = 2003^{4 \times 500 + 3}$; the units digit is 7; $C = 7$.

- I3** If $x + y + z = 10$, $x^2 + y^2 + z^2 = 10$ and $xy + yz + zx = m$, find the value of m .
 $(x + y + z)^2 = 10^2 \Rightarrow x^2 + y^2 + z^2 + 2(xy + yz + zx) = 100 \Rightarrow 10 + 2m = 100 \Rightarrow m = 45$

- I4** Arrange the natural numbers in the following order. In this arrangement, 9 is in the row 3 and the column 2. If the number 2003 is in the row x and the column y , find the value of xy .

1	2	4	7	11	16	...
3	5	8	12	17	...	
6	9	13	18	...		
10	14	19	...			
15	20	...				
21	...					

Reference: 2003 FI1.4

Consider the integers in the first column of each row: 1, 3, 6, 10, ...

They are equivalent to 1, $1 + 2$, $1 + 2 + 3$, $1 + 2 + 3 + 4$, ...

$$\text{The first integer in the } n^{\text{th}} \text{ row} = 1 + 2 + 3 + \dots + n = \frac{n(n+1)}{2}$$

$$\frac{n(n+1)}{2} < 2003 \Rightarrow n(n+1) < 4006$$

$$\therefore 62 \times 63 = 3906, 63 \times 64 = 4032$$

$$\therefore \text{The greatest possible } n = 62$$

$$3906 \div 2 = 1953$$

$$\text{The } 63^{\text{rd}} \text{ element of the first row} = 1954$$

$$\text{The } 62^{\text{nd}} \text{ element of the second row} = 1955, \dots \text{ and so on.}$$

$$2003 = 1953 + 50; 63 - 50 + 1 = 14$$

$$\text{The } 14^{\text{th}} \text{ element of the } 50^{\text{th}} \text{ row is } 2003; x = 50, y = 14$$

$$xy = 50 \times 14 = 700$$

- 15** Let $E = \sqrt{12+6\sqrt{3}} + \sqrt{12-6\sqrt{3}}$, find the value of E .

Reference: 1993 FI1.4, 1999 HG3, 2001 FG2.1, 2011 HI7, 2015 FI4.2, 2015 FG3.1

$$\sqrt{12+6\sqrt{3}} = \sqrt{9+3+2\sqrt{9 \times 3}} = \sqrt{a+b+2\sqrt{ab}} = \sqrt{a} + \sqrt{b} = 3 + \sqrt{3}$$

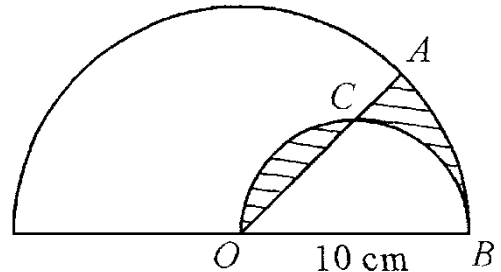
$$\sqrt{12-6\sqrt{3}} = \sqrt{9+3-2\sqrt{9 \times 3}} = \sqrt{a+b-2\sqrt{ab}} = \sqrt{a} - \sqrt{b} = 3 - \sqrt{3}$$

$$\sqrt{12+6\sqrt{3}} + \sqrt{12-6\sqrt{3}} = 3 + \sqrt{3} + 3 - \sqrt{3} = 6$$

- 16** In the figure, O is the centre of the bigger semicircle with radius 10 cm, OB is the diameter of the smaller semicircle and C is the midpoint of arc OB and it lies on the segment OA . Let the area of the shaded region be $K \text{ cm}^2$, find the value of K . (Take $\pi = 3$)

