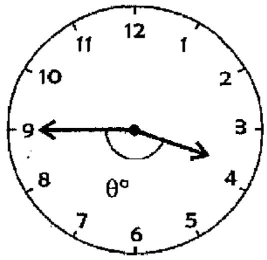
06-07 Individual	**1**	157.5	**2**	30	**3**	57	**4**	2006	**5**	500
	6	5	**7**	$\frac{4}{3}$	**8**	30	**9**	82	**10**	$3\sqrt{2}$

06-07 Group	**1**	20	**2**	$\frac{14049}{8}$	**3**	$\frac{4}{9}$	**4**	2	**5**	12
	6	3	**7**	$\frac{3}{2}$	**8**	$10\sqrt{2}$	**9**	$\frac{4}{3}$	**10**	2011

Individual Events

I1 In Figure 1, a clock indicates the time 3:45. If the angle between the hour-hand and the minute-hand is θ°, find the value of θ.

Reference 1984 FG7.1, 1985 FI3.1, 1987 FG7.1, 1989 FI1.1, 1990 FG6.3



At 3:00 pm, the minute-hand lags the hour-hand by $360°\times\frac{1}{4} = 90°$.

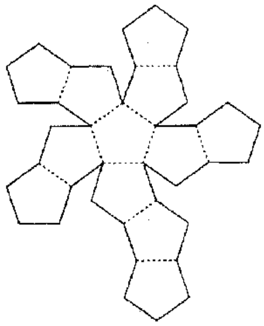
At 3:45 pm, the minute-hand has moved $360°\times\frac{3}{4} = 270°$;

the hour-hand has moved $360°\times\frac{1}{12}\times\frac{3}{4} = 22.5°$.

The angle between the hour-hand and the minute-hand is

$270° - 90° - 22.5° = 157.5° \Rightarrow \theta = 157.5$

I2 In Figure 2, there is a paper net that can be wrapped into a regular polyhedron. If this polyhedron has y edges, find the value of y.



There are altogether 12 pentagons. Each pentagon has 5 edges. The total number of edges is $12\times5 = 60$.

When the paper net is wrapped to form a polyhedron, every 2 edges are stuck together to form 1 edge.

∴ The number of edges in the polyhedron $= 60 \div 2 = 30$

I3 Among 4 English books, 6 Chinese books and 9 Japanese books, two books are selected. It is found that they are of the same language. If there are X such choices, find the value of X.

$X = {}_4C_2 + {}_6C_2 + {}_9C_2 = 6 + 15 + 36 = 57$

I4 Let r_1 and r_2 be the two real roots of the equation $(x - 2006)(x - 2007) = 2007$

If r is the smaller real root of the equation $(x - r_1)(x - r_2) = -2007$, find the value of r.

$(x - 2006)(x - 2007) = 2007 \Rightarrow x^2 - 4013x + 2005\times2007 = 0$

$\therefore r_1 + r_2 = 4013,\ r_1r_2 = 2005\times2007$...............(*)

$(x - r_1)(x - r_2) = -2007 \Rightarrow x^2 - (r_1 + r_2)x + r_1r_2 + 2007 = 0$

$x^2 - 4013x + 2005\times2007 + 2007 = 0$ by (*)

$x^2 - 4013x + 2006\times2007 = 0$

$(x - 2006)(x - 2007) = 0$

$x = 2006$ or $x = 2007$

r = the smaller real root = 2006

I5 Given that α and β are the roots of the equation $x^2 - 5^{2007}x + 5^{1000} = 0$.

If $s = \log_{25}\frac{\alpha^2}{\beta} + \log_{25}\frac{\beta^2}{\alpha}$, find the value of s.

