

Individual Events

| SI | <i>A</i> | 15 | I1 | <i>R</i> | 30 | I2 | <i>a</i> | 16 | I3 | <i>m</i> | 3 | I4 | <i>m</i> | 3 |
|----|----------|----|----|----------|-----|----|----------|---------------|----|----------|------|----|----------|---------------|
| | <i>B</i> | 3 | | <i>S</i> | 120 | | <i>b</i> | $\frac{3}{2}$ | | <i>n</i> | 9 | | <i>n</i> | $\frac{9}{4}$ |
| | <i>C</i> | 4 | | <i>T</i> | 11 | | <i>c</i> | 36 | | <i>p</i> | 2 | | <i>p</i> | 9 |
| | <i>D</i> | 8 | | <i>U</i> | 72 | | <i>d</i> | 42 | | <i>q</i> | 1141 | | <i>q</i> | 8 |

Group Events

| SG | <i>z</i> | 540 | G1 | <i>q</i> | 3 | G2 | <i>A</i> | $-\frac{17}{13}$ | G3 | <i>A</i> | 5 | G4 | <i>P</i> | $\frac{3}{8}$ |
|----|------------|-----|----|----------|---------------|----|----------|------------------|----|----------|-----------------------|----|----------|---------------|
| | <i>R</i> | 6 | | <i>k</i> | 1 | | <i>B</i> | 13 | | <i>R</i> | 4018 | | <i>R</i> | $\frac{1}{2}$ |
| | <i>k</i> | 5 | | <i>w</i> | 25 | | <i>C</i> | 46 | | <i>Q</i> | $\frac{4\sqrt{5}}{5}$ | | <i>S</i> | 320 |
| | <i>xyz</i> | 1 | | <i>p</i> | $\frac{3}{2}$ | | <i>D</i> | 30 | | <i>T</i> | $5-2\sqrt{3}$ | | <i>Q</i> | -1 |

Sample Individual Event (2008 Final Individual Event 1)

SI.1 Let $A = 15 \times \tan 44^\circ \times \tan 45^\circ \times \tan 46^\circ$, find the value of A .

Similar question 2012 G2.1

$$A = 15 \times \tan 44^\circ \times 1 \times \frac{1}{\tan 44^\circ} = 15$$

SI.2 Let n be a positive integer and $\overbrace{20082008 \cdots 2008}^{n \text{ 2008's}}15$ is divisible by A . If the least possible value of n is B , find the value of B .

The given number is divisible by 15. Therefore it is divisible by 3 and 5.

The last 2 digits of the given number is 15, which is divisible by 15.

The necessary condition is: $\overbrace{20082008 \cdots 2008}^{n \text{ 2008's}}$ must be divisible by 3.

$2 + 0 + 0 + 8 = 10$ which is not divisible by 3.

The least possible n is 3: $2+0+0+8+2+0+0+8+2+0+0+8 = 30$ which is divisible by 3.

SI.3 Given that there are C integers that satisfy the equation $|x-2| + |x+1| = B$, find the value of C
Reference: 1994 HG1, 2001 HG9, 2004 FG4.2, 2008 HI8, 2008 FI1.3, 2010 HG6, 2012 FG2.3

$$|x-2| + |x+1| = 3$$

If $x < -1$, $2-x-x-1 = 3 \Rightarrow x = -1$ (rejected)

If $-1 \leq x \leq 2$, $2-x+x+1 = 3 \Rightarrow 3 = 3$, always true $-1 \leq x \leq 2$

If $2 < x$, $x-2+x+1 = 3 \Rightarrow x = 2$ (reject)

$-1 \leq x \leq 2$ only

$\therefore x$ is an integer, $x = -1, 0, 1, 2$; $C = 4$

SI.4 In the coordinate plane, the distance from the point $(-C, 0)$ to the straight line $y = x$ is \sqrt{D} , find the value of D .

The distance from $P(x_0, y_0)$ to the straight line $Ax + By + C = 0$ is $\left| \frac{Ax_0 + By_0 + C}{\sqrt{A^2 + B^2}} \right|$.

The distance from $(-4, 0)$ to $x - y = 0$ is $\sqrt{D} = \left| \frac{-4-0+0}{\sqrt{1^2 + (-1)^2}} \right| = \frac{4}{\sqrt{2}} = 2\sqrt{2} = \sqrt{8}$; $D = 8$

Individual Event 1

II.1 Let a, b, c and d be the distinct roots of the equation $x^4 - 15x^2 + 56 = 0$.

If $R = a^2 + b^2 + c^2 + d^2$, find the value of R .

$$x^4 - 15x^2 + 56 = 0 \Rightarrow (x^2 - 7)(x^2 - 8) = 0$$

$$a = \sqrt{7}, b = -\sqrt{7}, c = \sqrt{8}, d = -\sqrt{8}$$

$$R = a^2 + b^2 + c^2 + d^2 = 7 + 7 + 8 + 8 = 30$$

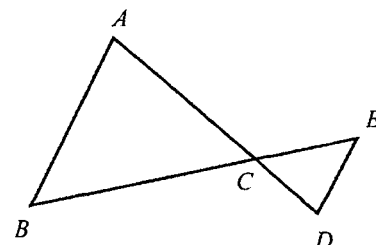
II.2 In Figure 1, AD and BE are straight lines with $AB = AC$ and $AB \parallel ED$. If $\angle ABC = R^\circ$ and $\angle ADE = S^\circ$, find the value of S .

$$\angle ABC = 30^\circ = \angle ACB \quad (\text{base } \angle \text{ isos. } \Delta)$$

$$\angle BAC = 120^\circ \quad (\angle \text{s sum of } \Delta)$$

$$\angle ADE = 120^\circ \quad (\text{alt. } \angle \text{s } AB \parallel ED)$$

$$S = 120$$



II.3 Let $F = 1 + 2 + 2^2 + 2^3 + \dots + 2^S$ and $T = \sqrt{\frac{\log(1+F)}{\log 2}}$, find the value of T .