Shaded area = area of sector OAB – area of $\triangle OCB$

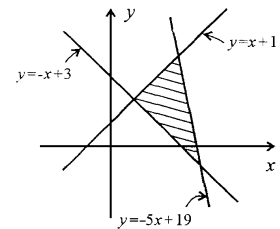
$$= \frac{1}{2} 10^2 \cdot \frac{\pi}{4} - \frac{1}{2} 10 \cdot 5 = 12.5$$



- 17** In the figure, let the shaded area formed by the three straight lines $y = -x + 3$, $y = x + 1$ and $y = -5x + 19$ be R , find the value of R .
Intersection points are $A(1, 2)$, $B(3, 4)$, $C(4, -1)$.

$$\angle CAB = 90^\circ$$

$$\text{Area} = \frac{1}{2} \sqrt{8} \sqrt{18} = 6 \text{ sq. unit}$$



- 18** If $t = \sin^4 \frac{\pi}{6} - \cos^2 \frac{2\pi}{6}$, find the value of t .

$$t = \sin^4 \frac{\pi}{6} - \cos^2 \frac{2\pi}{6}$$

$$= \left(\frac{1}{2}\right)^4 - \left(\frac{1}{2}\right)^2$$

$$= \frac{1}{16} - \frac{1}{4}$$

$$= -\frac{3}{16}$$

- 19** In the figure, C lies on AE , $\triangle ABC$ and $\triangle CDE$ are equilateral triangles, F and G are the mid-points of BC and DE respectively. If the area of $\triangle ABC$ is 24 cm^2 , the area of $\triangle CDE$ is 60 cm^2 , and the area of $\triangle AFG$ is $Q \text{ cm}^2$, find the value of Q .

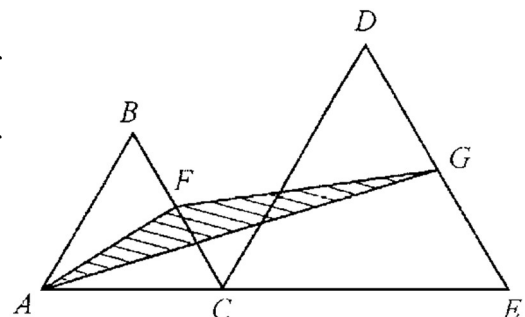
Reference: 2000 FI4.2, 2005 HG7, 2018 HI12

$$\angle FAC = \angle GCE = 30^\circ$$

$AF \parallel CG$ (corr. \angle s eq.)

$$\text{Area of } \triangle AFG = \text{Area of } \triangle ACF = 12 \text{ cm}^2$$

(They have the same bases AF and the same height)



- I10** If α and β are the roots of the quadratic equation $4x^2 - 10x + 3 = 0$ and $k = \alpha^2 + \beta^2$, find the value of k .

$$\begin{aligned}k &= \alpha^2 + \beta^2 \\&= (\alpha + \beta)^2 - 2\alpha\beta \\&= \left(\frac{5}{2}\right)^2 - 2 \cdot \left(\frac{3}{4}\right) \\&= \frac{19}{4}\end{aligned}$$

Group Events

G1 If $x = \frac{1}{2} + \left(\frac{1}{3} + \frac{2}{3}\right) + \left(\frac{1}{4} + \frac{2}{4} + \frac{3}{4}\right) + \left(\frac{1}{5} + \frac{2}{5} + \frac{3}{5} + \frac{4}{5}\right) + \cdots + \left(\frac{1}{100} + \frac{2}{100} + \cdots + \frac{99}{100}\right)$,

find the value of x . **Reference: 1995 HG3, 1996 FG9.4, 2018 HG9**

$$x = \frac{1}{2} + \frac{2}{2} + \frac{3}{2} + \frac{4}{2} + \cdots + \frac{99}{2} = \frac{100}{2} \times 99 = 2475$$

G2 If z is the positive root of the equation $6 \times 4^x - 13 \times 6^x + 6 \times 9^x = 0$, find the value of z .

