Individual Events

SI			I1			I2			I3			I4			IS		
	R	30		***P***	20		***P***	3		***P***	7		***a***	2		***P***	95
	S	120		***Q***	36		***Q***	5		***Q***	13		***b***	1		***Q***	329
	T	11		***R***	8		***R***	6		***R***	5		***c***	2		****R*** see the remark	6
	U	72		****S*** see the remark	5040		***S***	$\frac{-95+3\sqrt{1505}}{10}$		***S***	$\sqrt{5}$		****d*** see the remark	2		***S***	198

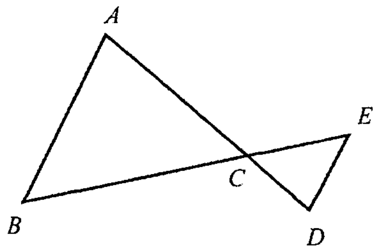
Group Events

SG			G1			G2			G3			G4			GS		
	q	3		***a***	2		***area***	40		***a***	1		***P***	20		****m*** see the remark	4
	k	1		***b***	3		****pairs*** see the remark	2550		***a+b+c***	1		$\frac{n}{m}$	$\frac{2}{3}$		***v***	6
	w	25		***c***	2		***x***	60		***y − x***	$\frac{1}{2}$		***r***	3		α	3
	p	$\frac{3}{2}$		***x***	3		***P***	−1		$\frac{P_1}{P_2}$	7		****BGHI*** see the remark	6		***F***	208

Sample Individual Event (2009 Final Individual Event 1)

S1.1 Let a, b, c and d be the distinct roots of the equation $x^4 - 15x^2 + 56 = 0$.
If $R = a^2 + b^2 + c^2 + d^2$, find the value of R.
$x^4 - 15x^2 + 56 = 0 \Rightarrow (x^2 - 7)(x^2 - 8) = 0$
$a = \sqrt{7}$, $b = -\sqrt{7}$, $c = \sqrt{8}$, $d = -\sqrt{8}$
$R = a^2 + b^2 + c^2 + d^2 = 7 + 7 + 8 + 8 = 30$

S1.2 In Figure 1, AD and BE are straight lines with $AB = AC$ and AB // ED.
If $\angle ABC = R°$ and $\angle ADE = S°$, find the value of S.
$\angle ABC = 30° = \angle ACB$ (base ∠ isos. Δ)
$\angle BAC = 120°$ (∠s sum of Δ)
$\angle ADE = 120°$ (alt. ∠s AB // ED)
$S = 120$



S1.3 Let $F = 1 + 2 + 2^2 + 2^3 + \ldots + 2^S$ and $T = \sqrt{\frac{\log(1+F)}{\log 2}}$, find the value of T.

$F = 1 + 2 + 2^2 + 2^3 + \ldots + 2^{120} = \frac{2^{121}-1}{2-1} = 2^{121} - 1$

$T = \sqrt{\frac{\log(1+F)}{\log 2}} = \sqrt{\frac{\log 2^{121}}{\log 2}} = 11$

S1.4 Let f(x) be a function such that $f(n) = (n - 1)f(n - 1)$ and $f(n) \neq 0$ hold for all integers $n \geq 6$.
If $U = \frac{f(T)}{(T-1)f(T-3)}$, find the value of U.

$f(n) = (n - 1)f(n - 1) = (n - 1)(n - 2)f(n - 2) = \cdots\cdots$

$U = \frac{f(11)}{(11-1)f(11-3)} = \frac{10\times 9\times 8\times f(8)}{10\times f(8)} = 8\times 9 = 72$

Individual Event 1

I1.1 If the average of a, b and c is 12, and the average of $2a + 1$, $2b + 2$, $2c + 3$ and 2 is P, find the value of P.

$a + b + c = 36$ …… (1)

$$P = \frac{2a+1+2b+2+2c+3+2}{4} = \frac{2(a+b+c)+8}{4} = \frac{2\times 36+8}{4} = 20$$

I1.2 Let $20112011 = aP^5 + bP^4 + cP^3 + dP^2 + eP + f$, where a, b, c, d, e and f are integers and $0 \le a, b, c, d, e, f < P$. If $Q = a + b + c + d + e + f$, find the value of Q.

Reference: 2008 FG4.3

$$\begin{array}{rl}
20 \,\big|\, & 2\;0\;1\;1\;2\;0\;1\;1 \\
20 \,\big|\, & 1\;0\;0\;5\;6\;0\;0 \quad \cdots\cdots\ 11 \\
20 \,\big|\, & 5\;0\;2\;8\;0 \quad \cdots\cdots\ 0 \\
20 \,\big|\, & 2\;5\;1\;4 \quad \cdots\cdots\ 0 \\
20 \,\big|\, & 1\;2\;5 \quad \cdots\cdots\ 14 \\
 & 6 \quad \cdots\cdots\ 5
\end{array}$$

$a = 6$, $b = 5$, $c = 14$, $d = 0$, $e = 0$, $f = 11$; $Q = 6 + 5 + 14 + 0 + 0 + 11 = 36$

I1.3 If R is the units digit of the value of $8^Q + 7^{10Q} + 6^{100Q} + 5^{1000Q}$, find the value of R.

$8^{36} \equiv 6 \pmod{10}$, $7^{360} \equiv 1 \pmod{10}$, $6^{360} \equiv 6 \pmod{10}$, $5^{36000} \equiv 5 \pmod{10}$

$8^{36} + 7^{360} + 6^{3600} + 5^{36000} \equiv 6 + 1 + 6 + 5 \equiv 8 \pmod{10}$

$R = 8$

I1.4 If S is the number of ways to arrange R persons in a circle, find the value of S.

Reference: 1998 FI5.3, 2000 FG4.4

First arrange the 8 persons in a row. Number of permutations $= P_8^8 = 8!$

Suppose the first and the last in the row are A and H respectively.