Reference: 2015 FI1.4, 2017 FI3.4

$$F = 1 + 2 + 2^2 + 2^3 + \dots + 2^{120}$$

$$= \frac{2^{121} - 1}{2 - 1} = 2^{121} - 1$$

$$T = \sqrt{\frac{\log(1+F)}{\log 2}}$$

$$= \sqrt{\frac{\log 2^{121}}{\log 2}} = 11$$

II.4 Let $f(x)$ be a function such that $f(n) = (n-1)f(n-1)$ and $f(n) \neq 0$ hold for all integers $n \geq 6$.

If $U = \frac{f(T)}{(T-1)f(T-3)}$, find the value of U .

Reference: 1999 FI5.4

$$f(n) = (n-1)f(n-1) = (n-1)(n-2)f(n-2) = \dots$$

$$U = \frac{f(11)}{(11-1)f(11-3)}$$

$$= \frac{10 \times 9 \times 8 \times f(8)}{10 \times f(8)}$$

$$= 8 \times 9 = 72$$

Individual Event 2

12.1 Let $[x]$ be the largest integer not greater than x . If $a = \left[(\sqrt{3} - \sqrt{2})^{2009} \right] + 16$, find the value of a .

$$0 < \sqrt{3} - \sqrt{2} = \frac{1}{\sqrt{3} + \sqrt{2}} < 1$$

$$0 < (\sqrt{3} - \sqrt{2})^{2009} < 1$$

$$\begin{aligned} \Rightarrow a &= \left[(\sqrt{3} - \sqrt{2})^{2009} \right] + 16 \\ &= 0 + 16 = 16 \end{aligned}$$

12.2 In the coordinate plane, if the area of the triangle formed by the x -axis, y -axis and the line $3x + ay = 12$ is b square units, find the value of b .

$$3x + 16y = 12$$

$$x\text{-intercept} = 4$$

$$y\text{-intercept} = \frac{3}{4}$$

$$\text{Area} = b = \frac{1}{2} \cdot 4 \cdot \frac{3}{4} = \frac{3}{2}$$

12.3 Given that $x - \frac{1}{x} = 2b$ and $x^3 - \frac{1}{x^3} = c$, find the value of c .

Reference: 1990 FI2.2

$$x - \frac{1}{x} = 3$$

$$\Rightarrow x^2 - 2 + \frac{1}{x^2} = 9$$

$$\Rightarrow x^2 + \frac{1}{x^2} = 11$$

$$c = x^3 - \frac{1}{x^3}$$

$$= \left(x - \frac{1}{x} \right) \left(x^2 + 1 + \frac{1}{x^2} \right)$$

$$= 3 \times (11 + 1) = 36$$

12.4 In Figure 1, $\alpha = c$, $\beta = 43$, $\gamma = 59$ and $\omega = d$, find the value of d .

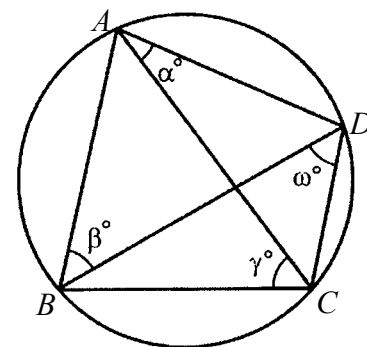
$\angle BAC = \omega^\circ$ (\angle s in the same seg.)

$\angle ACD = \beta^\circ$ (\angle s in the same seg.)

$\angle BAD + \angle BCD = 180^\circ$ (opp. \angle s cyclic quad.)

$$c + d + 43 + 59 = 180$$

$$d = 180 - 43 - 59 - 36 = 42 \quad (\because c = 36)$$



Individual Event 3

- 13.1** Given that $\frac{4}{\sqrt{6}+\sqrt{2}} - \frac{1}{\sqrt{3}+\sqrt{2}} = \sqrt{a}-\sqrt{b}$. If $m = a - b$, find the value of m .

$$\begin{aligned}\frac{4}{\sqrt{6}+\sqrt{2}} - \frac{1}{\sqrt{3}+\sqrt{2}} &= \frac{4}{\sqrt{6}+\sqrt{2}} \cdot \frac{\sqrt{6}-\sqrt{2}}{\sqrt{6}-\sqrt{2}} - \frac{1}{\sqrt{3}+\sqrt{2}} \cdot \frac{\sqrt{3}-\sqrt{2}}{\sqrt{3}-\sqrt{2}} \\ &= \frac{4(\sqrt{6}-\sqrt{2})}{6-2} - \frac{\sqrt{3}-\sqrt{2}}{3-2} \\ &= \sqrt{6}-\sqrt{2} - (\sqrt{3}-\sqrt{2}) \\ &= \sqrt{6}-\sqrt{3}\end{aligned}$$

$$a = 6, b = 3; m = 6 - 3 = 3$$

- 13.2** In figure 1, PQR is a right-angled triangle and $RSTU$ is a rectangle. Let A , B and C be the areas of the corresponding regions. If $A : B = m : 2$ and $A : C = n : 1$, find the value of n .

