

Individual Events

I1	a	2	I2	a	16	I3	a	-1	I4	A	16
	b	1		b	160		b	17		b	$-\frac{1}{2}$
	c	-6		c	3		c	8		c	$\frac{3}{2}$
	d	$\frac{50}{11}$		d	$\frac{8}{27}$		d	18		d	6

Group Events

G1	W	$\frac{1+\sqrt{5}}{2}$	G2	R	18434	G3	b	40	G4	x	137
	T	29		x	6		t	$\frac{12}{5} (= 2.4)$		R	$\frac{1}{2}$
	S	106		y	12100		x	$10\sqrt{3}$		z	77
	k	4		Q	9		S	25		r	6

Individual Event 1

I1.1 Let a be a real number and $\sqrt{a} = \sqrt{7+\sqrt{13}} - \sqrt{7-\sqrt{13}}$. Find the value of a .

Reference: 2016 FI1.2, 2019 FI2.1

$$(\sqrt{a})^2 = (\sqrt{7+\sqrt{13}} - \sqrt{7-\sqrt{13}})^2$$

$$a = 7 + \sqrt{13} - 2\sqrt{7^2 - \sqrt{13}^2} + 7 - \sqrt{13}$$

$$= 14 - 2\sqrt{36} = 2$$

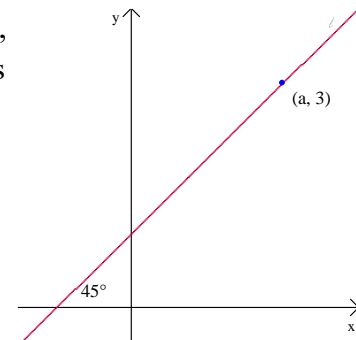
I1.2 In Figure 1, the straight line ℓ passes through the point $(a, 3)$, and makes an angle 45° with the x -axis. If the equation of ℓ is $x + my + n = 0$ and $b = |1 + m + n|$, find the value of b .

$$\ell: \frac{y-3}{x-2} = \tan 45^\circ$$

$$y - 3 = x - 2$$

$$x - y + 1 = 0, m = -1, n = 1$$

$$b = |1 - 1 + 1| = 1$$



I1.3 If $x - b$ is a factor of $x^3 - 6x^2 + 11x + c$, find the value of c .

$$f(x) = x^3 - 6x^2 + 11x + c$$

$$f(1) = 1 - 6 + 11 + c = 0$$

$$c = -6$$

I1.4 If $\cos x + \sin x = -\frac{c}{5}$ and $d = \tan x + \cot x$, find the value of d .

Reference: 1992 HI20, 1993 HG10, 1995 HI5, 2007 HI7, 2014 HG3

$$\cos x + \sin x = \frac{6}{5}$$

$$(\cos x + \sin x)^2 = \frac{36}{25}$$

$$1 + 2 \sin x \cos x = \frac{36}{25}$$

$$2 \sin x \cos x = \frac{11}{25} \Rightarrow \sin x \cos x = \frac{11}{50}$$

$$d = \tan x + \cot x = \frac{\sin x}{\cos x} + \frac{\cos x}{\sin x} = \frac{\sin^2 x + \cos^2 x}{\sin x \cos x} = \frac{1}{\sin x \cos x} = \frac{50}{11}$$

Individual Event 2

- 12.1** Let $n = 1 + 3 + 5 + \dots + 31$ and $m = 2 + 4 + 6 + \dots + 32$. If $a = m - n$, find the value of a .

$$\begin{aligned} a &= 2 + 4 + 6 + \dots + 32 - (1 + 3 + 5 + \dots + 31) \\ &= (2 - 1) + (4 - 3) + \dots + (32 - 31) \\ &= 1 + 1 + \dots + 1 = 16 \end{aligned}$$