$\alpha\beta = 5^{1000}$

$s = \log_{25}\frac{\alpha^2}{\beta} + \log_{25}\frac{\beta^2}{\alpha} = \log_{25}\left(\frac{\alpha^2}{\beta}\cdot\frac{\beta^2}{\alpha}\right)$

$= \log_{25}(\alpha\beta) = \frac{\log 5^{1000}}{\log 25} = \frac{1000\log 5}{2\log 5} = 500$

I6 For any real number a, b, c and d, we define the operation *:

$$(a, b)*(c, d) = (ad + bc, bd)$$

If $(x, y) = \left(1, \frac{3}{7-\sqrt{5}}\right) * \left(8+\sqrt{5}, 3\right)$ and $a = \frac{x}{y}$, find the value of a.

$$(x, y) = \left(3 + \frac{3\left(8+\sqrt{5}\right)}{7-\sqrt{5}}, \frac{9}{7-\sqrt{5}}\right)$$

$$= \left(\frac{21-3\sqrt{5}+24+3\sqrt{5}}{7-\sqrt{5}}, \frac{9}{7-\sqrt{5}}\right)$$

$$= \left(\frac{45}{7-\sqrt{5}}, \frac{9}{7-\sqrt{5}}\right)$$

$$a = \frac{x}{y} = 5$$

I7 Given that $\sin\alpha - \cos\alpha = \frac{1}{5}$ and $0° < \alpha < 180°$. If $\tan\alpha = B$, find the value of B.

Reference: 1992 HI20, 1993 HG10, 1995 HI5, 2007 FI1.4, 2014 HG3

$$(\sin\alpha - \cos\alpha)^2 = \frac{1}{25}$$

$$\sin^2\alpha - 2\sin\alpha\cos\alpha + \cos^2\alpha = \frac{1}{25}$$

$$1 - 2\sin\alpha\cos\alpha = \frac{1}{25}$$

$$\sin\alpha\cos\alpha = \frac{12}{25}$$

$$25\sin\alpha\cos\alpha = 12(\sin^2\alpha + \cos^2\alpha)$$

$$12\sin^2\alpha - 25\sin\alpha\cos\alpha + 12\cos^2\alpha = 0$$

$$(3\sin\alpha - 4\cos\alpha)(4\sin\alpha - 3\cos\alpha) = 0$$

$$\tan\alpha = \frac{4}{3} \text{ or } \frac{3}{4}$$

Check when $\tan\alpha = \frac{4}{3}$, then $\sin\alpha = \frac{4}{5}$, $\cos\alpha = \frac{3}{5}$

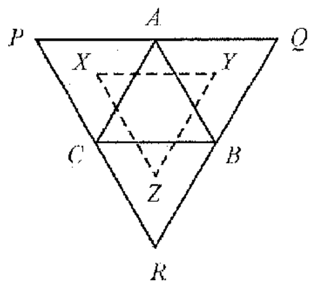
LHS $= \sin\alpha - \cos\alpha = \frac{4}{5} - \frac{3}{5} = \frac{1}{5} =$ RHS

When $\tan\alpha = \frac{3}{4}$, then $\sin\alpha = \frac{3}{5}$, $\cos\alpha = \frac{4}{5}$

LHS $= \sin\alpha - \cos\alpha = \frac{3}{5} - \frac{4}{5} = -\frac{1}{5} \neq$ RHS

$\therefore B = \tan\alpha = \frac{4}{3}$

I8 In Figure 3, ΔPAC, ΔQBA, ΔRCB and ΔABC are equilateral triangles. The points X, Y and Z are the incentres of ΔPAC, ΔQBA, ΔRCB respectively. If the length of PA is 10 cm and the perimeter of ΔXYZ is w cm, find the value of w. (Remark: the incentre of a triangle is the point of intersection of the three interior angle bisectors of the triangle.)



It can be easily proved that ΔXYZ is an equilateral triangle whose side is half of PQ.

$\therefore w = 3\times20 \div 2 = 30$

I9 Let $f(x) = \frac{1}{2}(4x^2 - 60x + 9 + |4x^2 - 60x + 9|)$.

If $k = f(1) + f(2) + f(3) + \ldots + f(15) + f(16)$, find the value of k.