$$A : B = 3 : 2, A : C = n : 1 \Rightarrow A : B : C = 3n : 2n : 3$$

$$\text{Let } TS = UR = x, QU = y$$

$$\Delta PTS \sim \Delta TQU \sim \Delta PQR \text{ (equiangular)}$$

$$S_{\Delta PTS} : S_{\Delta TQU} : S_{\Delta PQR} = A : C : (A + B + C) = 3n : 3 : (5n + 3)$$

$$x^2 : y^2 : (x + y)^2 = 3n : 3 : (5n + 3)$$

$$\frac{y}{x} = \frac{1}{\sqrt{n}} \quad \dots (1), \quad \frac{x+y}{y} = \frac{\sqrt{5n+3}}{\sqrt{3}} \quad \dots (2)$$

$$\text{From (2): } \frac{x}{y} + 1 = \frac{\sqrt{5n+3}}{\sqrt{3}} \quad \dots (3)$$

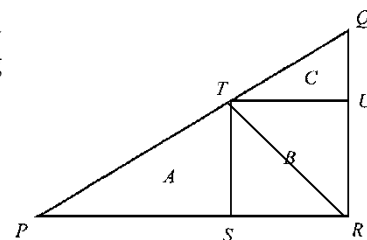
$$\text{Sub. (1) into (3): } \sqrt{n} + 1 = \frac{\sqrt{5n+3}}{\sqrt{3}}$$

$$\sqrt{3n} + \sqrt{3} = \sqrt{5n+3}$$

$$(\sqrt{3n} + \sqrt{3})^2 = 5n + 3$$

$$3n + 6\sqrt{n} + 3 = 5n + 3$$

$$6\sqrt{n} = 2n \Rightarrow n = 9$$



Method 2

$$\text{Let } SR = x, PS = z$$

Join TR which bisects the area of the rectangle.

$$\frac{A}{B} = \frac{3}{2} \Rightarrow \frac{A}{B} = \frac{3}{2}$$

$$\frac{S_{\Delta TPS}}{S_{\Delta TSR}} = \frac{3}{1} \Rightarrow \frac{z}{x} = \frac{3}{1} \quad \dots (4)$$

$$\therefore \Delta PTS \sim \Delta TQU \text{ (equiangular)}$$

$$\therefore \frac{A}{C} = \frac{n}{1} \Rightarrow \left(\frac{z}{x}\right)^2 = \frac{n}{1}$$

$$n = 3^2 = 9 \text{ (by (4))}$$

- 13.3** Let x_1, x_2, x_3, x_4 be real numbers and $x_1 \neq x_2$. If $(x_1 + x_3)(x_1 + x_4) = (x_2 + x_3)(x_2 + x_4) = n - 10$ and $p = (x_1 + x_3)(x_2 + x_3) + (x_1 + x_4)(x_2 + x_4)$, find the value of p .

Reference: 2002 HI7, 2006HG6, 2014 HG7

$$x_1^2 + x_1x_3 + x_1x_4 + x_3x_4 = x_2^2 + x_2x_3 + x_2x_4 + x_3x_4 = -1$$

$$x_1^2 - x_2^2 + x_1x_3 - x_2x_3 + x_1x_4 - x_2x_4 = 0$$

$$(x_1 - x_2)(x_1 + x_2 + x_3 + x_4) = 0 \Rightarrow x_1 + x_2 + x_3 + x_4 = 0 \quad \dots (1)$$

$$p = (x_1 + x_3)(x_2 + x_3) + (x_1 + x_4)(x_2 + x_4)$$

$$\begin{aligned}&= -(x_2 + x_3)(x_2 + x_4) - (x_1 + x_3)(x_1 + x_4) \text{ (by (1), } x_1 + x_3 = -(x_2 + x_4), x_2 + x_4 = -(x_1 + x_3)) \\ &= 1 + 1 = 2 \text{ (given } (x_1 + x_3)(x_1 + x_4) = (x_2 + x_3)(x_2 + x_4) = -1)\end{aligned}$$

- 13.4** The total number of students in a school is a multiple of 7 and not less than 1000.

Given that the same remainder 1 will be obtained when the number of students is divided by $p + 1$, $p + 2$ and $p + 3$. Let q be the least of the possible numbers of students in the school, find the value of q .

$$p + 1 = 3, p + 2 = 4, p + 3 = 5; \text{HCF} = 1, \text{LCM} = 60$$

$$q = 60m + 1, \text{ where } m \text{ is an integer.}$$

$$\therefore q \geq 1000 \text{ and } q = 7k, k \text{ is an integer.}$$

$$60m + 1 = 7k$$

$$7k - 60m = 1$$

$$k = 43, m = 5 \text{ satisfies the equation}$$

$$k = 43 + 60t; 7k \geq 1000 \Rightarrow 7(43 + 60t) \geq 1000 \Rightarrow t \geq 2 \Rightarrow \text{Least } q = 7 \times (43 + 60 \times 2) = 1141$$

Individual Event 4

I4.1 Given that $x_0^2 + x_0 - 1 = 0$. If $m = x_0^3 + 2x_0^2 + 2$, find the value of m .