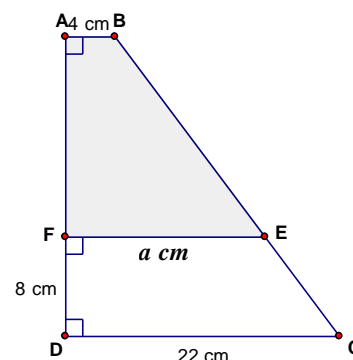
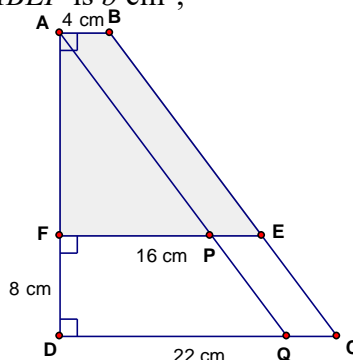
- 12.2** If Figure 1, $ABCD$ is a trapezium, $AB = 4$ cm, $EF = a$ cm, $CD = 22$ cm and $FD = 8$ cm, if the area of $ABEF$ is b cm², find the value of b .

From A , draw a line APQ parallel to BC , cutting FE at P and DC at Q .
 $FP = 12$ cm, $DQ = 18$ cm

$$\triangle AFP \sim \triangle ADQ$$

$$\text{Let } AF = x \text{ cm} \Rightarrow \frac{x}{12} = \frac{x+8}{18}, x = 16$$

$$b = \frac{(4+16) \times 16}{2} = 160$$



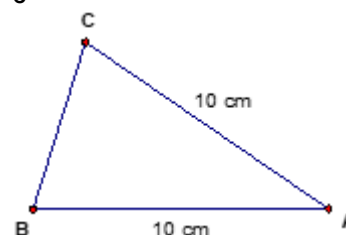
- 12.3** In Figure 2, $\triangle ABC$ is a triangle, $AB = AC = 10$ cm and $\angle ABC = b^\circ - 100^\circ$.

If $\triangle ABC$ has c axis of symmetry, find the value of c .

$$\angle ABC = 160^\circ - 100^\circ = 60^\circ = \angle ACB = \angle BAC$$

$\triangle ABC$ is an equilateral triangle.

It has 3 axis of symmetry.



- 12.4** Let d be the least real root of the $cx^{\frac{2}{3}} - 8x^{\frac{1}{3}} + 4 = 0$, find the value of d .

$$3x^{\frac{2}{3}} - 8x^{\frac{1}{3}} + 4 = 0 \Rightarrow \left(3x^{\frac{1}{3}} - 2 \right) \left(x^{\frac{1}{3}} - 2 \right) = 0$$

$$x^{\frac{1}{3}} = \frac{2}{3} \text{ or } 2$$

$$x = \frac{8}{27} \text{ or } 8, \text{ the least real root is } \frac{8}{27}.$$

Individual Event 3

- 13.1** Suppose that $a = \cos^4 \theta - \sin^4 \theta - 2 \cos^2 \theta$, find the value of a .

$$\begin{aligned} a &= (\cos^2 \theta + \sin^2 \theta)(\cos^2 \theta - \sin^2 \theta) - 2 \cos^2 \theta \\ &= \cos^2 \theta - \sin^2 \theta - 2 \cos^2 \theta = -(\sin^2 \theta + \cos^2 \theta) = -1 \end{aligned}$$

- 13.2** If $x^y = 3$ and $b = x^{3y} + 10a$, find the value of b .

$$b = (x^y)^3 - 10 = 3^3 - 10 = 27 - 10 = 17$$

- 13.3** If there is (are) c positive integer(s) n such that $\frac{n+b}{n-7}$ is also a positive integer,

find the value of c .

$$\frac{n+17}{n-7} = 1 + \frac{24}{n-7}$$

$$n-7 = 1, 2, 3, 4, 6, 8, 12, 24$$

$$c = 8$$

- 13.4** Suppose that $d = \log_4 2 + \log_4 4 + \log_4 8 + \dots + \log_4 2^c$, find the value of d .

$$\begin{aligned} d &= \log_4 2 + \log_4 4 + \log_4 8 + \dots + \log_4 2^8 \\ &= \log_4 (2 \times 4 \times 8 \times \dots \times 2^8) = \log_4 (2^{1+2+3+\dots+8}) \end{aligned}$$

$$= \log_4 (2^{36}) = \frac{\log 2^{36}}{\log 4} = \frac{36 \log 2}{2 \log 2} = 18$$