Now join the first and the last persons to form a ring.

A can be in any position of the ring. Each pattern is repeated 8 times.

$\therefore$ Number of permutations $= \dfrac{8!}{8} = 5040$

Remark: the original version was …"arrange R people" …

Note that the word "people" is an uncountable noun, whereas the word "persons" is a countable noun.

Individual Event 2

I2.1 If the solution of the system of equations $\begin{cases} x+y=P \\ 3x+5y=13 \end{cases}$ are positive integers, find the value of P.

5(1) – (2): $2x = 5P - 13 \Rightarrow x = \dfrac{5P-13}{2}$

(2) – 3(1): $2y = 13 - 3P \Rightarrow y = \dfrac{13-3P}{2}$

$\because$ x and y are positive integers $\therefore$ $\dfrac{5P-13}{2} > 0$ and $\dfrac{13-3P}{2} > 0$ and P is odd

$\dfrac{13}{5} < P < \dfrac{13}{3}$ and P is odd $\Rightarrow P = 3$

I2.2 If $x + y = P$, $x^2 + y^2 = Q$ and $x^3 + y^3 = P^2$, find the value of Q.

Reference: 2002 FG1.2

$x + y = 3$, $x^2 + y^2 = Q$ and $x^3 + y^3 = 9$

$(x + y)^2 = 3^2 \Rightarrow x^2 + y^2 + 2xy = 9 \Rightarrow Q + 2xy = 9$ …… (1)

$(x + y)(x^2 + y^2 - xy) = 9 \Rightarrow 3(Q - xy) = 9 \Rightarrow Q - xy = 3$ …… (2)

(1) + 2(2): $3Q = 15 \Rightarrow Q = 5$

I2.3 If a and b are distinct prime numbers and $a^2 - aQ + R = 0$ and $b^2 - bQ + R = 0$, find the value of R.

$a^2 - 5a + R = 0$ and $b^2 - 5b + R = 0$

a, b are the (prime numbers) roots of $x^2 - 5x + R = 0$

$a + b = 5$ …… (1), $ab = R$ …… (2)

$a = 2$, $b = 3 \Rightarrow R = 6$

I2.4 If $S > 0$ and $\dfrac{1}{S(S-1)} + \dfrac{1}{(S+1)S} + \cdots + \dfrac{1}{(S+20)(S+19)} = 1 - \dfrac{1}{R}$, find the value of S.

$$\left(\frac{1}{S-1} - \frac{1}{S}\right) + \left(\frac{1}{S} - \frac{1}{S+1}\right) + \cdots + \left(\frac{1}{S+19} - \frac{1}{S+20}\right) = \frac{5}{6}$$

$$\frac{1}{S-1} - \frac{1}{S+20} = \frac{5}{6}$$

$$\frac{21}{(S-1)(S+20)} = \frac{5}{6}$$

$5(S^2 + 19S - 20) = 126$

$5S^2 + 95S - 226 = 0$

$$S = \frac{-95 + \sqrt{13545}}{10}$$

$$= \frac{-95 + 3\sqrt{1505}}{10}$$

Individual Event 3

I3.1 If P is a prime number and the roots of the equation $x^2 + 2(P + 1)x + P^2 - P - 14 = 0$ are integers, find the least value of P.

Reference: 2000 FI5.2, 2001 FI2.1, 2010 FI2.2, 2013 HG1

$\Delta = 4(P + 1)^2 - 4(P^2 - P - 14) = m^2$

$\left(\frac{m}{2}\right)^2 = P^2 + 2P + 1 - P^2 + P + 14 = 3P + 15$

The possible square numbers are 16, 25, 36, $\cdots$

$3P + 15 = 16$ (no solution); $3P + 15 = 25$ (not an integer); $3P + 15 = 36 \Rightarrow P = 7$

The least possible $P = 7$

I3.2 Given that $x^2 + ax + b$ is a common factor of $2x^3 + 5x^2 + 24x + 11$ and $x^3 + Px - 22$.
If $Q = a + b$, find the value of Q.

Reference 1992 HI5, 1993 FI5.2, 2001 FI1.2

Let $f(x) = 2x^3 + 5x^2 + 24x + 11$; $g(x) = x^3 + 7x - 22$

$g(2) = 8 + 14 - 22 = 0 \Rightarrow x - 2$ is a factor

By division $g(x) = (x - 2)(x^2 + 2x + 11)$; $f(x) = (2x + 1)(x^2 + 2x + 11)$

$a = 2, b = 11; Q = a + b = 13$

Method 2

Let $f(x) = 2x^3 + 5x^2 + 24x + 11 = (x^2 + ax + b)(cx + d)$

$g(x) = x^3 + 7x - 22 = (x^2 + ax + b)(px + q)$

$f(x) - 2g(x) = 2x^3 + 5x^2 + 24x + 11 - 2(x^3 + 7x - 22) \equiv (x^2 + ax + b)[(c - 2p)x + d - 2q]$

$5x^2 + 10x + 55 \equiv (x^2 + ax + b)[(c - 2p)x + d - 2q]$

By comparing coefficients of x^3 and x^2 on both sides:

$c = 2p$ and $d - 2q = 5$

$5x^2 + 10x + 55 \equiv 5(x^2 + ax + b)$

$a = 2, b = 11$

$Q = a + b = 13$

I3.3 If R is a positive integer and $R^3 + 4R^2 + (Q - 93)R + 14Q + 10$ is a prime number, find the value of R.