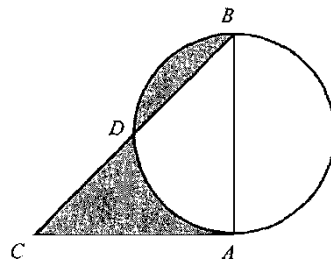
$$m = x_0^3 + 2x_0^2 + 2 = x_0^3 + x_0^2 - x_0 + x_0^2 + x_0 - 1 + 3 = 3$$

I4.2 In Figure 1, $\triangle BAC$ is a right-angled triangle, $AB = AC = m$ cm. Suppose that the circle with diameter AB intersects the line BC at D , and the total area of the shaded region is n cm². Find the value of n .

$$AB = AC = 3 \text{ cm}; \angle ADB = 90^\circ (\angle \text{ in semi-circle})$$

$$\text{Shaded area} = \text{area of } \triangle ACD = \frac{1}{2} \text{ area of } \triangle ABC = \frac{1}{2} \cdot \frac{1}{2} \cdot 3 \cdot 3 \text{ cm}^2$$

$$n = \frac{9}{4}$$



I4.3 Given that $p = 4n \left(\frac{1}{2^{2009}} \right)^{\log(1)}$, find the value of p .

$$p = 4n \left(\frac{1}{2^{2009}} \right)^0 = 4 \cdot \frac{9}{4} = 9$$

I4.4 Let x and y be real numbers satisfying the equation $(x - \sqrt{p})^2 + (y - \sqrt{p})^2 = 1$.

If $k = \frac{y}{x-3}$ and q is the least possible values of k^2 , find the value of q .

$$(x-3)^2 + (y-3)^2 = 1 \dots (1)$$

$$\text{Sub. } y = k(x-3) \text{ into (1): } (x-3)^2 + [k(x-3)-3]^2 = 1$$

$$(x-3)^2 + k^2(x-3)^2 - 6k(x-3) + 9 = 1$$

$$(1+k^2)(x-3)^2 - 6k(x-3) + 8 = 0 \Rightarrow (1+k^2)t^2 - 6kt + 8 = 0; \text{ where } t = x-3$$

$$\text{For real values of } t: \Delta = 4[3^2k^2 - 8(1+k^2)] \geq 0$$

$$k^2 \geq 8$$

The least possible value of $k^2 = q = 8$.

Method 2

The circle $(x-3)^2 + (y-3)^2 = 1$ intersects with the variable line $y = k(x-3)$ which passes through a fixed point $(3, 0)$, where k is the slope of the line.

From the graph, the extreme points when the variable line touches the circle at B and C .

$H(3, 3)$ is the centre of the circle.

In $\triangle ABH$, $\angle ABH = 90^\circ$ (\angle in semi-circle)

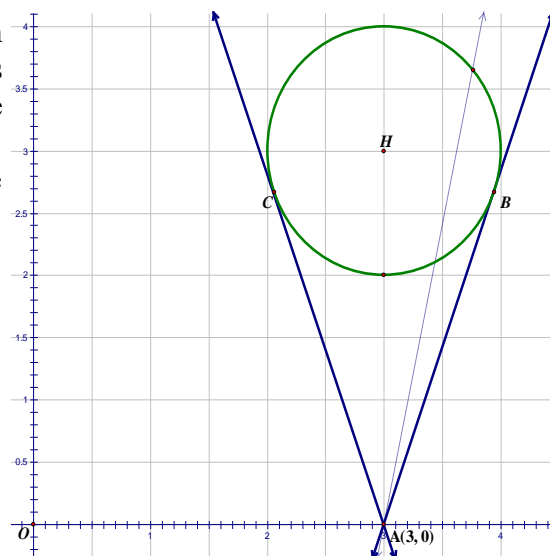
Let $\angle HAB = \theta$, $AH = 3$, $HB = 1$,

$AB = \sqrt{8} = 2\sqrt{2}$ (Pythagoras' theorem)

$$\tan \theta = \frac{1}{\sqrt{8}}$$

Slope of $AB = \tan(90^\circ - \theta) = \sqrt{8}$

Least possible $k^2 = q = 8$



Sample Group Event (2008 Final Group Event 2)

SG.1 In Figure 1, BD , FC , GC and FE are straight lines.

If $z = a + b + c + d + e + f + g$, find the value of z .

$$a^\circ + b^\circ + g^\circ + \angle BHG = 360^\circ \quad (\angle \text{sum of polygon } ABHG)$$

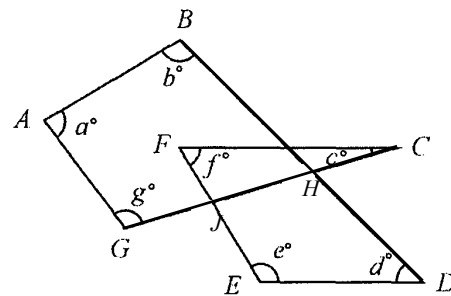
$$c^\circ + f^\circ = \angle CJE \quad (\text{ext. } \angle \text{ of } \triangle CFJ)$$

$$c^\circ + f^\circ + e^\circ + d^\circ + \angle JHD = 360^\circ \quad (\angle \text{sum of polygon } JHDE)$$

$$a^\circ + b^\circ + g^\circ + \angle BHG + c^\circ + f^\circ + e^\circ + d^\circ + \angle JHD = 720^\circ$$

$$a^\circ + b^\circ + c^\circ + d^\circ + e^\circ + f^\circ + g^\circ + 180^\circ = 720^\circ$$

$$z = 540$$



SG.2 If R is the remainder of $1^6 + 2^6 + 3^6 + 4^6 + 5^6 + 6^6$ divided by 7, find the value of R .

$$x^6 + y^6 = (x + y)(x^5 - x^4y + x^3y^2 - x^2y^3 + xy^4 - y^5) + 2y^6$$

$$6^6 + 1^6 = 7Q_1 + 2; 5^6 + 2^6 = 7Q_2 + 2 \times 2^6; 4^6 + 3^6 = 7Q_3 + 2 \times 3^6$$

$$2 + 2 \times 2^6 + 2 \times 3^6 = 2(1 + 64 + 729) = 1588 = 7 \times 226 + 6; R = 6$$

Method 2 $1^6 + 2^6 + 3^6 + 4^6 + 5^6 + 6^6 \equiv 1^6 + 2^6 + 3^6 + (-3)^6 + (-2)^6 + (-1)^6 \pmod{7}$

$$\equiv 2(1^6 + 2^6 + 3^6) \equiv 2(1 + 64 + 729) \pmod{7}$$

$$\equiv 2(1 + 1 + 1) \pmod{7} \equiv 6 \pmod{7}$$

SG.3 If $14!$ is divisible by 6^k , where k is an integer, find the largest possible value of k .