Similar question: 2004 FI3.2

Note that if $a \geq 0$, $\frac{1}{2}(a + |a|) = a$; if $a < 0$, $\frac{1}{2}(a + |a|) = 0$

Now, $4x^2 - 60x + 9 = (2x - 15)^2 - 216 = (2x - 15 + 6\sqrt{6})(2x - 15 - 6\sqrt{6})$

$= 4(x - 7.5 + 3\sqrt{6})(x - 7.5 - 3\sqrt{6})$

If $7.5 - 3\sqrt{6} < x < 7.5 + 3\sqrt{6}$, then $4x^2 - 60x + 9 < 0$

If $x \leq 7.5 - 3\sqrt{6}$ or $7.5 + 3\sqrt{6} \leq x$, then $4x^2 - 60x + 9 \geq 0$

$7.5 - 3\sqrt{6} = \sqrt{56.25} - \sqrt{54} > 0$, $\sqrt{56.25} - \sqrt{54} < \sqrt{56.25} - 7 = 0.5$

$0 < 7.5 - 3\sqrt{6} < 0.5$

$7.5 + 3\sqrt{6} = \sqrt{56.25} + \sqrt{54} > 7.5 + 7 = 14.5$, $\sqrt{56.25} + \sqrt{54} < 7.5 + 7.5 = 15$

$14.5 < 7.5 + 3\sqrt{6} < 15$

Let $g(x) = 4x^2 - 60x + 9$, $\therefore$ $g(1) < 0, g(2) < 0, \ldots, g(14) < 0, g(15) > 0, g(16) > 0$

$f(1) = 0, f(2) = 0, \ldots, f(14) = 0, f(15) = g(15), f(16) = g(16)$

$k = f(1) + f(2) + f(3) + \ldots + f(15) + f(16) = f(15) + f(16) = g(15) + g(16)$

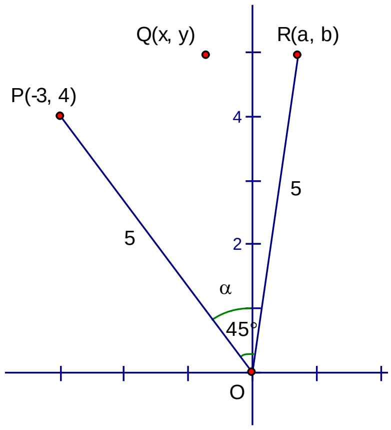
$= 4\times 15^2 - 60\times 15 + 9 + 4\times 16^2 - 60\times 16 + 9$

$= 60\times 15 - 60\times 15 + 9 + 64\times 16 - 60\times 16 + 9$

$= 9 + 4\times 16 + 9$

$= 82$

I10 The coordinates of point P on the plane is $(-3, 4)$. After rotating 45° clockwise about the centre $(0, 0)$ and reflecting along the y-axis, the point P reaches the point $Q = (x, y)$. If $z = x + y$, find the value of z. **Reference: 2015 HG3**



Let $P(-3, 4)$ makes an angle α with the positive y-axis.

Then $\sin\alpha = \frac{3}{5}$, $\cos\alpha = \frac{4}{5}$, $OP = 5$.

Let $R(a, b)$ be the point after rotating P clockwise about O.

Then $OR = OP = 5$

$a = 5\sin(45° - \alpha) = 5\sin 45°\cos\alpha - 5\cos 45°\sin\alpha$

$= 5\times\frac{\sqrt{2}}{2}\times\frac{4}{5} - 5\times\frac{\sqrt{2}}{2}\times\frac{3}{5} = \frac{\sqrt{2}}{2}$

$b = 5\cos(45° - \alpha) = 5\cos 45°\cos\alpha + 5\sin 45°\sin\alpha$

$= 5\times\frac{\sqrt{2}}{2}\times\frac{4}{5} + 5\times\frac{\sqrt{2}}{2}\times\frac{3}{5} = \frac{7\sqrt{2}}{2}$

$Q = (-a, b) = \left(-\frac{\sqrt{2}}{2}, \frac{7\sqrt{2}}{2}\right)$

$z = x + y = -\frac{\sqrt{2}}{2} + \frac{7\sqrt{2}}{2} = 3\sqrt{2}$

Group Events

G1 If there are N integers from 1 to 50 that are relatively prime to 50, find the value of N.
(Remark: positive integers a and b are said to be relatively prime if their greatest common divisor is 1.)