$$(3 \cdot 2^x - 2 \cdot 3^x)(2 \cdot 2^x - 3 \cdot 3^x) = 0$$

$$3 \cdot 2^x = 2 \cdot 3^x \text{ or } 2 \cdot 2^x = 3 \cdot 3^x$$

$$\frac{2^x}{3^x} = \frac{2}{3} \text{ or } \frac{2^x}{3^x} = \frac{3}{2} = \left(\frac{2}{3}\right)^{-1}$$

$$x = 1 \text{ or } -1 \text{ (rejected)}$$

$$z = \text{positive root} = 1$$

G3 If there are at most k mutually non-congruent isosceles triangles whose perimeter is 25 cm and the lengths of the three sides are positive integers when expressed in cm, find the value of k .

Possible triangles are $\{7, 7, 11\}$, $\{8, 8, 9\}$, $\{9, 9, 7\}$, $\{10, 10, 5\}$, $\{11, 11, 3\}$, $\{12, 12, 1\}$

$$k = 6$$

G4 Given that a, b are positive real numbers satisfying $a^3 = 2004$ and $b^2 = 2004$.

If the number of integers x that satisfy the inequality $a < x < b$ is h , find the value of h .

$$12^3 = 1728, 44^2 = 1936$$

$$a^3 = 2004 \Rightarrow 12 < a < 13; b^2 = 2004 \Rightarrow 44 < b < 45$$

$$a < x < b \Rightarrow 12 < x < 45 \Rightarrow \text{number of integral values of } x = 32$$

G5 If the sum of R consecutive positive integers is 1000 (where $R > 1$), find the least value of R .

Let the smallest positive integer be x . (**Reference: 2006 HG5**)

$$x + (x + 1) + \cdots + (x + R - 1) = 1000$$

$$\frac{R}{2} \times (2x + R - 1) = 1000$$

$$R(2x + R - 1) = 2000 \Rightarrow 2x + R - 1 = \frac{2000}{R}, \text{ which is an integer.}$$

Possible R are: 1, 2, 4, 5, 8, 10, 16, 20, 25, 40, 50, 80, 100, 125, 250, 400, 500, 1000, 2000.

When $R = 4m + 2$, where m is an integer.

$$(4m + 2)(2x + 4m + 1) = 2000 \Rightarrow (2m + 1)(2x + 4m + 1) = 1000$$

L.H.S. is odd, R.H.S. is even \therefore reject 2, 10, 50, 250.

When $R = 4m$, where m is an integer.

$$4m(2x + 4m - 1) = 2000 \Rightarrow m(2x + 4m - 1) = 500 = 4 \times 125 \Rightarrow m \text{ is a multiple of } 4$$

$\therefore R = \text{multiple of } 16 \Rightarrow$ reject 4, 8, 20, 40, 100, 500, 1000

$$2x + R - 1 = \frac{2000}{R} > R - 1 \Rightarrow 2000 > R(R - 1) \Rightarrow \sqrt{2000} > R - 1 \Rightarrow 45 > R$$

The possible values of R are 1, 5, 16, 25.

$$\text{When } R = 1, 1(2x) = 2000 \Rightarrow x = 1000$$

$$\text{When } R = 5, 5(2x + 4) = 2000 \Rightarrow x = 198$$

$$\text{When } R = 16, 16(2x + 15) = 2000 \Rightarrow x = 55$$

$$\text{When } R = 25, 25(2x + 24) = 2000 \Rightarrow x = 28$$

The least value of $R > 1$ is 5, $x = 198$.

$$198 + 199 + 200 + 201 + 202 = 1000$$

- G6** If a, b and c are positive integers such that $abc + ab + bc + ac + a + b + c = 2003$, find the least value of abc .

