Individual Events

SI	k	2	I1	A	15	I2	P	3	I3	A	2	I4	P	4	IS	a	2
	d	1		B	3		Q	4		B	3		Q	300		b	2
	a	–6		C	4		R	10		C	45		R	2		c	–3
	t	$\frac{50}{11}$		D	8		S	112.5		D	7		S	$2\sqrt{3}$		d	35

Group Events

SG	W	$\frac{1+\sqrt{5}}{2}$	G1	m	8	G2	z	540	G3	k	$\sqrt{33}$	G4	m	13	GS	value	–1
	T	29		h	$\sqrt{13}$		R	6		v	6		n	6		$x^4+\frac{1}{x^4}$	4036079
	S	106		$x+y+z$	11		k	5		value	106		$abc+def$	72		$\cot\alpha$	$\frac{99}{20}$
	k	4		Number	72		xyz	1		r	27405		$p+q$	2		value	$\frac{6023}{6022}$

Sample Individual Event (2007 Final Individual Event 1)

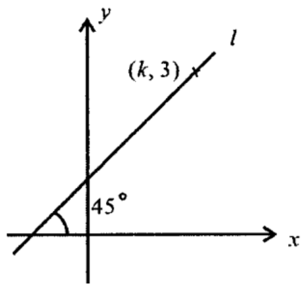
SI.1 Let $\sqrt{k}=\sqrt{7+\sqrt{13}}-\sqrt{7-\sqrt{13}}$, find the value of k.

$$\sqrt{k}^2=\left(\sqrt{7+\sqrt{13}}-\sqrt{7-\sqrt{13}}\right)^2$$

$$k=7+\sqrt{13}-2\sqrt{7^2-\sqrt{13}^2}+7-\sqrt{13}$$

$$=14-2\sqrt{36}=2$$

SI.2 In Figure 1, the straight line ℓ passes though the point $(k, 3)$ and makes an angle 45° with the x-axis. If the equation of ℓ is $x + by + c = 0$ and $d = |1 + b + c|$, find the value of d.



$\ell: \dfrac{y-3}{x-2}=\tan 45^\circ$

$y - 3 = x - 2$

$x - y + 1 = 0$, $b = -1$, $c = 1$

$d = |1 - 1 + 1| = 1$

SI.3 If $x - d$ is a factor of $x^3 - 6x^2 + 11x + a$, find the value of a.

$f(x) = x^3 - 6x^2 + 11x + a$

$f(1) = 1 - 6 + 11 + a = 0$

$a = -6$

SI.4 If $\cos x + \sin x = -\dfrac{a}{5}$ and $t = \tan x + \cot x$, find the value of t.

$\cos x + \sin x = \dfrac{6}{5}$

$(\cos x + \sin x)^2 = \dfrac{36}{25}$

$1 + 2\sin x\cos x = \dfrac{36}{25}$

$2\sin x\cos x = \dfrac{11}{25} \Rightarrow \sin x\cos x = \dfrac{11}{50}$

$d = \tan x + \cot x$

$=\dfrac{\sin x}{\cos x}+\dfrac{\cos x}{\sin x}$

$=\dfrac{\sin^2 x+\cos^2 x}{\sin x\cos x}$

$\dfrac{1}{\sin x\cos x}=\dfrac{50}{11}$

Individual Event 1

I1.1 Let A = 15×tan 44°×tan45°×tan46°, find the value of A.

Similar question: 2012 FG2.1

$$A = 15\times \tan 44^\circ \times 1 \times \frac{1}{\tan 44^\circ} = 15$$

I1.2 Let n be a positive integer and $\overbrace{20082008\cdots2008}^{n\text{ 2008's}}15$ is divisible by A.

If the least possible value of n is B, find the value of B.

Reference: 2010 HG2

The given number is divisible by 15. Therefore it is divisible by 3 and 5.

The last 2 digits of the given number is 15, which is divisible by 15.

The necessary condition is: $\overbrace{20082008\cdots2008}^{n\text{ 2008's}}$ must be divisible by 3.

2 + 0 + 0 + 8 = 10 which is not divisible by 3.

The least possible n is 3: 2+0+0+8+2+0+0+8+2+0+0+8 = 30 which is divisible by 3.

I1.3 Given that there are C integers that satisfy the equation $|x-2|+|x+1|=B$, find the value of C

Reference: 1994 HG1, 2001 HG9, 2004 FG4.2, 2008 HI8, 2008 FI1.3, 2010 HG6, 2012 FG2.3

$|x-2|+|x+1|=3$

If $x<-1$, $2-x-x-1=3$

$\Rightarrow x=-1$ (rejected)

If $-1\le x\le 2$, $2-x+x+1=3$

$\Rightarrow 3=3$, always true $\therefore$

$-1\le x\le 2$

If $2<x$, $x-2+x+1=3$

$\Rightarrow x=2$ (reject)

$\therefore -1\le x\le 2$ only

$\because$ x is an integer

$\therefore x=-1, 0, 1, 2$; $C=4$

I1.4 In the coordinate plane, the distance from the point $(-C, 0)$ to the straight line $y = x$ is $\sqrt{D}$, find the value of D.

The distance from P(x_0, y_0) to the straight line $Ax + By + C = 0$ is $\left|\frac{Ax_0+By_0+C}{\sqrt{A^2+B^2}}\right|$.