We count the number of factors of 3 in $14!$. They are 3, 6, 9, 12. So there are 5 factors of 3.

$$k = 5$$

SG.4 Let x , y and z be real numbers that satisfy $x + \frac{1}{y} = 4$, $y + \frac{1}{z} = 1$ and $z + \frac{1}{x} = \frac{7}{3}$.

Find the value of xyz . (Reference 2010 FG2.2)

Method 1

From (1), $x = 4 - \frac{1}{y} = \frac{4y-1}{y}$

$$\Rightarrow \frac{1}{x} = \frac{y}{4y-1} \quad \dots\dots (4)$$

Sub. (4) into (3): $z + \frac{y}{4y-1} = \frac{7}{3}$

$$z = \frac{7}{3} - \frac{y}{4y-1} \quad \dots\dots (5)$$

From (2): $\frac{1}{z} = 1 - y$

$$z = \frac{1}{1-y} \quad \dots\dots (6)$$

$$(5) = (6): \frac{1}{1-y} = \frac{7}{3} - \frac{y}{4y-1}$$

$$\frac{1}{1-y} = \frac{28y-7-3y}{3(4y-1)}$$

$$3(4y-1) = (1-y)(25y-7)$$

$$12y-3 = -25y^2-7+32y$$

$$25y^2-20y+4=0$$

$$(5y-2)^2=0$$

$$y = \frac{2}{5}$$

Sub. $y = \frac{2}{5}$ into (6): $z = \frac{1}{1-\frac{2}{5}} = \frac{5}{3}$

Sub. $y = \frac{2}{5}$ into (1): $x + \frac{5}{2} = 4 \Rightarrow x = \frac{3}{2}$

$$xyz = \frac{3}{2} \times \frac{5}{3} \times \frac{2}{5} = 1$$

Method 3 (1) \times (2) \times (3) $-$ (1) $-$ (2) $-$ (3):

$$xyz + x + y + z + \frac{1}{x} + \frac{1}{y} + \frac{1}{z} + \frac{1}{xyz} - \left(x + y + z + \frac{1}{x} + \frac{1}{y} + \frac{1}{z} \right) = \frac{28}{3} - \frac{22}{3} \Rightarrow xyz + \frac{1}{xyz} = 2$$

$$xyz = 1$$

Method 2

$$\begin{cases} x + \frac{1}{y} = 4 \dots\dots (1) \\ y + \frac{1}{z} = 1 \dots\dots (2) \\ z + \frac{1}{x} = \frac{7}{3} \dots\dots (3) \end{cases}$$

$$\begin{cases} y + \frac{1}{z} = 1 \dots\dots (2) \\ z + \frac{1}{x} = \frac{7}{3} \dots\dots (3) \end{cases}$$

$$\begin{cases} z + \frac{1}{x} = \frac{7}{3} \dots\dots (3) \end{cases}$$

$$(1) \times (2): xy + 1 + \frac{x}{z} + \frac{1}{yz} = 4$$

$$x \left(y + \frac{1}{z} \right) + \frac{1}{yz} = 3$$

Sub. (2) into the eqt.: $x + \frac{x}{xyz} = 3$

Let $a = xyz$, then $x + \frac{x}{a} = 3 \dots\dots (4)$

$$(2) \times (3): y \left(\frac{7}{3} \right) + \frac{y}{a} = \frac{4}{3} \Rightarrow y \left(\frac{7}{3} + \frac{1}{a} \right) = \frac{4}{3} \dots\dots (5)$$

$$(1) \times (3): z(4) + \frac{z}{a} = \frac{25}{3} \Rightarrow z \left(4 + \frac{1}{a} \right) = \frac{25}{3} \dots\dots (6)$$

$$(4) \times (5) \times (6): a \left(1 + \frac{1}{a} \right) \left(\frac{7}{3} + \frac{1}{a} \right) \left(4 + \frac{1}{a} \right) = \frac{100}{3}$$

$$\frac{(a+1)(7a+3)(4a+1)}{3a^2} = \frac{100}{3}$$

which reduces to $28a^3 - 53a^2 + 22a + 3 = 0$

$$\Rightarrow (a-1)^2(28a+3) = 0$$

$$\therefore a = 1$$

Group Event 1

G1.1 Given some triangles with side lengths a cm, 2 cm and b cm, where a and b are integers and $a \leq 2 \leq b$. If there are q non-congruent classes of triangles satisfying the above conditions, find the value of q .

When $a = 1$, possible $b = 2$

When $a = 2$, possible $b = 2$ or 3

$\therefore q = 3$

G1.2 Given that the equation $|x| - \frac{4}{x} = \frac{3|x|}{x}$ has k distinct real root(s), find the value of k .