Individual Event 4

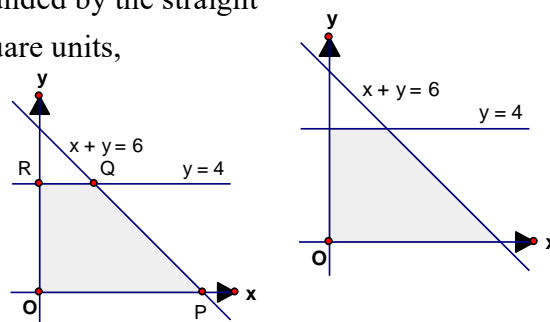
- I4.1** In Figure 1, let the area of the closed region bounded by the straight line $x + y = 6$ and $y = 4$, $x = 0$ and $y = 0$ be A square units, find the value of A .

As shown in the figure, the intersection points

$$P(6, 0), Q(2, 4), R(0, 6)$$

$$OP = 6, OR = 4, QR = 2$$

$$\text{Area} = A = \frac{1}{2}(6 + 2) \cdot 4 = 16$$



- I4.2** Let $[x]$ be the largest integer not greater than x . For example, $[2.5] = 2$.

If b satisfies the system of equations $\begin{cases} Ax^2 - 4 = 0 \\ 3 + 2(x + [x]) = 0 \end{cases}$, find the value of b .

$$\begin{cases} 16x^2 - 4 = 0 \\ 3 + 2(x + [x]) = 0 \end{cases} \text{ from the first equation } x = \frac{1}{2} \text{ or } -\frac{1}{2}.$$

Substitute $x = \frac{1}{2}$ into the second equation: $\text{LHS} = 3 + 2(\frac{1}{2} + 0) = 4 \neq \text{RHS}$

Substitute $x = -\frac{1}{2}$ into the second equation: $\text{LHS} = 3 + 2(-\frac{1}{2} - 1) = 0 = \text{RHS}$

$$\therefore b = -\frac{1}{2}$$

- I4.3** Let c be the constant term in the expansion of $\left(2x + \frac{b}{\sqrt{x}}\right)^3$. Find the value of c .

$$\left(2x + \frac{b}{\sqrt{x}}\right)^3 = 8x^3 + 12bx\sqrt{x} + 6b^2 + \frac{b^3}{x\sqrt{x}}$$

c = the constant term

$$= 6b^2$$

$$= 6\left(-\frac{1}{2}\right)^2 = \frac{3}{2}$$

- I4.4** If the number of integral solutions of the inequality $\left|\frac{x}{2} - \sqrt{2}\right| < c$ is d , find the value of d .

$$\left|\frac{x}{2} - \sqrt{2}\right| < \frac{3}{2}$$

$$-\frac{3}{2} < \frac{x}{2} - \sqrt{2} < \frac{3}{2}$$

$$2\sqrt{2} - 3 < x < 2\sqrt{2} + 3$$

$$2(1.4) - 3 < x < 2(1.4) + 3$$

$$-0.2 < x < 5.8$$

$$x = 0, 1, 2, 3, 4, 5$$

$$d = 6$$

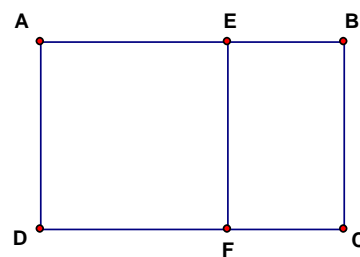
Group Event 1

G1.1 In Figure 1, $AEFD$ is a unit square. The ratio of the length of the rectangle $ABCD$ to its width is equal to the ratio of the length of the rectangle $BCFE$ to its width.

If the length of AB is W units, find the value of W .

$$\frac{W}{1} = \frac{1}{W-1}$$

$$W^2 - W - 1 = 0 \Rightarrow W = \frac{1 + \sqrt{5}}{2}$$



G1.2 On the coordinate plane, there are T points (x, y) , where x, y are integers, satisfying $x^2 + y^2 < 10$, find the value of T . **Reference: 2002 FI4.3**

T = number of integral points inside the circle $x^2 + y^2 = 10$.