Reference: 2018 FI4.3

$$(a+1)(b+1)(c+1) = 2004 = 2^2 \times 3 \times 167$$

abc is the least when the difference between a, b and c are the greatest.

$$a+1=2, b+1=2, c+1=501$$

$$a=1, b=1, c=500$$

$$abc = 500$$

- G7** In the figure, $ABCD$ is a trapezium, the segments AB and CD are both perpendicular to BC and the diagonals AC and BD intersect at X . If $AB = 9$ cm, $BC = 12$ cm and $CD = 16$ cm, and the area of $\triangle BXC$ is W cm², find the value of W .

Reference: 1993 HI2, 1997 HG3, 2000 FI2.2, 2002 FI1.3, 2010HG4, 2013 HG2

$$\triangle ABX \sim \triangle CDX$$

$$AX : CX = AB : CD = 9 : 16$$

$$S_{\triangle ABX} : S_{\triangle CDX} = 9^2 : 16^2 = 81 : 256$$

$$\text{Let } S_{\triangle ABX} = 81y, S_{\triangle CDX} = 256y$$

$$\text{Let } AX = 9t, CX = 16t (\because \triangle ABX \sim \triangle CDX)$$

$\triangle ABX$ and $\triangle BCX$ have the same height.

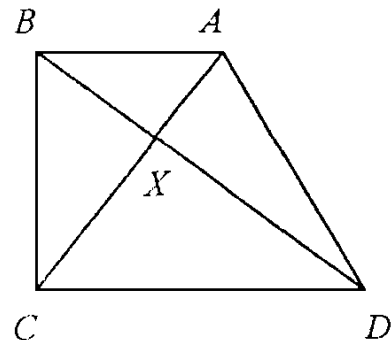
$$S_{\triangle BCX} = S_{\triangle ABX} \times \frac{16t}{9t} = 81y \times \frac{16}{9} = 144y$$

$$S_{\triangle ABC} = S_{\triangle ABX} + S_{\triangle BCX}$$

$$\frac{9 \times 12}{2} = 81y + 144y$$

$$y = \frac{6}{25}$$

$$\Rightarrow S_{\triangle BCX} = 144y = 144 \times \frac{6}{25} = 34.56$$



- G8** Let $y = \log_{1400} \sqrt{2} + \log_{1400} \sqrt[3]{5} + \log_{1400} \sqrt[6]{7}$, find the value of y .

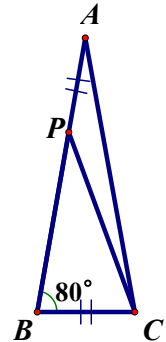
$$y = \frac{\log \sqrt{2} \times \log \sqrt[3]{5} \times \log \sqrt[6]{7}}{\log 1400} = \frac{\frac{1}{2} \log 2 + \frac{1}{3} \log 5 + \frac{1}{6} \log 7}{\log 1400} = \frac{3 \log 2 + 2 \log 5 + \log 7}{6 \log 1400}$$

$$y = \frac{\log 8 + \log 25 + \log 7}{6 \log 1400} = \frac{\log(2 \times 4 \times 25 \times 7)}{6 \log 1400} = \frac{\log 1400}{6 \log 1400} = \frac{1}{6}$$

- G9** In the figure, $\triangle ABC$ is an isosceles triangle with $AB = AC$ and $\angle ABC = 80^\circ$.
If P is a point on the AB such that $AP = BC$, $\angle ACP = k^\circ$, find the value of k .

Reference: 《數學教育》第八期（一九九九年六月），2010 HG10

$$\begin{aligned}\angle ACB &= 80^\circ = \angle ABC && (\text{base } \angle\text{s isos. } \triangle) \\ \angle BAC &= 20^\circ && (\angle \text{ sum of } \triangle) \\ \angle BPC &= (20 + k)^\circ && (\text{ext. } \angle \text{ of } \triangle APC) \\ \frac{AP}{\sin k^\circ} &= \frac{CP}{\sin 20^\circ} \dots\dots (1) && (\text{sine rule on } \triangle APC) \\ \frac{BC}{\sin(20 + k)^\circ} &= \frac{CP}{\sin 80^\circ} \dots\dots (2) && (\text{sine rule on } \triangle BCP)\end{aligned}$$



$$(1) \div (2): \frac{\sin(20 + k)^\circ}{\sin k^\circ} = \frac{\sin 80^\circ}{\sin 20^\circ} = \frac{\cos 10^\circ}{2 \sin 10^\circ \cos 10^\circ} = \frac{1}{2 \sin 10^\circ}$$