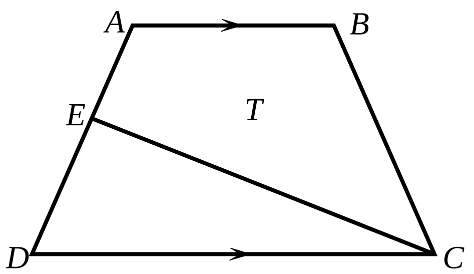
We first find the number of positive integers less than or equal to 50 that are **not** relatively prime to 50.

They are 2, 4, 6, …, 50 (There are 25 multiples of 2). Also, 5, 15, 25, 35, 45 are integers which are not relatively prime to 50 (There are 5 of them).

$\therefore$ The number of integers which are not relatively prime to 50 is 30.

The number of integers which are relatively prime to 50 is 20.

G2 In Figure 2, $ABCD$ is a trapezium, $AB \parallel CD$, $\angle BCE = \angle ECD$, $CE \perp AD$ and $DE = 2AE$.
If the area of ΔDEC is 2007 cm^2 and the area of quadrilateral $ABCE$ is T cm^2, find the value of T.



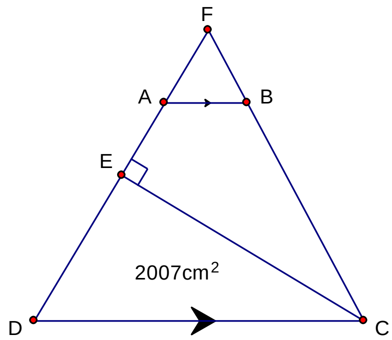
Produce DA and CB to meet at F.

Then it is easy to prove that $\Delta CDE \cong \Delta CFE$ (ASA)

$\therefore$ Area of $\Delta CDF = 2\times 2007\text{ cm}^2 = 4014\text{ cm}^2$

Again, $\Delta FAB \sim \Delta FDC$ (equiangular)

$FA : FD = (FE - AE) : 2DE$
$= (DE - AE) : 2DE$
$= (2AE - AE) : 4AE$
$= 1 : 4$



By the ratio of areas of similar triangles,

$$\frac{\text{Area of } \Delta ABF}{\text{Area of } \Delta DCF} = \left(\frac{1}{4}\right)^2$$

$$\frac{\text{Area of } \Delta ABF}{4014\text{cm}^2} = \frac{1}{16}$$

Area of $\Delta ABF = \dfrac{2007}{8}\text{ cm}^2$

$T = 2007 - \dfrac{2007}{8} = \dfrac{14049}{8}\text{ cm}^2$ ($= 1756.125\text{ cm}^2$)

G3 Given that $a^2 - 3a + 1 = 0$. If $A = \dfrac{2a^5 - 5a^4 + 2a^3 - 8a^2 + 7a}{3a^2 + 3}$, find the value of A.

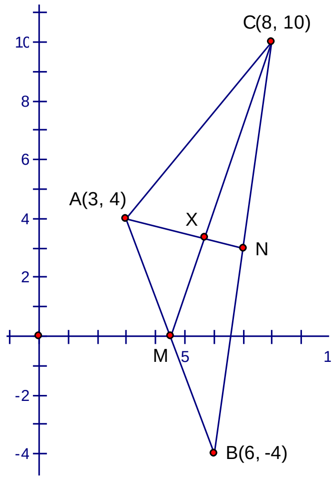
Reference 1993 HI9, 2000 HG1, 2001 FG2.1, 2009 HG2

By division algorithm, $2a^5 - 5a^4 + 2a^3 - 8a^2 + 7a = (a^2 - 3a + 1)(2a^3 + a^2 + 3a) + 4a = 4a$

$3a^2 + 3 = 3(a^2 + 1) = 3(3a) = 9a$

$A = \dfrac{4a}{9a} = \dfrac{4}{9}$

G4 Given that the coordinates of the points A, B and C are (3, 4), (6, –4) and (8,10) respectively. M and N are the midpoints of AB and BC respectively. X is a point on AN such that $AX : XN = 2 : 1$. If $r = \dfrac{CX}{XM}$, find the value of r.