(Reference: 2004 FI4.2)

Let $f(R) = R^3 + 4R^2 - 80R + 192$

$f(4) = 64 + 64 - 320 + 192 = 0 \Rightarrow x - 4$ is a factor

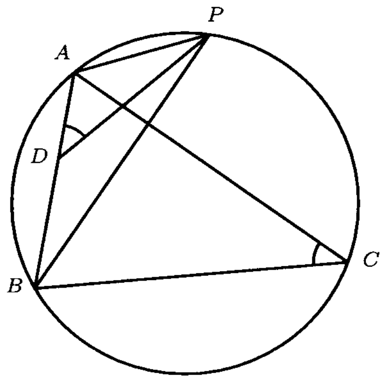
By division, $f(R) = (R - 4)(R^2 + 8R - 48) = (R - 4)^2(R + 12)$

$\because$ $f(R)$ is a prime number

$\therefore R - 4 = 1$

$\Rightarrow R = 5$ and $R + 12 = 17$, which is a prime.

I3.4 In Figure 1, AP, AB, PB, PD, AC and BC are line segments and D is a point on AB. If the length of AB is R times that of AD, $\angle ADP = \angle ACB$ and $S = \frac{PB}{PD}$, find the value of S.



Consider ΔADP and ΔABP.

$\angle ADP = \angle ACB = \angle APB$ (given, $\angle$s in the same segment AB)

$\angle DAP = \angle PAB$ (Common)

$\angle APD = \angle ABP$ ($\angle$s sum of Δ)

$\therefore \Delta ADP \sim \Delta APB$ (equiangular)

Let $AD = k$, $AB = 5k$, $AP = y$

$$\frac{PB}{PD} = \frac{AB}{AP} = \frac{AP}{AD} \quad \text{(Ratio of sides, } \sim\Delta\text{'s)}$$

$$\frac{PB}{PD} = \frac{5k}{y} = \frac{y}{k}$$

$$\therefore \left(\frac{y}{k}\right)^2 = 5 \Rightarrow \frac{y}{k} = \sqrt{5}$$

$$\frac{PB}{PD} = \sqrt{5}$$

Individual Event 4

I4.1 Consider the function $y = \sin x + \sqrt{3}\cos x$. Let a be the maximum value of y. Find the value of a.

$y = \sin x + \sqrt{3}\cos x$

$= 2\left(\sin x \cdot \frac{1}{2} + \cos x \cdot \frac{\sqrt{3}}{2}\right)$

$= 2(\sin x \cdot \cos 60^\circ + \cos x \cdot \sin 60^\circ)$

$= 2\sin(x + 60^\circ)$

a = maximum value of $y = 2$

I4.2 Find the value of b if b and y satisfy $|b - y| = b + y - a$ and $|b + y| = b + a$.

From the first equation :

$(b - y = b + y - 2$ or $y - b = b + y - 2)$ and $b + y - 2 \ge 0$

$(y = 1$ or $b = 1)$ and $b + y - 2 \ge 0$

When $y = 1 \Rightarrow b \ge 1$ ⋯⋯ (3)

When $b = 1 \Rightarrow y \ge 1$ ⋯⋯ (4)

From the second equation:

$(b + y = b + 2$ or $b + y = -b - 2)$ and $b + 2 \ge 0$

$(y = 2$ or $2b + y = -2)$ and $b \ge -2$

When $y = 2$ and $b \ge -2$ ⋯⋯ (5)

When $2b + y = -2$ and $b \ge -2 \Rightarrow (y \le 2$ and $b \ge -2$ and $2b + y = -2)$ ⋯⋯ (6)

(3) and (5): $y = 1$, $b \ge 1$ and $y = 2$ and $b \ge -2 \Rightarrow$ contradiction

(3) and (6): $y = 1$, $b \ge 1$ and $(y \le 2, b \ge -2, 2b + y = -2) \Rightarrow y = 1$ and $b = -1.5$ and $b \ge 1$!!!

(4) and (6): $(y \ge 1, b = 1)$ and $(y \le 2, b \ge -2, 2b + y = -2) \Rightarrow y = -4$, $b = 1$ and $y \ge 1$!!!

(4) and (5): $(b = 1, y \ge 1)$ and $(y = 2, b \ge -2) \Rightarrow b = 1$ and $y = 2$

$\therefore b = 1$

I4.3 Let x, y and z be positive integers. If $|x - y|^{2010} + |z - x|^{2011} = b$ and $c = |x - y| + |y - z| + |z - x|$, find the value of c.

Reference: 1996 FI2.3, 2005FI4.1, 2006 FI4.2, 2013 FI1.4, 2015 HG4, 2015 FI1.1

Clearly $|x - y|$ and $|z - x|$ are non-negative integers

$|x - y|^{2010} + |z - x|^{2011} = 1$

$\Rightarrow (|x - y| = 0$ and $|z - x| = 1)$ or $(|x - y| = 1$ and $|z - x| = 0)$

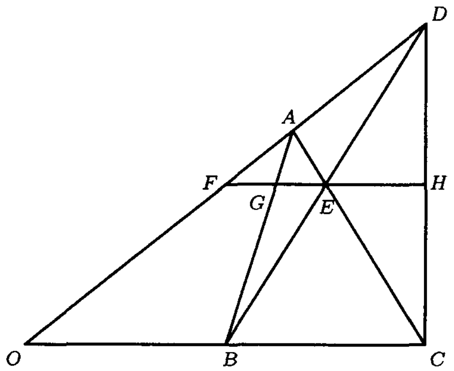
When $x = y$ and $|z - x| = 1$,

$c = 0 + |y - z| + |z - x| = 2|z - x| = 2$

When $|x - y| = 1$ and $|z - x| = 0$,

$c = 1 + |y - z| + 0 = 1 + |y - x| = 1 + 1 = 2$

I4.4 In Figure 1, let ODC be a triangle. Given that FH, AB, AC and BD are line segments such that AB intersects FH at G, AC, BD and FH intersect at E, $GE = 1$, $EH = c$ and $FH \parallel OC$. If $d = EF$, find the value of d.



$\Delta AGE \sim \Delta ABC$ (equiangular)

Let $\frac{CE}{AE} = k$, $AE = x$, $AG = t$.