We first count the number of integral points in the first quadrant:

$$x = 1; y = 1, 2$$

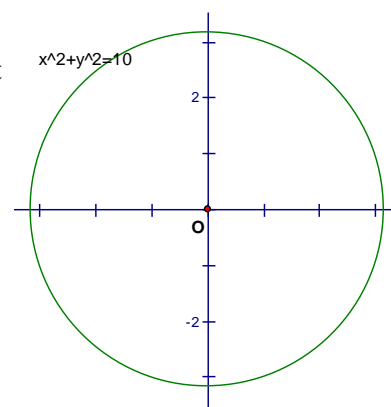
$$x = 2; y = 1, 2$$

Next, the number of integral points on the x -axis and y -axis

$$= 3 + 3 + 3 + 3 + 1 = 13$$

$$T = 4 \times 4 + 3 + 3 + 3 + 3 + 1$$

$$= 29$$



G1.3 Let P and $P + 2$ be both prime numbers satisfying $P(P + 2) \leq 2007$.

If S represents the sum of such possible values of P , find the value of S .

$$P^2 + 2P - 2007 \leq 0$$

$$(P + 1)^2 - 2008 \leq 0$$

$$(P + 1 + \sqrt{2008})(P + 1 - \sqrt{2008}) \leq 0$$

$$(P + 1 + 2\sqrt{502})(P + 1 - 2\sqrt{502}) \leq 0$$

$$-1 - 2\sqrt{502} \leq P \leq -1 + 2\sqrt{502}$$

$$P \text{ is a prime} \Rightarrow 0 < P \leq -1 + 2\sqrt{502}$$

$$22 = \sqrt{484} < \sqrt{502} < \sqrt{529} = 23$$

$$43 < -1 + 2\sqrt{502} < 45$$

$$\therefore (P, P + 2) = (3, 5), (5, 7), (11, 13), (17, 19), (29, 31), (41, 43)$$

$$S = 3 + 5 + 11 + 17 + 29 + 41 = 106$$

G1.4 It is known that $\log_{10}(2007^{2006} \times 2006^{2007}) = a \times 10^k$, where $1 \leq a < 10$ and k is an integer.

Find the value of k .

$$a \times 10^k = 2006 \log 2007 + 2007 \log 2006 = 2006 \times (\log 2007 + \log 2006) + \log 2006$$

$$2006 \times (\log 2006 + \log 2006) + \log 2006 < a \times 10^k < 2006 \times (\log 2007 + \log 2007) + \log 2007$$

$$4013 \log 2006 < a \times 10^k < 4013 \log 2007$$

$$4013 \log(2.006 \times 10^3) < a \times 10^k < 4013 \log(2.007 \times 10^3)$$

$$4013 (\log 2.006 + 3) < a \times 10^k < 4013 (\log 2.007 + 3)$$

$$4013 \log 2 + 4013 \times 3 < a \times 10^k < 4013 \log 3 + 3$$

$$1.32429 \times 10^4 = 4013 \times 0.3 + 4013 \times 3 < a \times 10^k < 4013 \times 0.5 + 4013 \times 3 = 1.40455 \times 10^4$$

$$k = 4$$

Group Event 2

G2.1 If $R = 1 \times 2 + 2 \times 2^2 + 3 \times 2^3 + \dots + 10 \times 2^{10}$, find the value of R .

Reference: 2005 HI7, 2005 FG2.4

$$2R = 1 \times 2^2 + 2 \times 2^3 + \dots + 9 \times 2^{10} + 10 \times 2^{11}$$

$$R - 2R = 2 + 2^2 + 2^3 + \dots + 2^{10} - 10 \times 2^{11}$$

$$-R = \frac{a(R^n - 1)}{R - 1} - 10 \times 2^{11} = \frac{2(2^{10} - 1)}{2 - 1} - 10 \times 2048$$

$$R = 20480 - 2(1023) = 18434$$

G2.2 If integer x satisfies $x \geq 3 + \sqrt{3 + \sqrt{3 + \sqrt{3 + \sqrt{3 + \sqrt{3}}}}}$, find the minimum value of x .