The distance from (–4, 0) to $x - y = 0$ is

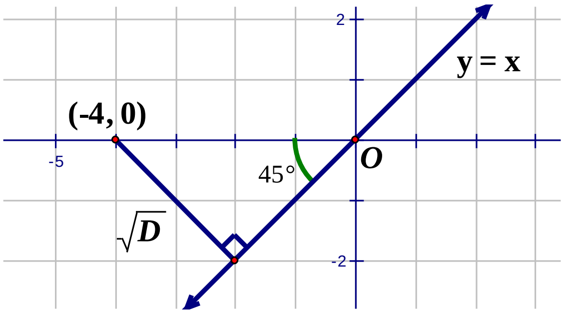
$$\sqrt{D}=\left|\frac{-4-0+0}{\sqrt{1^2+(-1)^2}}\right|=\frac{4}{\sqrt{2}}=2\sqrt{2}=\sqrt{8}$$

$D = 8$

Method 2

$\sqrt{D} = 4\sin 45^\circ = \frac{4}{\sqrt{2}}$

$D=\frac{16}{2}=8$



Individual Event 2

I2.1 Given that $P=\left[\sqrt[3]{6}\times\left(\sqrt[3]{\frac{1}{162}}\right)\right]^{-1}$, find the value of P.

$P=\left[\sqrt[3]{\frac{6}{162}}\right]^{-1}$

$=\sqrt[3]{\frac{162}{6}}$

$=\sqrt[3]{27}=3$

I2.2 Let a, b and c be real numbers with ratios $b:(a+c)=1:2$ and $a:(b+c)=1:P$.

If $Q=\frac{a+b+c}{a}$, find the value of Q.

$2b=a+c$ …… (1), $3a=b+c$ …… (2)

(1) – (2): $2b-3a=a-b\Rightarrow 3b=4a\Rightarrow a:b=3:4$

Let $a=3k$, $b=4k$, sub. into (1): $2(4k)=3k+c\Rightarrow c=5k$

$Q=\frac{a+b+c}{a}=\frac{3k+4k+5k}{3k}=4$

I2.3 Let $R=\left(\sqrt{\sqrt{3}+\sqrt{2}}\right)^{Q}+\left(\sqrt{\sqrt{3}-\sqrt{2}}\right)^{Q}$. Find the value of R.

$R=\left(\sqrt{\sqrt{3}+\sqrt{2}}\right)^{4}+\left(\sqrt{\sqrt{3}-\sqrt{2}}\right)^{4}$

$=\left(\sqrt{3}+\sqrt{2}\right)^{2}+\left(\sqrt{3}-\sqrt{2}\right)^{2}$

$=3+2\sqrt{6}+2+3-2\sqrt{6}+2=10$

I2.4 Let $S=(x-R)^2+(x+5)^2$, where x is a real number. Find the minimum value of S.

$S=(x-10)^2+(x+5)^2$

$=x^2-20x+100+x^2+10x+25$

$=2x^2-10x+125$

$=2(x^2-5x)+125$

$=2(x-2.5)^2+125-2\times2.5^2$

$=2(x-2.5)^2+112.5\geq112.5$

The minimum value of S is 112.5.

Method 2

$S=(10-x)^2+(x+5)^2$

Let $a=10-x$, $b=x+5$

$a+b=15$, which is a constant

$\therefore a^2+b^2$ reaches its minimum when $a=b=7.5$

$\therefore$ Minimum $S=7.5^2+7.5^2$

$=112.5$

Individual Event 3

I3.1 Given that $\frac{1-\sqrt{3}}{2}$ satisfies the equation $x^2 + px + q = 0$, where p and q are rational numbers. If $A = |p| + 2|q|$, find the value of A.

For an equation with rational coefficients, conjugate roots occur in pairs.

That is, the other root is $\frac{1+\sqrt{3}}{2}$.

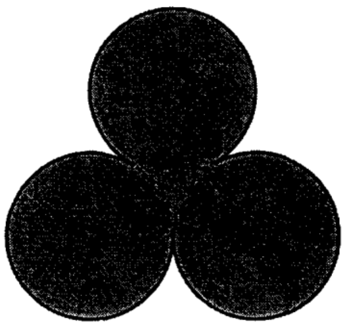
$$\frac{1-\sqrt{3}}{2}+\frac{1+\sqrt{3}}{2}=-p$$
$$\Rightarrow p=-1$$
$$\frac{1-\sqrt{3}}{2}\times\frac{1+\sqrt{3}}{2}=q$$
$$\Rightarrow q=-\frac{1}{2}$$
$$A = |-1| + 2\left|-\frac{1}{2}\right| = 2$$

I3.2 Two bags U_1 and U_2 contain identical red and white balls. U_1 contains A red balls and 2 white balls. U_2 contains 2 red balls and B white balls. Take two balls out of each bag. If the probability of all four balls are red is $\frac{1}{60}$, find the value of B.