$BC = k + 1$, $EC = kx$, $GB = kt$ (corr. sides, ~Δs)

$\Delta DEH \sim \Delta DBC$ (equiangular)

$\frac{BC}{EH} = \frac{k+1}{2} = \frac{DB}{DE}$ (corr. sides, ~Δs)

Let $DE = 2y \Rightarrow DB = (k + 1)y$

$EB = DB - DE = (k - 1)y$

$\Delta AFG \sim \Delta AOB$ (equiangular)

$FG = d - 1$, $\frac{OB}{FG} = \frac{AB}{AG}$ (corr. sides, ~Δs)

$OB = (d-1)\cdot\frac{(k+1)t}{t} = (d - 1)(k + 1)$

$\Delta DFE \sim \Delta DOB$ (equiangular)

$\frac{FE}{OB} = \frac{DE}{DB}$ (corr. sides, ~Δs)

$\Rightarrow d = (d-1)(k+1)\cdot\frac{2y}{(k+1)y}$

$\Rightarrow d = 2$

Method 2

$\Delta AFG \sim \Delta AOB$ and $\Delta AGE \sim \Delta ABC$

$\frac{d-1}{1} = \frac{OB}{BC}$ (corr. sides, ~Δs)

$\Delta DFE \sim \Delta DOB$ and $\Delta DEH \sim \Delta DBC$

$\frac{d}{2} = \frac{OB}{BC}$ (corr. sides, ~Δs)

Equating the two equations

$\frac{d-1}{1} = \frac{d}{2}$

$d = 2$

Remark: There are some typing mistakes in the Chinese old version:

$\cdots$ *AC* 及 *AD* 為綫段 $\cdots$ *FH* // *BC* $\cdots$

Individual Spare

IS.1 Let P be the number of triangles whose side lengths are integers less than or equal to 9. Find the value of P.

The sides must satisfy triangle inequality. i.e. $a + b > c$.

Possible order triples are (1, 1, 1), (2, 2, 2), … , (9, 9, 9),
(2, 2, 1), (2, 2, 3), (3, 3, 1), (3, 3, 2), (3, 3, 4), (3, 3, 5),
(4, 4, 1), (4, 4, 2), (4, 4, 3), (4, 4, 5), (4, 4, 6), (4, 4, 7),
(5, 5, 1), … , (5, 5, 4), (5, 5, 6), (5, 5, 7), (5, 5, 8), (5, 5, 9),
(6, 6, 1), … , (6, 6, 9) (except (6, 6, 6))
(7, 7, 1), … , (7, 7, 9) (except (7, 7, 7))
(8, 8, 1), … , (8, 8, 9) (except (8, 8, 8))
(9, 9, 1), … , (9, 9, 8)
(2, 3, 4), (2, 4, 5), (2, 5, 6), (2, 6, 7), (2, 7, 8), (2, 8, 9),
(3, 4, 5), (3, 4, 6), (3, 5, 6), (3, 5, 7), (3, 6, 7), (3, 6, 8), (3, 7, 8), (3, 7, 9), (3, 8, 9),
(4, 5, 6), (4, 5, 7), (4, 5, 8), (4, 6, 7), (4, 6, 8), (4, 6, 9), (4, 7, 8), (4, 7, 9), (4, 8, 9),
(5, 6, 7), (5, 6, 8), (5, 6, 9), (5, 7, 8), (5, 7, 9), (5, 8, 9), (6, 7, 8), (6, 7, 9), (6, 8, 9), (7, 8, 9).

Total number of triangles = 9 + 6 + 6 + 8×5 + 6 + 9 + 9 + 6 + 4 = 95

Method 2 First we find the number of order triples.

Case 1 All numbers are the same: (1, 1, 1), … , (9, 9, 9).

Case 2 Two of them are the same, the third is different: (1, 1, 2), … , (9, 9, 1)

There are $C_1^9 \times C_1^8 = 72$ possible triples.

Case 3 All numbers are different. There are $C_3^9 = 84$ possible triples.

$\therefore$ Total 9 + 72 + 84 = **165** possible triples.

Next we find the number of triples which **cannot form a triangle**, i.e. $a + b \le c$.

Possible triples are (1, 1, 2), … (1, 1, 9) (8 triples)
(1, 2, 3), … , (1, 2, 9) (7 triples)
(1, 3, 4), … , (1, 3, 9) (6 triples)
(1, 4, 5), … , (1, 4, 9) (5 triples)
(1, 5, 6), … , (1, 5, 9) (4 triples)
(1, 6, 7), (1, 6, 8), (1, 6, 9), (1, 7, 8), (1, 7, 9), (1, 8, 9),
(2, 2, 4), … , (2, 2, 9) (6 triples)
(2, 3, 5), … , (2, 3, 9) (5 triples)
(2, 4, 6), … , (2, 4, 9) (4 triples)
(2, 5, 7), (2, 5, 8), (2, 5, 9), (2, 6, 8), (2, 6, 9), (2, 7, 9),
(3, 3, 6), … , (3, 3, 9) (4 triples)
(3, 4, 7), (3, 4, 8), (3, 4, 9), (3, 5, 8), (3, 5, 9), (3, 6, 9), (4, 4, 8), (4, 4, 9), (4, 5, 9).

Total number of triples which cannot form a triangle
= (8 + 7 + … + 1) + (6 + 5 + … + 1) + (4 + 3 + 2 + 1) + (2 + 1) = 36 + 21 + 10 + 3 = **70**

$\therefore$ Number of triangles = **165 – 70** = **95**

IS.2 Let $Q = \log_{128} 2^3 + \log_{128} 2^5 + \log_{128} 2^7 + \ldots + \log_{128} 2^P$. Find the value of Q.