$$\text{Let } y = 3 + \sqrt{3 + \sqrt{3 + \sqrt{3 + \sqrt{3 + \dots}}}} \text{ (to infinity)}$$

$$(y - 3)^2 = 3 + \sqrt{3 + \sqrt{3 + \sqrt{3 + \sqrt{3 + \dots}}}} = y$$

$$y^2 - 7y + 9 = 0$$

$$y = \frac{7 + \sqrt{13}}{2} \text{ or } \frac{7 - \sqrt{13}}{2}$$

$$\text{Clearly } y > 3 \text{ and } \frac{7 - \sqrt{13}}{2} < 3$$

$$\therefore y = \frac{7 + \sqrt{13}}{2} \text{ only}$$

$$5 = \frac{7 + \sqrt{9}}{2} < \frac{7 + \sqrt{13}}{2} < \frac{7 + \sqrt{16}}{2} = 5.5$$

$$3 + \sqrt{3 + \sqrt{3}} > 3 + \sqrt{3 + 1.7} > 3 + \sqrt{4.41} = 3 + 2.1 = 5.1$$

$$5.1 < 3 + \sqrt{3 + \sqrt{3}} < 3 + \sqrt{3 + \sqrt{3 + \sqrt{3 + \sqrt{3 + \sqrt{3}}}}} < 3 + \sqrt{3 + \sqrt{3 + \sqrt{3 + \sqrt{3 + \dots}}}} < 5.5$$

$$x = 6$$

G2.3 Let $y = \frac{146410000 - 12100}{12099}$, find the value of y .

$$y = \frac{12100^2 - 12100}{12100 - 1}$$

$$= \frac{12100(12100 - 1)}{12100 - 1}$$

$$= 12100$$

G2.4 On the coordinate plane, a circle with centre $T(3, 3)$ passes through the origin $O(0, 0)$. If A is a point on the circle such that $\angle AOT = 45^\circ$ and the area of $\triangle AOT$ is Q square units, find the value of Q .

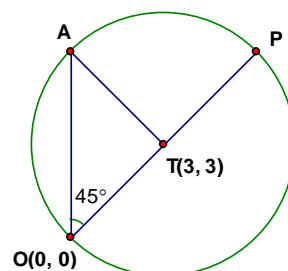
$$OT = \sqrt{3^2 + 3^2} = 3\sqrt{2}$$

$$OT = AT = \text{radii}$$

$$\angle OAT = 45^\circ \text{ (side opp. eq. } \angle\text{s)}$$

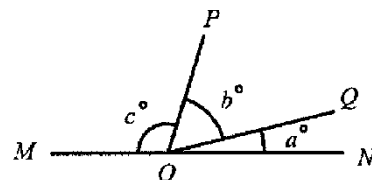
$$\angle ATO = 90^\circ \text{ (}\angle\text{s sum of } \Delta\text{)}$$

$$Q = \frac{1}{2} OT \cdot AT = \frac{1}{2} \cdot (3\sqrt{2})^2 = 9$$



Group Event 3

- G3.1** In figure 1, MN is a straight line, $\angle QON = a^\circ$, $\angle POQ = b^\circ$ and $\angle POM = c^\circ$. If $b : a = 2 : 1$ and $c : b = 3 : 1$, find the value of b .
- $b = 2a, c = 3b = 6a$
 $a + b + c = 180$ (adj. \angle s on st. line)
 $a + 2a + 6a = 180 \Rightarrow a = 20$
 $b = 2a = 40$



- G3.2** It is known that $\sqrt{\frac{50+120+130}{2} \times (150-50) \times (150-120) \times (150-130)} = \frac{50 \times 130 \times k}{2}$.

If $t = \frac{k}{\sqrt{1-k^2}}$, find the value of t .