U_1 contains 2 red and 2 white, total 4 balls. U_2 contains 2 red and B white, total $2 + B$ balls.

$$\text{P(all 4 are red)} = \frac{2}{4}\times\frac{1}{3}\times\frac{2}{2+B}\times\frac{1}{1+B}=\frac{1}{60}$$
$$20 = (2 + B)(1 + B)$$
$$B^2 + 3B - 18 = 0$$
$$(B - 3)(B + 6) = 0$$
$$B = 3$$

I3.3 Figure 1 is formed by three identical circles touching one another, the radius of each circle is B cm. If the perimeter of the shaded region is C cm, find the value of C. (Take $\pi = 3$)

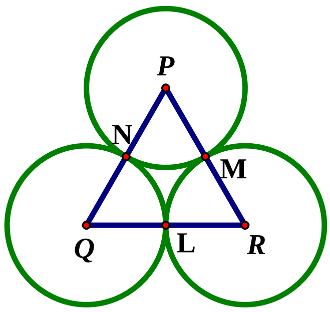


Let the centres of the circles be P, Q and R respectively.

Then $PQ = 2B = 6 = QR = PR$

ΔPQR is an equilateral Δ.

$\angle P = \angle Q = \angle R = 60°$



$$\text{Perimeter} = 3\times2\pi\times3 - 3\times\overset{\frown}{MN}$$
$$= 18\pi - 3\times2\pi\times3\times\frac{60}{360}$$
$$= 15\pi = 45$$

I3.4 Let D be the integer closest to $\sqrt{C}$, find the value of D.

$6=\sqrt{36}<\sqrt{45}<\sqrt{49}=7$

$6 < D < 7$

$6.5^2 = 42.25 < 45$

$6.5 < D < 7$

$D = 7$

Method 2

$6=\sqrt{36}<\sqrt{45}<\sqrt{49}=7 \Rightarrow 6 < D < 7$

$\because |45 - 36| = 9, |45 - 49| = 4$

$\therefore$ 45 is closer to 49

$\therefore \sqrt{45}$ is closer to 7

$D = 7$

Individual Event 4

I4.1 Given that x and y are real numbers such that $|x| + x + y = 10$ and $|y| + x - y = 10$. If $P = x + y$, find the value of P.

If $x \ge 0, y \ge 0$; $\begin{cases} 2x + y = 10 \\ x = 10 \end{cases}$, solving give $x = 10, y = -10$ (reject)

If $x \ge 0, y < 0$, $\begin{cases} 2x + y = 10 \\ x - 2y = 10 \end{cases}$, solving give $x = 6, y = -2$

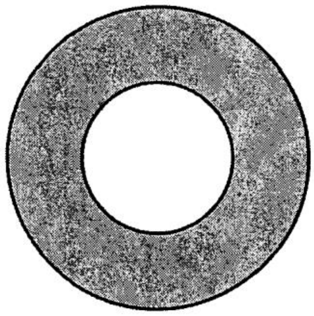
If $x < 0, y \ge 0$, $\begin{cases} y = 10 \\ x = 10 \end{cases}$, reject

If $x < 0, y < 0$; $\begin{cases} y = 10 \\ x - 2y = 10 \end{cases}$, solving give $x = 30, y = 10$ (reject)

$\therefore x = 6, y = -2$

$P = x + y = 6 - 2 = 4$

I4.2 In Figure 1, the shaded area is formed by two concentric circles and has area 96π cm^2. If the two radii differ by $2P$ cm and the large circle has area Q cm^2, find the value of Q. (Take $\pi = 3$)



Let the radii of the large and small circles be R and r respectively.

$\pi(R^2 - r^2) = 96\pi$ and $R - r = 8$ … (1)

$(R + r)(R - r) = 96 \Rightarrow (R + r)\times 8 = 96 \Rightarrow R + r = 12$ … (2)

(2) + (1): $2R = 20 \Rightarrow R = 10$

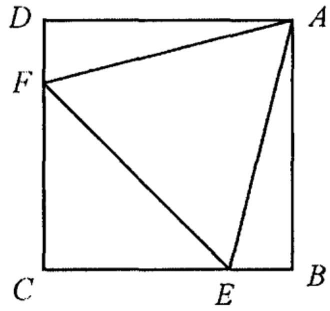
$\Rightarrow Q = \pi(10)^2 = 300$

I4.3 Let R be the largest integer such that $R^Q < 5^{200}$, find the value of R.

Reference: 1996 HI4, 1999 FG5.3, 2008 FG2.4

$R^{300} < 5^{200} \Rightarrow R^3 < 25$, the largest integer is 2.

I4.4 In Figure 2, there are a square $ABCD$ with side length $(R - 1)$ cm and an equilateral triangle AEF. (E and F are points on BC and CD respectively). If the area of ΔAEF is $(S - 3)$ cm^2, find the value of S.