$Q = 3 \log_{128} 2 + 5 \log_{128} 2 + 7 \log_{128} 2 + \ldots + 95 \log_{128} 2$

$= (3 + 5 + \ldots + 95) \log_{128} 2 = \frac{3+95}{2} \times 47 \times \log_{128} 2 = \log_{128} 2^{2303} = \log_{128} (2^7)^{329} = 329$

IS.3 Consider the line $12x - 4y + (Q - 305) = 0$. If the area of the triangle formed by the x-axis, the y-axis and this line is R square units, what is the value of R?

$12x - 4y + 24 = 0 \Rightarrow$ Height = 6, base = 2; area $R = \frac{1}{2} \cdot 6 \cdot 2 = 6$

Remark: the original question is …$12x - 4y + Q = 0$….

The answer is very difficult to carry forward to next question.

IS.4 If $x + \frac{1}{x} = R$ and $x^3 + \frac{1}{x^3} = S$, find the value of S.

$S = \left(x + \frac{1}{x}\right)\left(x^2 - 1 + \frac{1}{x^2}\right)$

$= R\left[\left(x + \frac{1}{x}\right)^2 - 3\right]$

$= R^3 - 3R$

$= 216 - 3(6)$

$= 198$

Sample Group Event (2009 Final Group Event 1)

SG.1 Given some triangles with side lengths a cm, 2 cm and b cm, where a and b are integers and $a \le 2 \le b$. If there are q non-congruent classes of triangles satisfying the above conditions, find the value of q.

When $a = 1$, possible $b = 2$

When $a = 2$, possible $b = 2$ or 3

$\therefore q = 3$

SG.2 Given that the equation $|x| - \frac{4}{x} = \frac{3|x|}{x}$ has k distinct real root(s), find the value of k.

When $x > 0$: $x^2 - 4 = 3x$

$\Rightarrow x^2 - 3x - 4 = 0$

$\Rightarrow (x + 1)(x - 4) = 0$

$\Rightarrow x = 4$

When $x < 0$: $-x^2 - 4 = -3x$

$\Rightarrow x^2 - 3x + 4 = 0$

$\Delta = 9 - 16 < 0$

$\Rightarrow$ no real roots.

$k = 1$ (There is only one real root.)

SG.3 Given that x and y are non-zero real numbers satisfying the equations $\frac{\sqrt{x}}{\sqrt{y}} - \frac{\sqrt{y}}{\sqrt{x}} = \frac{7}{12}$ and $x - y = 7$. If $w = x + y$, find the value of w.

The first equation is equivalent to $\frac{x-y}{\sqrt{xy}} = \frac{7}{12} \Rightarrow \sqrt{xy} = 12 \Rightarrow xy = 144$

Sub. $y = \frac{144}{x}$ into $x - y = 7$: $x - \frac{144}{x} = 7 \Rightarrow x^2 - 7x - 144 = 0 \Rightarrow (x + 9)(x - 16) = 0$

$x = -9$ or 16; when $x = -9$, $y = -16$ (rejected $\because \sqrt{x}$ is undefined); when $x = 16$; $y = 9$

$w = 16 + 9 = 25$

Method 2 The first equation is equivalent to $\frac{x-y}{\sqrt{xy}} = \frac{7}{12} \Rightarrow \sqrt{xy} = 12 \Rightarrow xy = 144$ …… (1)

$\because x - y = 7$ and $x + y = w$

$\therefore x = \frac{w+7}{2}, y = \frac{w-7}{2}$

Sub. these equations into (1): $\left(\frac{w+7}{2}\right)\left(\frac{w-7}{2}\right) = 144$

$w^2 - 49 = 576 \Rightarrow w = \pm 25$

$\because$ From the given equation $\frac{\sqrt{x}}{\sqrt{y}} - \frac{\sqrt{y}}{\sqrt{x}} = \frac{7}{12}$, we know that both $x > 0$ and $y > 0$

$\therefore w = x + y = 25$ only

SG.4 Given that x and y are real numbers and $\left|x-\frac{1}{2}\right|+\sqrt{y^2-1}=0$.

Let $p = |x| + |y|$, find the value of p.

Reference: 2006 FI4.2 $\cdots$ $y^2 + 4y + 4 + \sqrt{x+y+k} = 0$. If $r = |xy|$, $\cdots$

Both $\left|x-\frac{1}{2}\right|$ and $\sqrt{y^2-1}$ are non-negative numbers.

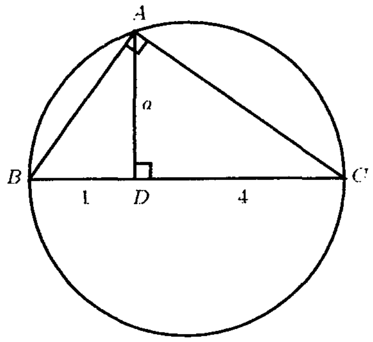
The sum of two non-negative numbers = 0 means each of them is zero

$x=\frac{1}{2}$, $y = \pm 1$

$p=\frac{1}{2}+1=\frac{3}{2}$

Group Event 1

G1.1 In Figure 1, BC is the diameter of the circle. A is a point on the circle, AB and AC are line segments and AD is a line segment perpendicular to BC. If $BD = 1$, $DC = 4$ and $AD = a$, find the value of a.



$\Delta ABD \sim \Delta CAD$ (equiangular)

$\frac{a}{1}=\frac{4}{a}$ (ratio of sides ~Δ's)

$a^2 = 1\times 4$

$a = 2$

G1.2 If $b = 1-\cfrac{1}{1-\cfrac{1}{1-\cfrac{1}{1-\cfrac{1}{-\frac{1}{2}}}}}$, find the value of b.

$1-\frac{1}{-\frac{1}{2}}=3$; $1-\cfrac{1}{1-\cfrac{1}{-\frac{1}{2}}}=\frac{2}{3}$; $1-\cfrac{1}{1-\cfrac{1}{1-\cfrac{1}{-\frac{1}{2}}}}=-\frac{1}{2}$; $b = 1 + 2 = 3$

G1.3 If x, y and z are real numbers, $xyz \neq 0$, $2xy = 3yz = 5xz$ and $c=\frac{x+3y-3z}{x+3y-6z}$, find the value of c.