$M = (4.5, 0)$, $N = (7, 3)$

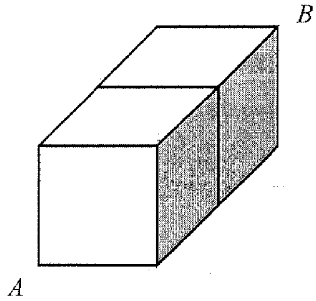
$$X = \left(\frac{2\times 7 + 3}{3}, \frac{2\times 3 + 4}{3}\right) = \left(\frac{17}{3}, \frac{10}{3}\right)$$

slope of $CM = \dfrac{10}{8 - 4.5} = \dfrac{20}{7}$, slope of $MX = \dfrac{\frac{10}{3}}{\frac{17}{3} - \frac{9}{2}} = \dfrac{20}{7}$

$\therefore CXM$ are collinear

$r = \dfrac{CX}{XM} \Rightarrow y\text{-coordinate of } X = \dfrac{10}{3} = \dfrac{10}{1+r} \Rightarrow r = 2$

G5 In Figure 2, a 1 cm × 1 cm × 2 cm rectangular box is made by two cubes with side length 1 cm. An ant is climbing along the box in a way that it must stay on the edges of the cubes through out the climbing. Starting from vertex *A* and climbing with a speed of 1 cm per minutes, it reaches vertex *B* after 4 minutes. If the total number of possible paths taken by the ant is *S*, find the value of *S*.

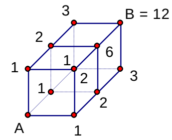


Reference: 1983 FI4.1, 1998 HG6, 2000 HI4

From *A* the ant can only climb upwards, or to the right, or towards *B*.

Add the numbers on the corners of the box. The numbers shows the number of possible ways for the ant to climb from *A* to reach there.

$\therefore$ The number of possible ways to reach *B* is 12.



G6 If the remainder of 7^{2007} when dividing by 5 is *R*, find the value of *R*.

$7 \div 5$ .... 2, $7^2 \div 5$ .... 4, $7^3 \div 5$ .... 3, $7^4 \div 5$ 1

The remainder repeats as the exponent increases.

$2007 = 4\times 501 + 3$, the remainder is 3.

Method 2

$7^2 = 49 = 50 - 1$

$7^{2007} = 7\cdot 7^{2006} = 7\cdot(7^2)^{1003} = 7(50 - 1)^{2003} = 7[50m + (-1)^{2003}]$, where m is an integer.

$= 5(70m) - 7 = 5(70m - 2) + 3$, the remainder is 3.

G7 Let $k = \sin 30° + \cos 60° + \sin 90° + \cos 120° + \ldots + \sin 1890° + \cos 1920°$, find the value of k.

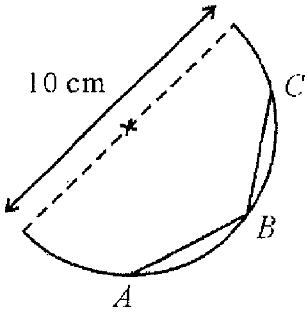
$\sin 30° + \cos 60° + \sin 90° + \cos 120° + \sin 150° + \cos 180° + \sin 210° + \cos 240° + \sin 270° + \cos 300° + \sin 330° + \cos 360°$

$$= \frac{1}{2}+\frac{1}{2}+1-\frac{1}{2}+\frac{1}{2}-1-\frac{1}{2}-\frac{1}{2}-1+\frac{1}{2}-\frac{1}{2}+1=0$$