Reference: 1995 HG7

Let $AF = x$ cm $= FE = AE$

$\angle FAE = 60°$, $\angle DAF = \angle BAE = 15°$

$AD = 1 = AF \cos 15° = x \cos 15° \Rightarrow x = \sec 15°$

$$S - 3 = \frac{1}{2}x^2 \sin 60° = \frac{1}{2}\cdot\frac{1}{\cos^2 15°}\cdot\frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{2}\cdot\frac{1}{1+\cos 30°} = \frac{\sqrt{3}}{2}\cdot\frac{1}{1+\frac{\sqrt{3}}{2}} = \frac{\sqrt{3}}{2}\cdot\frac{2}{2+\sqrt{3}} = 2\sqrt{3} - 3$$

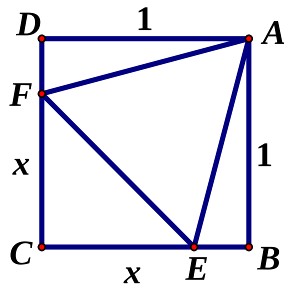
$S = 2\sqrt{3}$

Method 2

Let $CE = CF = x$ cm

Then $BE = DF = (1 - x)$ cm

By Pythagoras' theorem,



$AF = FE \Rightarrow 1 + (1 - x)^2 = x^2 + x^2$

$2 - 2x + x^2 = 2x^2$

$x^2 + 2x - 2 = 0 \Rightarrow x^2 = 2 - 2x$

$x = -1 + \sqrt{3}$

Area of ΔAFE = Area of square – area of ΔCEF – 2 area of ΔADF

$$= 1 - \frac{x^2}{2} - 2\times\frac{1\times(1-x)}{2} = x - \frac{x^2}{2} = x - \frac{2-2x}{2} = 2x - 1$$

$$= 2\left(-1+\sqrt{3}\right) - 1 = 2\sqrt{3} - 3$$

$S - 3 = 2\sqrt{3} - 3$

$S = 2\sqrt{3}$

Individual Spare

IS.1 If all the positive factors of 28 are $d_1, d_2, \ldots, d_n$ and $a = \frac{1}{d_1} + \frac{1}{d_2} + \cdots + \frac{1}{d_n}$, find the value of a.

Positive factors of 28 are 1, 2, 4, 7, 14, 28. $a = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{7} + \frac{1}{14} + \frac{1}{28} = 2$

IS.2 Given that x is a negative real number that satisfy $\dfrac{1}{x + \frac{1}{x+2}} = a$. If $b = x + \frac{7}{2}$, find the value of b

$\dfrac{1}{x + \frac{1}{x+2}} = 2 \Rightarrow \dfrac{x+2}{x(x+2)+1} = 2$

$\Rightarrow x + 2 = 2x^2 + 4x + 2$

$\Rightarrow 2x^2 + 3x = 0$

$\Rightarrow x = -1.5$ or 0 (reject)

$b = -1.5 + 3.5 = 2$

IS.3 Let α and β be the two roots of the equation $x^2 + cx + b = 0$, where $c < 0$ and $\alpha - \beta = 1$. Find the value of c.

Reference: 2016 FI1.3

$\alpha\beta = b = 2$; $(\alpha - \beta)^2 = 1$

$\Rightarrow (\alpha + \beta)^2 - 4\alpha\beta = 1$

$\Rightarrow (-c)^2 - 4\times2 = 1$

$c = -3$

IS.4 Let d be the remainder of $(196c)^{2008}$ divided by 97. Find the value of d.

$[196\times(-3)]^{2008} = 588^{2008} = (97\times6 + 6)^{2008}$

$= (97\times6)^{2008} + {}_{2008}C_1\cdot(97\times6)^{2007}\times6 + \ldots + 6^{2008}$

$= 97m + 6^{2008}$, where m is an integer.

Note that $2^5\times3 = 96 = 97 - 1 \equiv -1 \pmod{97}$;

$2\times3^5 = 486 = 97\times5 + 1 \equiv 1 \pmod{97}$;

$\therefore\ 6^6 = (2^5\times3)\times(2\times3^5) \equiv -1 \pmod{97}$

$6^{2008} = (6^6)^{334}\times6^4$

$\equiv (-1)^{334}\times1296$

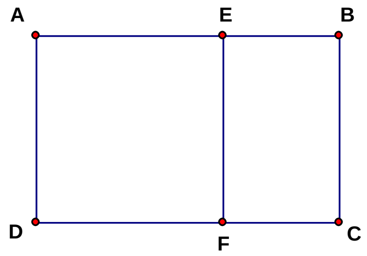
$\equiv 97\times13 + 35$

$\equiv 35 \pmod{97}$

$\therefore\ d = 35$

Sample Group Event (2007 Final Group Event 1)

SG.1 In Figure 1, *AEFD* is a unit square. The ratio of the length of the rectangle *ABCD* to its width is equal to the ratio of the length of the rectangle *BCFE* to its width. If the length of *AB* is *W* units, find the value of *W*.



$\frac{W}{1}=\frac{1}{W-1}$

$W^2 - W - 1 = 0$

$\Rightarrow W=\frac{1+\sqrt{5}}{2}$

SG.2 On the coordinate plane, there are *T* points (x, y), where x, y are integers, satisfying $x^2+y^2 < 10$, find the value of *T*. **Reference: 2002 FI4.3**

T = number of integral points inside the circle $x^2 + y^2 = 10$.