When $x > 0$: $x^2 - 4 = 3x \Rightarrow x^2 - 3x - 4 = 0 \Rightarrow (x + 1)(x - 4) = 0 \Rightarrow x = 4$

When $x < 0$: $-x^2 - 4 = -3x \Rightarrow x^2 - 3x + 4 = 0$; $D = 9 - 16 < 0 \Rightarrow$ no real roots.

$k = 1$ (There is only one real root.)

G1.3 Given that x and y are non-zero real numbers satisfying the equations $\frac{\sqrt{x}}{\sqrt{y}} - \frac{\sqrt{y}}{\sqrt{x}} = \frac{7}{12}$ and $x -$

$y = 7$. If $w = x + y$, find the value of w .

The first equation is equivalent to $\frac{x-y}{\sqrt{xy}} = \frac{7}{12} \Rightarrow \sqrt{xy} = 12 \Rightarrow xy = 144$

Sub. $y = \frac{144}{x}$ into $x - y = 7$: $x - \frac{144}{x} = 7 \Rightarrow x^2 - 7x - 144 = 0 \Rightarrow (x + 9)(x - 16) = 0$

$x = -9$ or 16; when $x = -9$, $y = -16$ (rejected $\because \sqrt{x}$ is undefined); when $x = 16$; $y = 9$

$w = 16 + 9 = 25$

Method 2 The first equation is equivalent to $\frac{x-y}{\sqrt{xy}} = \frac{7}{12} \Rightarrow \sqrt{xy} = 12 \Rightarrow xy = 144 \dots\dots (1)$

$\therefore x - y = 7$ and $x + y = w$

$\therefore x = \frac{w+7}{2}, y = \frac{w-7}{2}$

Sub. these equations into (1): $\left(\frac{w+7}{2}\right)\left(\frac{w-7}{2}\right) = 144$

$w^2 - 49 = 576 \Rightarrow w = \pm 25$

\therefore From the given equation $\frac{\sqrt{x}}{\sqrt{y}} - \frac{\sqrt{y}}{\sqrt{x}} = \frac{7}{12}$, we know that both $x > 0$ and $y > 0$

$\therefore w = x + y = 25$ only

G1.4 Given that x and y are real numbers and $\left|x - \frac{1}{2}\right| + \sqrt{y^2 - 1} = 0$.

Let $p = |x| + |y|$, find the value of p .

Reference: 2005 FI4.1, 2006 FI4.2, 2011 FI4.3, 2013 FI1.4, 2015 HG4, 2015 FI1.1

Both $\left|x - \frac{1}{2}\right|$ and $\sqrt{y^2 - 1}$ are non-negative numbers.

The sum of two non-negative numbers = 0 means each of them is zero

$x = \frac{1}{2}, y = \pm 1; p = \frac{1}{2} + 1 = \frac{3}{2}$

Group Event 2

G2.1 Given $\tan \theta = \frac{5}{12}$, where $180^\circ \leq \theta \leq 270^\circ$. If $A = \cos \theta + \sin \theta$, find the value of A .

$$\cos \theta = -\frac{12}{13}, \sin \theta = -\frac{5}{13}$$

$$A = -\frac{12}{13} - \frac{5}{13} = -\frac{17}{13}$$

G2.2 Let $[x]$ be the largest integer not greater than x .

If $B = \left[10 + \sqrt{10 + \sqrt{10 + \sqrt{10 + \dots}}} \right]$, find the value of B .

Reference: 2007 FG2.2 ... $x \geq 3 + \sqrt{3 + \sqrt{3 + \sqrt{3 + \sqrt{3 + \dots}}}}$...

$$\text{Let } y = \sqrt{10 + \sqrt{10 + \sqrt{10 + \dots}}}$$

$$y^2 = 10 + \sqrt{10 + \sqrt{10 + \sqrt{10 + \dots}}} = 10 + y$$

$$y^2 - y - 10 = 0$$

$$y = \frac{1 + \sqrt{41}}{2} \quad \text{or} \quad \frac{1 - \sqrt{41}}{2} \quad (\text{rejected})$$

$$6 < \sqrt{41} < 7 \Rightarrow \frac{7}{2} < \frac{1 + \sqrt{41}}{2} < 4$$

$$13.5 < 10 + \sqrt{10 + \sqrt{10 + \sqrt{10 + \dots}}} < 14; B = 13$$

G2.3 Let $a \oplus b = ab + 10$. If $C = (1 \oplus 2) \oplus 3$, find the value of C .

$$1 \oplus 2 = 2 + 10 = 12; C = 12 \oplus 3 = 36 + 10 = 46$$

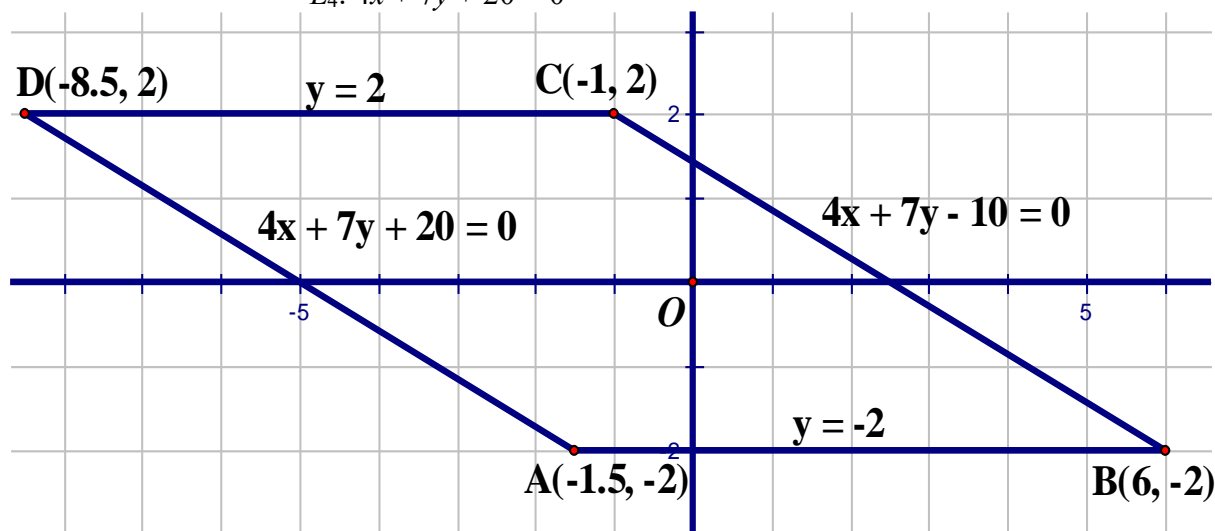
G2.4 In the coordinate plane, the area of the region bounded by the following lines is D square units, find the value of D .