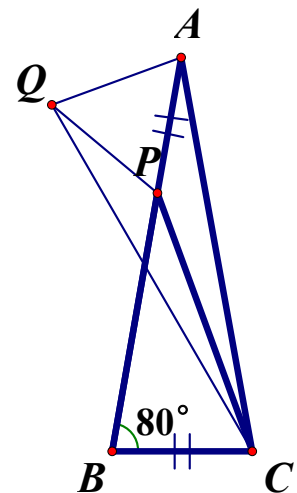
$$\begin{aligned}2 \sin(20 + k)^\circ \sin 10^\circ &= \sin k^\circ \\ \cos(10 + k)^\circ - \cos(30 + k)^\circ &= \sin k^\circ \\ \cos(10 + k)^\circ &= \sin(60 - k)^\circ + \sin k^\circ \\ \cos(10 + k)^\circ &= 2 \sin 30^\circ \cos(30 - k)^\circ \\ \cos(10 + k)^\circ &= \cos(30 - k)^\circ \\ 10 + k &= 30 - k \\ k &= 10\end{aligned}$$

Method 2

Rotate A 60° in anti-clockwise direction about P as shown.

$\triangle APQ$ is an equilateral triangle. Join QC .

$$\begin{aligned}\angle ACB &= 80^\circ = \angle ABC && (\text{base } \angle\text{s isos. } \triangle) \\ \angle BAC &= 20^\circ && (\angle \text{ sum of } \triangle) \\ \angle QAP &= 60^\circ = \angle AQP && (\angle\text{s of an equilateral } \triangle) \\ \angle QAC &= 60^\circ + 20^\circ = 80^\circ = \angle ACB \\ \therefore QA &= AP = BC && (\text{given}) \\ AC &= AC && (\text{common}) \\ \therefore \triangle ACB &\cong \triangle CAQ && (\text{S.A.S.}) \\ \therefore \angle AQC &= \angle ABC = 80^\circ && (\text{corr. } \angle\text{s } \cong \triangle\text{s}) \\ \angle CQP &= 80^\circ - 60^\circ = 20^\circ = \angle CAP \\ CP &= CP && (\text{common}) \\ AP &= QP && (\text{by construction}) \\ \triangle APC &\cong \triangle QPC && (\text{S.S.S.}) \\ \therefore \angle ACQ &= \angle BAC = 20^\circ && (\text{corr. } \angle\text{s } \triangle ACB \cong \triangle CAQ) \\ \angle ACP &= \angle QCP = 10^\circ && (\text{corr. } \angle\text{s } \triangle APC \cong \triangle QPC)\end{aligned}$$



G10 Suppose $P(a, b)$ is a point on the straight line $x - y + 1 = 0$ such that the sum of the distance between P and the point $A(1,0)$ and the distance between P and the point $B(3,0)$ is the least, find the value of $a + b$.

Regard $x - y + 1 = 0$ as mirror.

$C(-1,2)$ is the mirror image of $A(1,0)$.

Sum of distance is the least

$\Rightarrow P(a, b)$ lies on BC .

$P(a, b)$ lies on $x - y + 1 = 0$

$\Rightarrow b = a + 1$

$m_{PB} = m_{BC}$

$$\frac{a+1}{a-3} = \frac{2}{-4}$$

$$-2a - 2 = a - 3$$

$$a = \frac{1}{3}, \quad b = \frac{4}{3}$$

$$a + b = \frac{5}{3}$$