We first count the number of integral points in the first quadrant:

$x = 1; y = 1, 2$

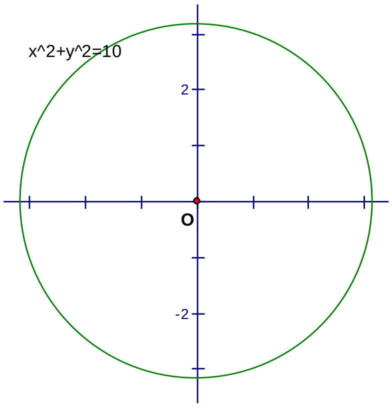
$x = 2; y = 1, 2$

Next, the number of integral points on the x-axis and y-axis

$= 3 + 3 + 3 + 3 + 1 = 13$

$T = 4\times4 + 3 + 3 + 3 + 3 + 1$

$= 29$



SG.3 Let P and $P + 2$ be both prime numbers satisfying $P(P + 2) \le 2007$. If S represents the sum of such possible values of P, find the value of S.

$P^2 + 2P - 2007 \le 0$

$(P + 1)^2 - 2008 \le 0$

$(P + 1 + \sqrt{2008})(P + 1 - \sqrt{2008}) \le 0$

$(P + 1 + 2\sqrt{502})(P + 1 - 2\sqrt{502}) \le 0$

$-1 - 2\sqrt{502} \le P \le -1 + 2\sqrt{502}$

P is a prime $\Rightarrow 0 < P \le -1 + 2\sqrt{502}$

$22 = \sqrt{484} < \sqrt{502} < \sqrt{529} = 23$

$43 < -1 + 2\sqrt{502} < 45$

$\therefore (P, P + 2) = (3, 5), (5, 7), (11, 13), (17, 19), (29, 31), (41, 43)$

$S = 3 + 5 + 11 + 17 + 29 + 41 = 106$

SG.4 It is known that $\log_{10}(2007^{2006} \times 2006^{2007}) = a\times10^k$, where $1 \le a < 10$ and k is an integer. Find the value of k.

$a\times10^k = 2006 \log 2007 + 2007 \log 2006 = 2006\times(\log 2007 + \log 2006) + \log 2006$

$2006\times(\log 2006 + \log 2006) + \log 2006 < a\times10^k < 2006\times(\log 2007 + \log 2007) + \log 2007$

$4013 \log 2006 < a\times10^k < 4013 \log 2007$

$4013 \log(2.006\times10^3) < a\times10^k < 4013 \log(2.007\times10^3)$

$4013 (\log 2.006 + 3) < a\times10^k < 4013(\log 2.007 + 3)$

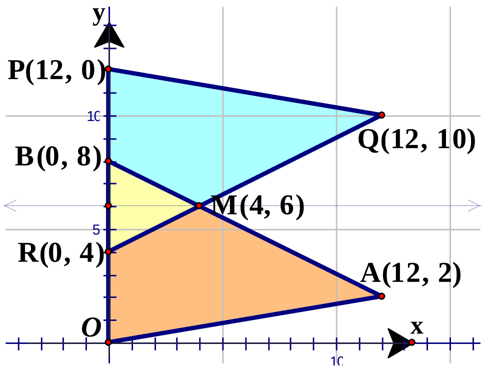
$4013 \log 2 + 4013\times3 < a\times10^k < 4013 \log 3 + 3$

$1.32429\times10^4 = 4013\times0.3 + 4013\times3 < a\times10^k < 4013\times0.5 + 4013\times3 = 1.40455\times10^4$

$k = 4$

Group Event 1

G1.1 Given that there are three points on the coordinate plane: $O(0, 0)$, $A(12, 2)$ and $B(0, 8)$. A reflection of ΔOAB along the straight line $y = 6$ creates ΔPQR. If the overlapped area of ΔOAB and ΔPQR is m square units, find the value of m.



$P(12, 0)$, $Q(12, 10)$, $R(0, 4)$.

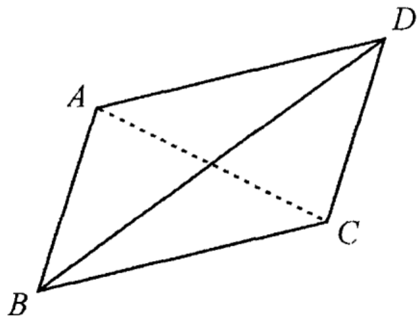
Suppose AB intersects QR at $M(x, 6)$.

Slope of MR = slope of QR

$\frac{6-4}{x} = \frac{10-4}{12} \Rightarrow x = 4$; $M(4, 6)$

Area of the overlap $\Delta BMR = \frac{1}{2}\cdot(8-4)\times 4 = 8$; $m = 8$

G1.2 In Figure 1, $ABCD$ is a parallelogram with $BA = 3$ cm, $BC = 4$ cm and $BD = \sqrt{37}$ cm. If $AC = h$ cm, find the value of h.



$CD = BA = 3$ cm (opp. sides, //-gram)

In ΔBCD, $\cos C = \frac{4^2+3^2-\sqrt{37}^2}{2\times 3\times 4} = -\frac{1}{2}$

$\cos B = \cos(180° - C) = -\cos C = \frac{1}{2}$ (int. $\angle$s AB // DC)

In ΔABC, $AC = \sqrt{3^2+4^2-2\cdot 3\cdot 4\cos B} = \sqrt{13}$; $h = \sqrt{13}$

Method 2

By Apollonius theorem,

$2BA^2 + 2BC^2 = AC^2 + BD^2$

$2\times 3^2 + 2\times 4^2 = h^2 + 37$

$h = \sqrt{13}$

G1.3 Given that x, y and z are positive integers and the fraction $\frac{151}{44}$ can be written in the form of $3+\cfrac{1}{x+\cfrac{1}{y+\cfrac{1}{z}}}$. Find the value of $x + y + z$.