The question is equivalent to: given a triangle with sides 50, 120, 130, find its area.

$$\cos C = \frac{50^2 + 130^2 - 120^2}{2 \cdot 50 \cdot 130} = \frac{5}{13}$$

$$\text{Using the formula } \frac{1}{2} ab \sin C = \frac{50 \times 130 \times k}{2}, k = \sin C = \sqrt{1 - \cos^2 C} = \frac{12}{13}$$

$$t = \frac{k}{\sqrt{1-k^2}} = \frac{\sin C}{\cos C} = \tan C = \frac{12}{5}$$

- G3.3** In Figure 2, an ant runs ahead straightly for 5 sec 15° cm from point A to point B . It then turns 30° to the right and run 5 sec 15° cm to point C . Again it repeatedly turns 30° to the right and run 5 sec 15° cm twice to reach the points D and E respectively. If the distance of AE is x cm, find the value of x .
- By symmetry $\angle BAE = \angle DEA = [180^\circ \times (5-2) - 150^\circ \times 3] \div 2 = 45^\circ$ (\angle sum of polygon)

Produce AB and ED to intersect at F .

$$\angle AFE = 180^\circ - 45^\circ - 45^\circ = 90^\circ \text{ (\angle s sum of Δ)}$$

By symmetry, $\angle BFC = \angle DFC = 45^\circ$

$$\angle BCF = \angle DCF = (360^\circ - 150^\circ) \div 2 = 105^\circ \text{ (\angle s at a pt.)}$$

Let $AB = y = 5 \text{ sec } 15^\circ \text{ cm} = CD = DE$, let $z = BF$.

$$\text{Apply Sine rule on } \Delta ABC, \frac{z}{\sin 105^\circ} = \frac{y}{\sin 45^\circ}$$

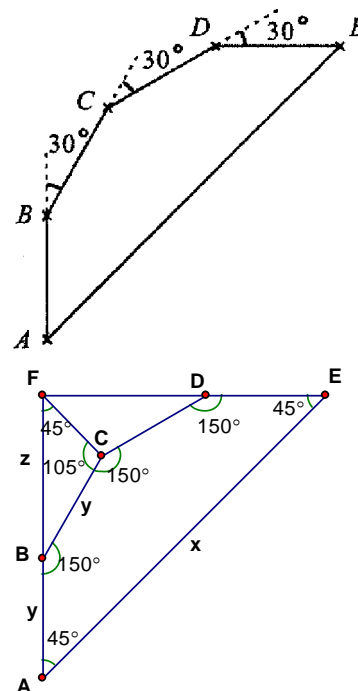
$$z = \sqrt{2} \sin 105^\circ y$$

$$\text{In } \Delta AEF, x = (y + z) \sec 45^\circ = \sqrt{2} (y + \sqrt{2} \sin 105^\circ y)$$

$$= y\sqrt{2}(1 + \sqrt{2} \sin 105^\circ)$$

$$= 5 \sec 15^\circ \cdot 2 \left(\frac{1}{\sqrt{2}} + \sin 105^\circ \right) = 10 \sec 15^\circ (\sin 105^\circ + \sin 45^\circ)$$

$$= 10 \sec 15^\circ (2 \sin 75^\circ \cos 30^\circ) = 20 \sec 15^\circ \cdot \cos 15^\circ \cdot \frac{\sqrt{3}}{2} = 10\sqrt{3}$$



Method 2

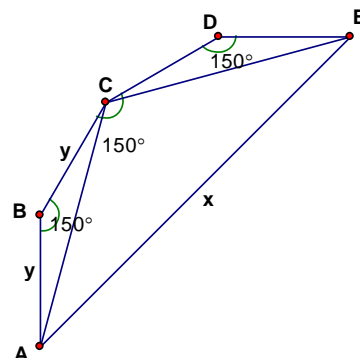
Join AC, CE . With similar working steps, $\angle BAE = \angle DEA = 45^\circ$

$$\angle BAC = \angle BCA = 15^\circ = \angle DCE = \angle DEC \text{ (\angle s sum of isos. Δ)}$$

$$\angle CAE = 45^\circ - 15^\circ = 30^\circ = \angle CEA$$

$$AC = CE = 2y \cos 15^\circ = 2 \times 5 \sec 15^\circ \times \cos 15^\circ = 10$$

$$x = 2 AC \cos 30^\circ = 20 \times \frac{\sqrt{3}}{2} = 10\sqrt{3}$$



G3.4 There are 4 problems in a mathematics competition. The scores of each problem are allocated in the following ways: 2 marks will be given for a correct answer, 1 mark will be deducted from a wrong answer and 0 mark will be given for a blank answer. To ensure that 3 candidates will have the same scores, there should be at least S candidates in the competition.