$$L_1: y - 2 = 0$$

$$L_2: y + 2 = 0$$

$$L_3: 4x + 7y - 10 = 0$$

$$L_4: 4x + 7y + 20 = 0$$



It is easy to show that the bounded region is a parallelogram $ABCD$ with vertices $A(-1.5, -2)$, $B(6, -2)$, $C(-1, 2)$, $D(-8.5, 2)$.

$$\text{The area } D = |6 - (-1.5)| \times |2 - (-2)| = 7.5 \times 4 = 30$$

Group Event 3

G3.1 Let $[x]$ be the largest integer not greater than x .

If $A = \left\lfloor \frac{2008 \times 80 + 2009 \times 130 + 2010 \times 180}{2008 \times 15 + 2009 \times 25 + 2010 \times 35} \right\rfloor$, find the value of A .

Reference: 2008 FGS.4 Calculate the value of $\frac{2008^3 + 4015^3}{2007^3 + 4015^3}$.

Let $a = 2009$, $b = 130$, $c = 25$

$$\begin{aligned} \frac{2008 \times 80 + 2009 \times 130 + 2010 \times 180}{2008 \times 15 + 2009 \times 25 + 2010 \times 35} &= \frac{(a-1)(b-50) + ab + (a+1)(b+50)}{(a-1)(c-10) + ac + (a+1)(c+10)} \\ &= \frac{ab - b - 50a + 50 + ab + ab + b + 50a + 50}{ac - c - 10a + 10 + ac + ac + c + 10a + 10} \\ &= \frac{3ab + 100}{3ac + 20} = \frac{3 \cdot 2009 \cdot 130 + 100}{3 \cdot 2009 \cdot 25 + 20} = \frac{783610}{150695} = 5 + d \end{aligned}$$

where $0 < d < 1$; $A = 5$

G3.2 There are R zeros at the end of $\underbrace{99\dots9}_{2009 \text{ of } 9\text{'s}} \times \underbrace{99\dots9}_{2009 \text{ of } 9\text{'s}} + 1 \underbrace{99\dots9}_{2009 \text{ of } 9\text{'s}}$, find the value of R .

$$\begin{aligned} \underbrace{99\dots9}_{2009 \text{ of } 9\text{'s}} \times \underbrace{99\dots9}_{2009 \text{ of } 9\text{'s}} + 1 \underbrace{99\dots9}_{2009 \text{ of } 9\text{'s}} &= \left(1 \underbrace{0\dots0}_{2009 \text{ of } 0\text{'s}} - 1 \right) \times \left(1 \underbrace{0\dots0}_{2009 \text{ of } 0\text{'s}} - 1 \right) + \left(2 \underbrace{0\dots0}_{2009 \text{ of } 0\text{'s}} - 1 \right) \\ &= (10^{2009} - 1)(10^{2009} - 1) + 2 \times 10^{2009} - 1 \\ &= 10^{4018} - 2 \times 10^{2009} + 1 + 2 \times 10^{2009} - 1 = 10^{4018} \end{aligned}$$

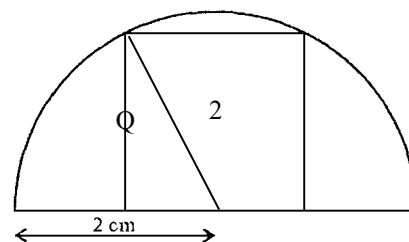
$R = 4018$

G3.3 In Figure 1, a square of side length Q cm is inscribed in a semi-circle of radius 2 cm. Find the value of Q .

$$Q^2 + \left(\frac{Q}{2} \right)^2 = 4 \quad (\text{Pythagoras' Theorem})$$

$$5Q^2 = 16$$

$$Q = \frac{4}{\sqrt{5}} = \frac{4\sqrt{5}}{5}$$



G3.4 In Figure 2, the sector OAB has radius 4 cm and $\angle AOB$ is a right angle. Let the semi-circle with diameter OB be centred at I with $IJ \parallel OA$, and IJ intersects the semi-circle at K . If the area of the shaded region is T cm², find the value of T . (Take $\pi = 3$)

$OI = 2$ cm, $OJ = 4$ cm

$$\cos \angle IOJ = \frac{OI}{OJ} = \frac{1}{2}$$

$$\angle IOJ = 60^\circ$$

$$S_{BIJ} = S_{\text{sector } OBJ} - S_{\triangle OIJ}$$

$$= \left(\frac{1}{2} \cdot 4^2 \cdot \frac{\pi}{3} - \frac{1}{2} \cdot 2 \cdot 4 \sin 60^\circ \right) \text{cm}^2$$