$\frac{151}{44} = 3 + \frac{19}{44} = 3 + \frac{1}{\frac{44}{19}}$

$= 3 + \frac{1}{2+\frac{6}{19}} = 3 + \cfrac{1}{2+\cfrac{1}{\frac{19}{6}}} = 3 + \cfrac{1}{2+\cfrac{1}{3+\frac{1}{6}}}$

$x = 2$, $y = 3$, $z = 6$; $x + y + z = 11$

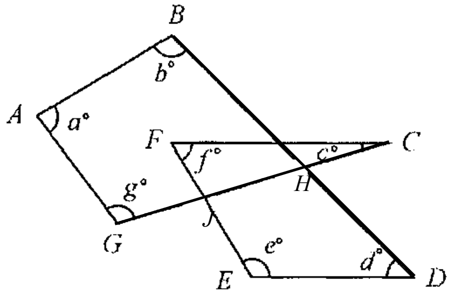
G1.4 When 491 is divided by a two-digit integer, the remainder is 59. Find this two-digit integer.

Let the number be $10x + y$, where $0 < x \le 9$, $0 \le y \le 9$.

$491 = (10x + y)\cdot Q + 59$; $59 < 10x + y$

$491 - 59 = 432 = (10x + y)\cdot Q$; $432 = 72\times 6$; the number is 72.

Group Event 2



G2.1 In Figure 1, BD, FC, GC and FE are straight lines. If $z = a + b + c + d + e + f + g$, find the value of z.

$a° + b° + g° + \angle BHG = 360°$ ($\angle$s sum of polygon $ABHG$)

$c° + f° = \angle CJE$ (ext. $\angle$ of ΔCFJ)

$c° + f° + e° + d° + \angle JHD = 360°$ ($\angle$s sum of polygon $JHDE$)

$a° + b° + g° + \angle BHG + c° + f° + e° + d° + \angle JHD = 720°$

$a° + b° + c° + d° + e° + f° + g° + 180° = 720°$

$z = 540$

G2.2 If R is the remainder of $1^6 + 2^6 + 3^6 + 4^6 + 5^6 + 6^6$ divided by 7, find the value of R.

$x^6 + y^6 = (x + y)(x^5 - x^4y + x^3y^2 - x^2y^3 + xy^4 - y^5) + 2y^6$

$6^6 + 1^6 = 7Q_1 + 2$; $5^6 + 2^6 = 7Q_2 + 2\times 2^6$; $4^6 + 3^6 = 7Q_3 + 2\times 3^6$

$2 + 2\times 2^6 + 2\times 3^6 = 2(1 + 64 + 729) = 1588 = 7\times 226 + 6$; $R = 6$

Method 2 $1^6 + 2^6 + 3^6 + 4^6 + 5^6 + 6^6 \equiv 1^6 + 2^6 + 3^6 + (-3)^6 + (-2)^6 + (-1)^6 \bmod 7$

$\equiv 2(1^6 + 2^6 + 3^6) \equiv 2(1 + 64 + 729) \bmod 7$

$\equiv 2(1 + 1 + 1) \bmod 7 \equiv 6 \bmod 7$

G2.3 If 14! is divisible by 6^k, where k is an integer, find the largest possible value of k.

We count the number of factors of 3 in 14!. They are 3, 6, 9, 12. So there are 5 factors of 3.

$k = 5$

G2.4 Let x, y and z be real numbers that satisfy $x + \frac{1}{y} = 4$, $y + \frac{1}{z} = 1$ and $z + \frac{1}{x} = \frac{7}{3}$.