$\frac{2xy}{xyz}=\frac{3yz}{xyz}=\frac{5xz}{xyz}\Rightarrow\frac{2}{z}=\frac{3}{x}=\frac{5}{y}\Rightarrow x:y:z=3:5:2$

Let $x = 3k$, $y = 5k$, $z = 2k$

$c=\frac{x+3y-3z}{x+3y-6z}=\frac{3k+15k-6k}{3k+15k-12k}=2$

G1.4 If x is an integer satisfying $\log_{1/4}(2x+1)<\log_{1/2}(x-1)$, find the maximum value of x.

$\frac{\log(2x+1)}{\log\frac{1}{4}}<\frac{\log(x-1)}{\log\frac{1}{2}}$

$\frac{\log(2x+1)}{-2\log 2}<\frac{\log(x-1)}{-\log 2}$

$\log(2x + 1) > 2\log(x - 1)$

$2x + 1 > (x - 1)^2$

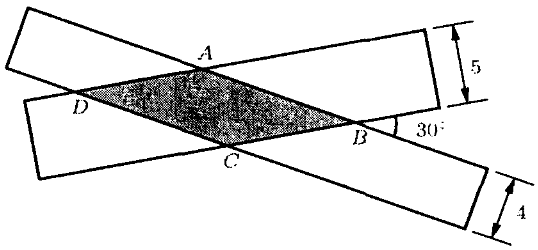
$x^2 - 4x < 0$

$0 < x < 4$

The maximum integral value of x is 3.

Group Event 2

G2.1 In Figure 1, two rectangles with widths 4 and 5 units cross each other at 30°. Find the area of the overlapped region.



Let $AB = x$, $BC = y$, $\angle ABC = 30°$

$x \sin 30° = 5 \Rightarrow x = 10$

$y \sin 30° = 4 \Rightarrow y = 8$

Area = $xy \sin 30° = 10\times8\times0.5 = 40$

G2.2 From 1 to 100, take a pair of integers (repetitions allowed) so that their sum is greater than 100. How many ways are there to pick such pairs?

Reference: 2002 FG3.3

(1, 100), (2, 100), … , (100, 100) (100 pairs)

(2, 99), … , (99, 99) (98 pairs)

(3, 98), … , (98, 98) (96 pairs)

....

(49, 52), (50, 52), (51, 52), (52, 52) (4 pairs)

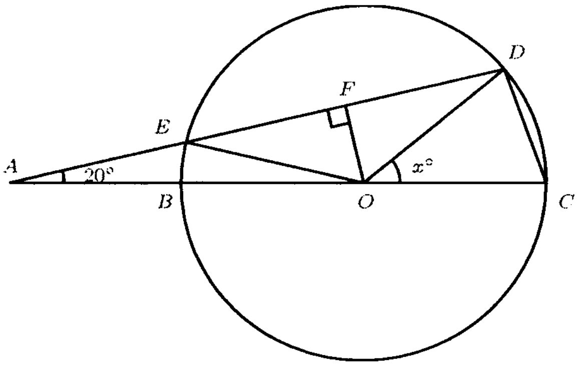
(50, 51), (51, 51) (2 pairs)

Total number of pairs = 2 + 4 + … + 100 = $\frac{2+100}{2}\times 50 = 2550$

Remark: the original version was …"take a pair of numbers"…從1到100選取兩數…

There are infinitely many ways if the numbers are not confined to be integers.

G2.3 In Figure 2, there is a circle with centre O and radius r. Triangle ACD intersects the circle at B, C, D and E. Line segment AE has the same length as the radius. If $\angle DAC = 20°$ and $\angle DOC = x°$, find the value of x.



$\angle AOE = 20°$ (Given $AE = OE$, base $\angle$s isos. Δ)

$\angle OED = 20° + 20° = 40°$ (ext. $\angle$ of ΔAOE)

$\angle ODE = \angle OED = 40°$ (base $\angle$s isos. Δ)

$x = 20 + 40 = 60$ (ext. $\angle$ of ΔAOD)

G2.4 Given that $\frac{1}{x}+\frac{2}{y}+\frac{3}{z}=0$ and $\frac{1}{x}-\frac{6}{y}-\frac{5}{z}=0$. If $P=\frac{x}{y}+\frac{y}{z}+\frac{z}{x}$, find the value of P.

(1) – (2): $\frac{8}{y}+\frac{8}{z}=0 \Rightarrow y=-z$

3(1) + (2): $\frac{4}{x}+\frac{4}{z}=0 \Rightarrow x=-z$

$P=\frac{x}{y}+\frac{y}{z}+\frac{z}{x}=\frac{-z}{-z}+\frac{-z}{z}+\frac{z}{-z}=-1$

Group Event 3

G3.1 If a is a positive integer and $a^2 + 100a$ is a prime number, find the maximum value of a.

$a(a + 100)$ is a prime number.

$a = 1$, $a^2 + 100a = 101$ which is a prime number

G3.2 Let a, b and c be real numbers. If 1 is a root of $x^2 + ax + 2 = 0$ and a and b be roots of $x^2 + 5x + c = 0$, find the value of $a + b + c$.

$1 + a + 2 = 0 \Rightarrow a = -3$

$-3 + b = -5 \Rightarrow b = -2$

$c = -3b = 6$

$a + b + c = 1$

G3.3 Let x and y be positive real numbers with $x < y$. If $\sqrt{x}+\sqrt{y}=1$, $\sqrt{\frac{x}{y}}+\sqrt{\frac{y}{x}}=\frac{10}{3}$ and $x < y$, find the value of $y - x$.