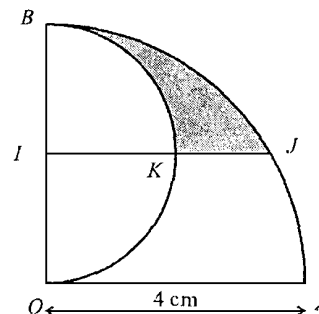
$$= \left(\frac{8\pi}{3} - 2\sqrt{3} \right) \text{cm}^2$$

$$\text{Shaded area} = S_{BIJ} - S_{BIK}$$

$$= \left(\frac{8\pi}{3} - 2\sqrt{3} - \frac{1}{4} \pi \cdot 2^2 \right) \text{cm}^2$$

$$= \left(\frac{5\pi}{3} - 2\sqrt{3} \right) \text{cm}^2$$

$$T = 5 - 2\sqrt{3}$$



Group Event 4

G4.1 Let P be a real number. If $\sqrt{3-2P} + \sqrt{1-2P} = 2$, find the value of P .

$$\begin{aligned}(\sqrt{3-2P})^2 &= (2 - \sqrt{1-2P})^2 \\ 3-2P &= 4 - 4\sqrt{1-2P} + 1-2P \\ 4\sqrt{1-2P} &= 2 \\ 4(1-2P) &= 1 \\ P &= \frac{3}{8}\end{aligned}$$

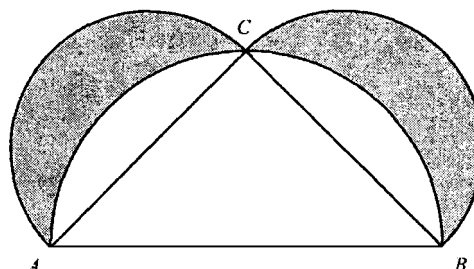
G4.2 In Figure 1, let AB , AC and BC be the diameters of the corresponding three semi-circles. If $AC = BC = 1$ cm and the area of the shaded region is R cm². Find the value of R .

Reference: 1994 HI9, 2019 FG4.1

$$AB = \sqrt{2}$$

$$\text{Shaded area} = R \text{ cm}^2 = S_{\text{circle with diameter } AC} - 2 S_{\text{segment } AC}$$

$$R = \pi \left(\frac{1}{2} \right)^2 - \left[\frac{1}{2} \pi \cdot \left(\frac{\sqrt{2}}{2} \right)^2 - \frac{1}{2} \cdot 1^2 \right] = \frac{1}{2}$$



G4.3 In Figure 2, AC , AD , BD , BE and CF are straight lines. If $\angle A + \angle B + \angle C + \angle D = 140^\circ$ and $a + b + c = S$, find the value of S .

$$\angle CFD = \angle A + \angle C \text{ (ext. } \angle \text{ of } \Delta)$$

$$\angle AEB = \angle B + \angle D \text{ (ext. } \angle \text{ of } \Delta)$$

$$a^\circ = \angle A + \angle AEB = \angle A + \angle B + \angle D \text{ (ext. } \angle \text{ of } \Delta)$$

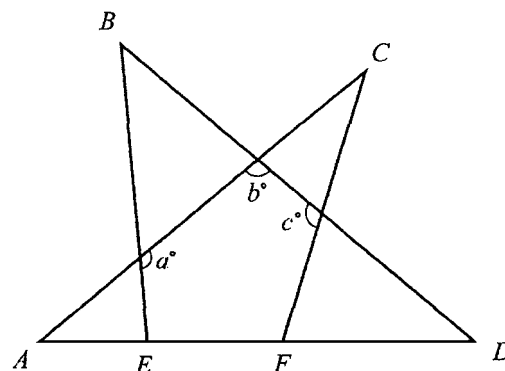
$$c^\circ = \angle D + \angle CFD = \angle D + \angle A + \angle C \text{ (ext. } \angle \text{ of } \Delta)$$

$$b^\circ + \angle A + \angle D = 180^\circ \text{ (}\angle\text{s sum of } \Delta)$$

$$\begin{aligned}a^\circ + b^\circ + c^\circ &= \angle A + \angle B + \angle D + 180^\circ - (\angle A + \angle D) \\ &\quad + \angle D + \angle A + \angle C\end{aligned}$$

$$= \angle A + \angle B + \angle C + \angle D + 180^\circ$$

$$S = a + b + c = 140 + 180 = 320$$



G4.4 Let $Q = \log_{2+\sqrt{2^2-1}}(2-\sqrt{2^2-1})$, find the value of Q .

$$\begin{aligned}Q &= \frac{\log(2-\sqrt{3})}{\log(2+\sqrt{3})} \\ &= \frac{\log(2-\sqrt{3})}{\log(2+\sqrt{3}) \cdot \frac{(2-\sqrt{3})}{(2-\sqrt{3})}} \\ &= \frac{\log(2-\sqrt{3})}{\log \frac{1}{2-\sqrt{3}}} \\ &= \frac{\log(2-\sqrt{3})}{-\log(2-\sqrt{3})} = -1\end{aligned}$$

$$\begin{aligned}\text{Method 2 } Q &= \log_{2+\sqrt{3}}(2-\sqrt{3}) \cdot \frac{(2+\sqrt{3})}{(2+\sqrt{3})} \\ &= \log_{2+\sqrt{3}} \frac{1}{(2+\sqrt{3})} \\ &= \log_{2+\sqrt{3}} (2+\sqrt{3})^{-1} = -1\end{aligned}$$