Find the value of xyz. **Reference 2010 FG2.2, 2017 FG1.1**

Method 1	**Method 2**
From (1), $x = 4 - \frac{1}{y} = \frac{4y-1}{y}$ $\Rightarrow \frac{1}{x} = \frac{y}{4y-1}$ …… (4) Sub. (4) into (3): $z + \frac{y}{4y-1} = \frac{7}{3}$ $z = \frac{7}{3} - \frac{y}{4y-1}$ …… (5) From (2): $\frac{1}{z} = 1 - y$ $z = \frac{1}{1-y}$ …… (6) (5) = (6): $\frac{1}{1-y} = \frac{7}{3} - \frac{y}{4y-1}$ $\frac{1}{1-y} = \frac{28y-7-3y}{3(4y-1)}$ $3(4y - 1) = (1 - y)(25y - 7)$ $12y - 3 = -25y^2 - 7 + 32y$ $25y^2 - 20y + 4 = 0$ $(5y - 2)^2 = 0 \Rightarrow y = \frac{2}{5}$ Sub. $y = \frac{2}{5}$ into (6): $z = \frac{1}{1-\frac{2}{5}} = \frac{5}{3}$ Sub. $y = \frac{2}{5}$ into (1): $x + \frac{5}{2} = 4 \Rightarrow x = \frac{3}{2}$ $xyz = \frac{2}{5}\times\frac{5}{3}\times\frac{3}{2} = 1$	$\begin{cases} x + \frac{1}{y} = 4 \cdots\cdots (1) \\ y + \frac{1}{z} = 1 \cdots\cdots (2) \\ z + \frac{1}{x} = \frac{7}{3} \cdots\cdots (3) \end{cases}$ (1)×(2): $xy + 1 + \frac{x}{z} + \frac{1}{yz} = 4$ $x\left(y + \frac{1}{z}\right) + \frac{1}{yz} = 3$ Sub. (2) into the eqt.: $x + \frac{x}{xyz} = 3$ Let $a = xyz$, then $x + \frac{x}{a} = 3 \cdots\cdots (4)$ (2)×(3): $y\left(\frac{7}{3}\right) + \frac{y}{a} = \frac{4}{3} \Rightarrow y\left(\frac{7}{3} + \frac{1}{a}\right) = \frac{4}{3} \cdots\cdots (5)$ (1)×(3): $z(4) + \frac{z}{a} = \frac{25}{3} \Rightarrow z\left(4 + \frac{1}{a}\right) = \frac{25}{3} \cdots\cdots (6)$ (4)×(5)×(6): $a\left(1 + \frac{1}{a}\right)\left(\frac{7}{3} + \frac{1}{a}\right)\left(4 + \frac{1}{a}\right) = \frac{100}{3}$ $\frac{(a+1)(7a+3)(4a+1)}{3a^2} = \frac{100}{3}$ which reduces to $28a^3 - 53a^2 + 22a + 3 = 0$ $\Rightarrow (a - 1)^2(28a + 3) = 0$ $\therefore a = 1$

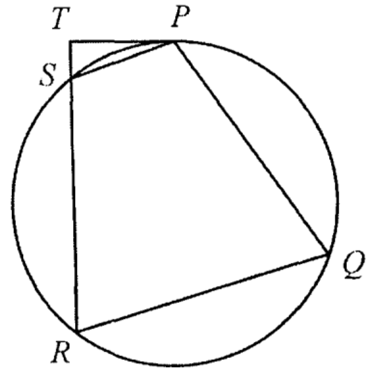
Method 3 (1)×(2)×(3) – (1) – (2) – (3):

$$xyz + x + y + z + \frac{1}{x} + \frac{1}{y} + \frac{1}{z} + \frac{1}{xyz} - \left(x + y + z + \frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right) = \frac{28}{3} - \frac{22}{3} \Rightarrow xyz + \frac{1}{xyz} = 2$$

$xyz = 1$

Group Event 3

G3.1 In Figure 1, $PQRS$ is a cyclic quadrilateral, where S is on the straight line RT and TP is tangent to the circle. If $RS = 8$ cm, $RT = 11$ cm and $TP = k$ cm, find the value of k.



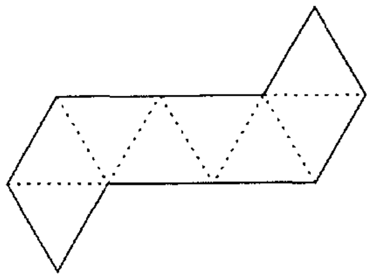
Join PR. $\angle SPT = \angle PRS$ ($\angle$ in alt. seg.)

$\angle STP = \angle PTR$ (common $\angle$)

$\Delta STP \sim \Delta PTR$ (equiangular)

$\frac{TP}{TR} = \frac{TS}{TP}$ (ratio of sides, ~ Δ) $\frac{k}{11} = \frac{11-8}{k}$; $k = \sqrt{33}$

G3.2 The layout in Figure 2 can be used to fold a polyhedron. If this polyhedron has v vertices, find the value of v.



There are 8 faces. $f = 8$.

There are 8 equilateral Δs, no of sides = $8\times3 = 24$

Each side is shared by 2 faces. Number of edge $e = 12$

By Euler formula, $v - e + f = 2$

$v - 12 + 8 = 2 \Rightarrow v = 6$

G3.3 For arbitrary real number x, define $[x]$ to be the largest integer less than or equal to x. For instance, $[2] = 2$ and $[3.4] = 3$. Find the value of $[1.008^8\times100]$.

$1.008^8\times100 = (1 + 0.008)^8\times100 = 100(1 + 8\times0.008 + 28\times0.008^2 + \ldots) \approx 106.4$

The integral value = 106

G3.4 When choosing, without replacement, 4 out of 30 labelled balls that are marked from 1 to 30, there are r combinations. Find the value of r.

$r = C_4^{30} = \frac{30\times29\times28\times27}{1\times2\times3\times4} = 27405$

Remark: If the question is changed to: Choose 4 out of 30 labelled balls that are marked from 1 to 30 with repetition is allowed, there are r combinations.

Find the value of r.

We shall divide into 5 different cases:

Case 1: All 4 balls are the same number, 30 combinations

Case 2: XXXY, where X, Y are different numbers, $2C_2^{30}$ = 870 combinations

Case 3 XXYY, where X, Y are different numbers, C_2^{30} = 435 combinations

Case 4 XXYZ, where X,Y,Z are different numbers, $30C_2^{29} = 30\times29\times28\div2 = 12180$ combinations

Case 5 XYZW, where X, Y, Z, W are different numbers, $C_4^{30} = 27405$ combinations

Total number of combinations = 30 + 870 + 435 + 12180 + 27405 = 40920

Group Event 4

G4.1 Regular tessellation is formed by identical regular m-polygons for some fixed m. Find the sum of all possible values of m .