$(1)^2$: $x + y + 2\sqrt{xy} = 1$

$\sqrt{xy}=\frac{1-(x+y)}{2}$ … (3)

(2): $\frac{x+y}{\sqrt{xy}}=\frac{10}{3}$ … (4)

Sub. (3) into (4): $\frac{x+y}{\frac{1-(x+y)}{2}}=\frac{10}{3}$

$6(x + y) = 10(1 - x - y)$

$16(x + y) = 10$

$x+y=\frac{5}{8}$

$\sqrt{xy}=\frac{1-(x+y)}{2}=\frac{1}{2}\left(1-\frac{5}{8}\right)=\frac{3}{16}$

$xy=\frac{9}{256}$

$(y-x)^2=(x+y)^2-4xy=\frac{25}{64}-\frac{9}{64}=\frac{1}{4}$

$y-x=\frac{1}{2}$

Method 2

Let $z=\sqrt{\frac{x}{y}}$, then $\frac{1}{z}=\sqrt{\frac{y}{x}}$

(2) becomes $z+\frac{1}{z}=\frac{10}{3}$

$3z^2 - 10z + 3 = 0$

$(3z - 1)(z - 3) = 0$

$z=\frac{1}{3}$ or 3

$\because x < y$

$\therefore z=\sqrt{\frac{x}{y}}<1 \Rightarrow z=\frac{1}{3}$ only

$\frac{\sqrt{y}-\sqrt{x}}{\sqrt{y}+\sqrt{x}}=\frac{1-\sqrt{\frac{x}{y}}}{1+\sqrt{\frac{x}{y}}}=\frac{1-\frac{1}{3}}{1+\frac{1}{3}}=\frac{1}{2}$

$\because \sqrt{x}+\sqrt{y}=1 \quad \therefore \sqrt{y}-\sqrt{x}=\frac{1}{2}$

$y-x=\left(\sqrt{y}-\sqrt{x}\right)\left(\sqrt{y}+\sqrt{x}\right)=\frac{1}{2}$

G3.4 Spilt the numbers 1, 2, … , 10 into two groups and let P_1 be the product of the first group and P_2 the product of the second group. If P_1 is a multiple of P_2, find the minimum value of $\frac{P_1}{P_2}$.

$P_1 = kP_2$, where k is a positive integer.

$\therefore$ All prime factors of P_2 can divide P_1.

$\frac{10}{5} = 2$, 10 must be a factor of the numerator and 5 must a factor of the denominator

7 is a prime which must be a factor of the numerator.

Among the even numbers 2, 4, 6, 8, 10, there are 8 factors of 2.

4 factors of 2 should be put in the numerator and 4 factors should be put in the denominator.

Among the number 3, 6, 9, there are 4 factors of 3.

2 factors of 3 should be put in the numerator and 2 factors should be put in the denominator.

Minimum value of $\frac{P_1}{P_2} = \frac{8\times 7\times 9\times 10}{1\times 2\times 3\times 4\times 5\times 6} = 7$

Group Event 4

G4.1 If $P = 2\sqrt[4]{2007\cdot 2009\cdot 2011\cdot 2013+10\cdot 2010\cdot 2010-9}-4000$, find the value of P.

Let $x = 2010$, $2007 = x - 3$, $2009 = x - 1$, $2011 = x + 1$, $2013 = x + 3$

$P = 2\sqrt[4]{(x-3)\cdot(x-1)\cdot(x+1)\cdot(x+3)+10x^2-9}-4000 = 2\sqrt[4]{(x^2-9)\cdot(x^2-1)+10x^2-9}-4000$

$= 2\sqrt[4]{x^4-10x^2+9+10x^2-9}-4000 = 2x - 4000 = 20$

G4.2 If $9x^2 + nx + 1$ and $4y^2 + 12y + m$ are squares with $n > 0$, find the value of $\frac{n}{m}$.

$9x^2 + nx + 1 = (3x + 1)^2 \Rightarrow n = 6$

$4y^2 + 12y + m = (2y + 3)^2 \Rightarrow m = 9$

$\frac{n}{m} = \frac{2}{3}$

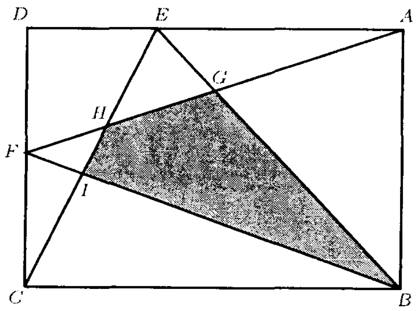
G4.3 Let n and $\frac{47}{5}\left(\frac{4}{47}+\frac{n}{141}\right)$ be positive integers. If r is the remainder of n divided by 15, find the value of r.

$\frac{47}{5}\left(\frac{4}{47}+\frac{n}{141}\right)=\frac{4}{5}+\frac{n}{15}=\frac{n+12}{15}$, which is an integer

$n + 12 = 15k$, where k is a positive integer

$r = 3$

G4.4 In figure 1, $ABCD$ is a rectangle, and E and F are points on AD and DC, respectively. Also, G is the intersection of AF and BE, H is the intersection of AF and CE, and I is the intersection of BF and CE. If the areas of AGE, $DEHF$ and CIF are 2, 3 and 1, respectively, find the area of the grey region $BGHI$. (**Reference: 2014 FI1.1, 2019 FG3.2)**



Let the area of $EGH = x$, area of $BCI = z$, area of $BGHI = w$

Area of $BCE = \frac{1}{2}$(area of $ABCD$) = area ADF + area BCF

$x + w + z = 3 + x + 2 + 1 + z$

$\Rightarrow w = 6$

$\therefore$ Area of the grey region $BGHI = 6$

Remark: there is a spelling mistake in the English version. old version: … gray region …

Group Spare

GS.1 Let α and β be the real roots of $y^2 - 6y + 5 = 0$. Let m be the minimum value of $|x - \alpha| + |x - \beta|$ over all real values of x. Find the value of m.