Find the value of S .

We shall tabulate different cases:

case no.	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	marks for each question
correct	4	3	3	2	2	2	1	1	1	1	0	0	0	0	0	2
blank	0	1	0	2	0	1	3	2	1	1	4	3	2	1	0	0
wrong	0	0	1	0	2	1	0	1	2	3	0	1	2	3	4	-1
Total	8	6	5	4	2	3	2	1	0	-1	0	-1	-2	-3	-4	

The possible total marks for one candidate to answer 4 questions are:

8, 6, 5, 4, 3, 2, 1, 0, -1, -2, -3, -4; altogether 12 possible combinations.

To ensure **3** candidates will have the same scores, we consider the worst scenario:

Given that there are 24 candidates. 2 candidates score 8 marks, 2 candidates score 6 marks, \dots ,

2 candidates score -4 marks, then the 25th candidate will score the same as the other two candidates.

Group Event 4

G4.1 Let x be the number of candies satisfies the inequalities $120 \leq x \leq 150$. 2 candies will be remained if they are divided into groups of 5 candies each; 5 candies will be remained if they are divided into groups of 6 candies each. Find the value of x .

$$x = 5m + 2 = 6n + 5, \text{ where } m \text{ and } n \text{ are integers.}$$

$$5m - 6n = 3$$

$$5 \times 3 - 6 \times 2 = 15 - 12 = 3$$

$$\therefore m = 3, n = 2 \text{ is a pair of solution}$$

The general solution is $m = 3 + 6t, n = 2 + 5t$, where t is any integer.

$$x = 5m + 2 = 5(3 + 6t) + 2 = 30t + 17$$

$$120 \leq x \leq 150 \Rightarrow 120 \leq 30t + 17 \leq 150$$

$$103 \leq 30t \leq 133$$

$$3.43 < t < 4.43 \Rightarrow t = 4$$

$$x = 30 \times 4 + 17 = 137$$

G4.2 On the coordinate plane, the points $A(3, 7)$ and $B(8, 14)$ are reflected about the line $y = kx + c$, where k and c are constants, their images are $C(5, 5)$ and $D(12, 10)$ respectively. If $R = \frac{k}{c}$, find the value of R .

By the property of reflection, the line $y = kx + c$ is the perpendicular bisector of A, C and B, D .

That is to say, the mid points of A, C and B, D lies on the line $y = kx + c$

$M = \text{mid point of } A, C = (4, 6), N = \text{mid point of } B, D = (10, 12)$

$$\text{By two points form, } \frac{y-6}{x-4} = \frac{12-6}{10-4}$$

$$y = x + 2 \Rightarrow k = 1, c = 2, R = \frac{1}{2}$$

G4.3 Given that $z = \sqrt[3]{456533}$ is an integer, find the value of z .

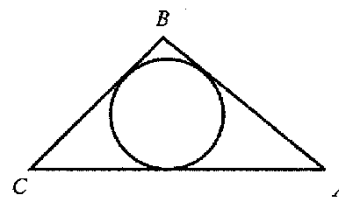
$$70 = \sqrt[3]{343000} < \sqrt[3]{456533} < \sqrt[3]{512000} = 80$$

By considering the cube of the unit digit, the only possible answer for z is 77.

G4.4 In Figure 1, $\triangle ABC$ is an isosceles triangle, $AB = BC = 20$ cm and

$\tan \angle BAC = \frac{4}{3}$. If the length of radius of the inscribed circle of

$\triangle ABC$ is r cm, find the value of r .



Reference: 2013 HG8, 2022 P1Q15

$$\angle BAC = \angle BCA; \sin \angle BAC = \frac{4}{5}, \cos \angle BAC = \frac{3}{5}.$$

$$AC = 2 \times 20 \cos \angle BAC = 40 \times \frac{3}{5} = 24, \text{ the height of } \triangle ABC \text{ from } B = 20 \sin \angle BAC = 16$$

$$\text{Area of } \triangle ABC = \frac{1}{2} \cdot 24 \cdot 16 = 192 = \frac{r}{2} (20 + 20 + 24)$$

$$r = 6$$