Each interior angle $= \dfrac{180^\circ(m-2)}{m}$ (∠s sum of polygon)

Suppose n m-polygons tessellate the space.

$\dfrac{180^\circ(m-2)}{m}\cdot n = 360^\circ$ (∠s at a point)

$n(m-2) = 2m$

$n(m-2) - 2m + 4 = 4$

$(n-2)(m-2) = 4$

$m - 2 = 1, 2$ or 4

$m = 3, 4$ or 6

Sum of all possible $m = 3 + 4 + 6 = 13$

G4.2 Amongst the seven numbers 3624, 36024, 360924, 3609924, 36099924, 360999924 and 3609999924, there are n of them that are divisible by 38. Find the value of n.

$38 = 2\times19$, we need to investigate which number is divisible by 19.

$19^2 = 361$

$3624 = 3610 + 13$

$36024 = 36100 - 76 = 100(19^2) - 19\times4$

$360924 = 361000 - 76$

$3609924 = 3610000 - 76$

$36099924 = 36100000 - 76$

$360999924 = 361000000 - 76$

$3609999924 = 3610000000 - 76$

$n = 6$

G4.3 If $208208 = 8^5a + 8^4b + 8^3c + 8^2d + 8e + f$, where a, b, c, d, e, and f are integers and $0 \le a, b, c, d, e, f \le 7$, find the value of $a\times b\times c + d\times e\times f$.

Reference: 2011 FI1.2

$$\begin{array}{r|l}
8 & 208208 \\ \hline
8 & 26026 \;\ldots\ldots 0 \\ \hline
8 & 3253 \;\ldots\ldots 2 \\ \hline
8 & 406 \;\ldots\ldots 5 \\ \hline
8 & 50 \;\ldots\ldots 6 \\ \hline
 & 6 \;\ldots\ldots 2
\end{array}$$

$a = 6, b = 2, c = 6, d = 5, e = 2, f = 0$

$a\times b\times c + d\times e\times f = 72$

G4.4 In the coordinate plane, rotate point $A(6, 8)$ about the origin $O(0, 0)$ counter-clockwise for 20070° to point $B(p, q)$. Find the value of $p + q$.

$20070^\circ = 360^\circ\times55 + 270^\circ$

$\therefore B(8, -6)$

$p + q = 2$

Group Spare

GS.1 Calculate the value of $\left(\sqrt{2008}+\sqrt{2007}\right)^{2007}\times\left(\sqrt{2007}-\sqrt{2008}\right)^{2007}$.

$$\left(\sqrt{2008}+\sqrt{2007}\right)^{2007}\times\left(\sqrt{2007}-\sqrt{2008}\right)^{2007}=(2007-2008)^{2007}=(-1)^{2007}=-1$$

GS.2 If $x-\frac{1}{x}=\sqrt{2007}$, find the value of $x^4+\frac{1}{x^4}$.

$$\left(x-\frac{1}{x}\right)^2=2007$$

$$x^2-2+\frac{1}{x^2}=2007$$

$$x^2+\frac{1}{x^2}=2009$$

$$\left(x^2+\frac{1}{x^2}\right)^2=2009^2=4036081$$

$$x^4+2+\frac{1}{x^4}=4036081$$

$$x^4+\frac{1}{x^4}=4036079$$

GS.3 Given that $\cos\alpha=-\frac{99}{101}$ and $180° < \alpha < 270°$. Find the value of $\cot\alpha$.

$$\sec\alpha=-\frac{101}{99}$$

$$\tan^2\alpha=\sec^2\alpha-1$$

$$=\left(-\frac{101}{99}\right)^2-1$$

$$=\frac{101^2-99^2}{99^2}$$

$$=\frac{(101-99)\cdot(101+99)}{99^2}$$

$$=\frac{400}{99^2}$$

$$\tan\alpha=\frac{20}{99}$$

$$\cot\alpha=\frac{99}{20}\quad(=4.95)$$

GS.4 Calculate the value of $\frac{2008^3+4015^3}{2007^3+4015^3}$.

Let $x = 2007.5$, then $2x = 4015$

$$\frac{2008^3+4015^3}{2007^3+4015^3}=\frac{(x+0.5)^3+(2x)^3}{(x-0.5)^3+(2x)^3}=\frac{8\left(x+\frac{1}{2}\right)^3+8(2x)^3}{8\left(x-\frac{1}{2}\right)^3+8(2x)^3}=\frac{(2x+1)^3+(4x)^3}{(2x-1)+(4x)^3}$$

$$=\frac{(2x+1+4x)\left[(2x+1)^2-4x(2x+1)+(4x)^2\right]}{(2x-1+4x)\left[(2x-1)^2-4x(2x-1)+(4x)^2\right]}$$

$$=\frac{(6x+1)\left(4x^2+4x+1-8x^2-4x+16x^2\right)}{(6x-1)\left(4x^2-4x+1-8x^2+4x+16x^2\right)}$$

$$=\frac{(6x+1)\left(12x^2+1\right)}{(6x-1)\left(12x^2+1\right)}=\frac{6x+1}{6x-1}$$

$$=\frac{6023}{6022}$$