Reference: 1994 HG1, 2001 HG9, 2004 FG4.2, 2008 HI8, 2008 FI1.3, 2010 HG6, 2012 FG2.3

$\alpha = 1$, $\beta = 5$

If $x < 1$, $|x - \alpha| + |x - \beta| = 1 - x + 5 - x = 6 - 2x > 4$

If $1 \le x \le 5$, $|x - \alpha| + |x - \beta| = x - 1 + 5 - x = 4$

If $x > 5$, $|x - \alpha| + |x - \beta| = x - 1 + x - 5 = 2x - 6 > 4$

$m = \text{min. of } |x - \alpha| + |x - \beta| = 4$

Method 2 Using the triangle inequality: $|a| + |b| \ge |a + b|$

$|x - \alpha| + |x - \beta| \ge |x - 1 + 5 - x| = 4 \Rightarrow m = 4$

Remark: there is a typing mistake in the English version. … minimum value *a* of …

GS.2 Let α, β, γ be real numbers satisfying $\alpha + \beta + \gamma = 2$ and $\alpha\beta\gamma = 4$. Let v be the minimum value of $|\alpha| + |\beta| + |\gamma|$. Find the value of v.

If at least one of $\alpha, \beta, \gamma = 0$, then $\alpha\beta\gamma \ne 4 \Rightarrow \alpha, \beta, \gamma \ne 0$

If $\alpha, \beta, \gamma > 0$, then

$$\frac{\alpha + \beta + \gamma}{3} \ge \sqrt[3]{\alpha\beta\gamma} \quad \text{(A.M.} \ge \text{G.M.)}$$

$$\frac{2}{3} \ge \sqrt[3]{4}$$

$2^3 \ge 27\times4 = 108$, which is a contradiction

If $\beta < 0$, in order that $\alpha\beta\gamma = 4 > 0$, WLOG let $\gamma < 0$, $\alpha > 0$

$\alpha = 2 - \beta - \gamma > 2$

$|\alpha| + |\beta| + |\gamma| = \alpha - (\beta + \gamma) = 2 + 2(-\beta - \gamma) \ge 2 + 4\sqrt{(-\beta)(-\gamma)}$, equality holds when $\beta = \gamma$

$4 = (2 - 2\beta)\beta^2$

$\beta^3 - \beta^2 + 2 = 0$

$(\beta + 1)(\beta^2 - 2\beta + 2) = 0$

$\beta = -1$ (For the 2nd equation, $\Delta = -4 < 0$, no real solution)

$\gamma = -1$, $\alpha = 4$

$|\alpha| + |\beta| + |\gamma| = 1 + 1 + 4 = 6$

$v = \text{min. of } |\alpha| + |\beta| + |\gamma| = 6$

GS.3 Let $y = |x + 1| - 2|x| + |x - 2|$ and $-1 \le x \le 2$. Let α be the maximum value of y. Find the value of α.

$y = x + 1 - 2|x| + 2 - x = 3 - 2|x|$

$0 \le |x| \le 2 \Rightarrow 3 \ge 3 - 2|x| \ge -1$

$\alpha = 3$

GS.4 Let F be the number of integral solutions of $x^2 + y^2 + z^2 + w^2 = 3(x + y + z + w)$.

Find the value of F.

$(x, y, z, w) = (0, 0, 0, 0)$ is a trivial solution.

$x^2 + y^2 + z^2 + w^2 - 3(x + y + z + w) = 0$

$$\left(x^2 - 3x + \frac{9}{4}\right) + \left(y^2 - 3y + \frac{9}{4}\right) + \left(z^2 - 3z + \frac{9}{4}\right) + \left(w^2 - 3w + \frac{9}{4}\right) = 9$$

$$\left(x - \frac{3}{2}\right)^2 + \left(y - \frac{3}{2}\right)^2 + \left(z - \frac{3}{2}\right)^2 + \left(w - \frac{3}{2}\right)^2 = 9$$

$(2x - 3)^2 + (2y - 3)^2 + (2z - 3)^2 + (2w - 3)^2 = 36$

Let $a = 2x - 3$, $b = 2y - 3$, $c = 2z - 3$, $d = 2w - 3$, the equation becomes $a^2 + b^2 + c^2 + d^2 = 36$

For integral solutions of (x, y, z, w), (a, d, c, d) must be odd integers.

In addition, the permutation of (a, b, c, d) is also a solution. (e.g. (b, d, c, a) is a solution)

$\because$ a, b, c, d are odd integers and $a^2 + b^2 + c^2 + d^2 \geq 0$

If one of the four unknowns, say, $a > 6$, then L.H.S. > 36, so L.H.S. ≠ R.H.S.

$\therefore$ $a, b, c, d = \pm 1, \pm 3, \pm 5$

When $a = \pm 5$, then $25 + b^2 + c^2 + d^2 = 36 \Rightarrow b^2 + c^2 + d^2 = 11$

The only integral solution to this equation is $b = \pm 3$, $c = \pm 1 = d$ or its permutations.

When the largest (in magnitude) of the 4 unknowns, say, a is ± 3, then $9 + b^2 + c^2 + d^2 = 36$

$\Rightarrow b^2 + c^2 + d^2 = 27$, the only solution is $b = \pm 3$, $c = \pm 3$, $d = \pm 3$ or its permutations.

$\therefore$ The integral solutions are $(a, b, c, d) = (5, 3, 1, 1)$ and its permutations … (1) $\times P^4_2 = 12$

$(3, 3, 3, 3)$ … (2) $\times 1$

If (a, b, c, d) is a solution, then $(\pm a, \pm b, \pm c, \pm d)$ are also solutions.

There are 16 solutions with different signs for $(\pm a, \pm b, \pm c, \pm d)$.

$\therefore$ $F = (12 + 1) \times 16$

$= 208$