The cycle repeats for every multiples of 360°, and $1800° = 360°\times 5$

$$k = \sin 1830° + \cos 1860° + \sin 1890° + \cos 1920° = \frac{1}{2}+\frac{1}{2}+1-\frac{1}{2}=\frac{3}{2}$$

G8 In figure 3, given that the diameter of the semicircle is 10 cm. *A*, *B* and *C* are three arbitrary points on the semi-circle where *B* is on the arc $\overset{\frown}{AC}$. If *x* is the sum of the length of the line segments *AB* and *BC*, find the greatest possible value of *x*. **(Reference: 2011 HG6)**



Let *O* be the centre, the radius is 5 cm.

Let *OD*, *OE* be the respective perpendicular bisectors.

It is easy to prove that $\angle COD = \angle BOD$, $\angle BOE = \angle AOE$.

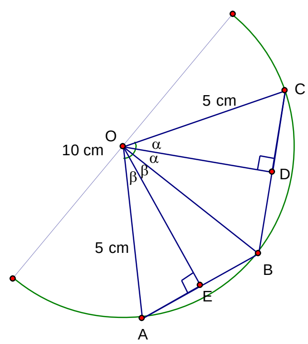
Let $\angle COD = \alpha$, $\angle AOE = \beta$

$\therefore 0° < 2\alpha + 2\beta < 180°$

$0° < \alpha + \beta < 90°$

$BD = 5\sin\alpha$, $BE = 5\sin\beta$

$$\begin{aligned} AB + BC &= 10\sin\alpha + 10\sin\beta \\ &= 10(\sin\alpha + \sin\beta) \\ &= 20\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2} \end{aligned}$$

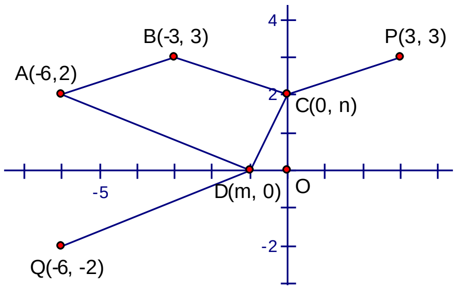


$\sin\frac{\alpha+\beta}{2} \le \sin 45°, \cos\frac{\alpha-\beta}{2} \le 1$, equality holds when $\alpha = \beta = 45°$

$x \le 20\sin 45° = 10\sqrt{2}$

G9 In the coordinate plane, the points $A = (-6, 2)$, $B = (-3, 3)$, $C = (0, n)$ and $D = (m, 0)$ form a quadrilateral $ABCD$. Find the value of n so that the perimeter of the quadrilateral $ABCD$ is the least.

In order that the perimeter is the least, $n > 0$, $m < 0$.
Reflect $B(-3, 3)$ along y-axis to $P(3, 3)$.
Reflect $A(-6, 2)$ along x-axis to $Q(-6, -2)$.
By the property of reflection, the perimeter is equal to
$QD + CD + CP + AB$
It is the least when P, C, D, Q are collinear.
In this case, slope of CP = slope of PQ.



$$\frac{3-n}{3}=\frac{3+2}{3+6}\Rightarrow n=\frac{4}{3}.$$

G10 Given that integers x and y satisfying the equation $3x + 5y = 1$. If $S = x - y$ and $S > 2007$, find the least possible value of S.

$3\times2 + 5\times(-1) = 1$, one possible pair solution is $(x, y) = (2, -1)$

The slope of $3x + 5y = 1$ is $-\frac{3}{5}$.

$\therefore$ The parametric equation of $3x + 5y = 1$ is $\begin{cases} x=2+5k \\ y=-1-3k \end{cases}$, where k is an integer.

$S = x - y = 2 + 5k - (-1 - 3k) = 3 + 8k$
$S > 2007 \Rightarrow 3 + 8k > 2007$
$k > 250.5$
The least possible $k = 251$
The least possible $S = 3 + 8(251) = 2011$