Individual Events

SI	*P*	30	I1	*a*	100	I2	*a*	3	I3	**a* see the remark	2	I4	*a*	1
	**Q* see the remark	120		*b*	5		*b*	600		*b*	7		*b*	7
	R	11		*c*	0		*c*	2		*c*	4		*c*	–61
	**S* see the remark	72		*d*	2		*d*	36		*d*	$4\frac{1}{3}$		*d*	69

Group Events

SG	*q*	3	G1	unit digit	5	G2	minimum *r*	1	G3	$m^3 - n^3$	1387	G4	no. of digits	34
	k	1		Integral part	1		*s*	40		Maximum	31			2000
	w	25			24		*t*	–0.5		*a*·*b*	$-\frac{1}{3}$			2519
	p	$\frac{3}{2}$		Greatest *A*	3		*u*	120		*BC*	9		*A*+*B*+*C*+*D*+*E*	15

Errata

FI1.2 "the remainder <u>of</u> .... divided by" is changed into "the remainder <u>when</u> .... <u>is</u> divided by"

FI1.4 "Find the <u>maximum possible</u> value of" is changed into "Find the value of"

FI2.2"增加 $(2b-a)$ $\underline{\text{cm}^2}$" is changed into "增加 $(2b-a)$ $\underline{\text{cm}^3}$"

FI3.1 "integer" is deleted, 求 a 的整數值更改為求 a 的值。。

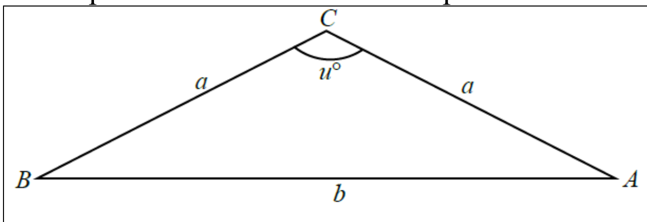
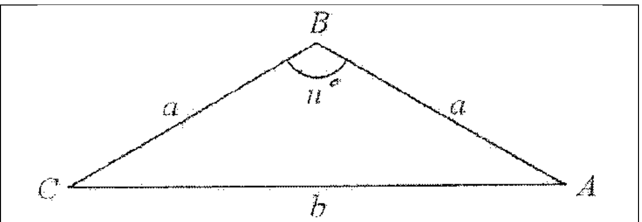
FI3.3 "The remainder <u>of</u> 392 divided by" is changed into "The remainder <u>when</u> 392 <u>is</u> divided by"

FI4.4 $\begin{cases} xy=6 \\ x^2y+\underline{yx^2}+x+y+c=2 \end{cases}$ is changed into $\begin{cases} xy=6 \\ x^2y+\underline{xy^2}+x+y+c=2 \end{cases}$

FG1.2 "integer" is changed into "integral"

FG1.3 "three-digit numbers how many" is changed into "three-digit numbers, how many".

FG2.4 wrong figure 1 on the internet http://www.edb.gov.hk/attachment/tc/curriculum-development/kla/ma/res/sa/2012d.pdf

	
Original wrong diagram	New diagram

Sample Individual Event (2009 Final Individual Event 1)

SI.1 Let a, b, c and d be the roots of the equation $x^4 - 15x^2 + 56 = 0$.
If $P = a^2 + b^2 + c^2 + d^2$, find the value of P.

$x^4 - 15x^2 + 56 = 0 \Rightarrow (x^2 - 7)(x^2 - 8) = 0$

$a = \sqrt{7}$, $b = -\sqrt{7}$, $c = \sqrt{8}$, $d = -\sqrt{8}$

$P = a^2 + b^2 + c^2 + d^2 = 7 + 7 + 8 + 8 = 30$

SI.2 In Figure 1, $AB = AC$ and $AB \parallel ED$.
If $\angle ABC = P°$ and $\angle ADE = Q°$, find the value of Q.

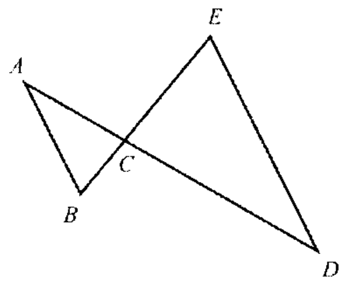


Figure 1

$\angle ABC = 30° = \angle ACB$ (base $\angle$s isos. Δ)

$\angle BAC = 120°$ ($\angle$s sum of Δ)

$\angle ADE = 120°$ (alt. $\angle$s, $AB \parallel ED$)

$Q = 120$

Remark: Original question … $AB \parallel DE$ ….
It is better for AB and ED to be oriented in the same direction.

SI.3 Let $F = 1 + 2 + 2^2 + 2^3 + \ldots + 2^Q$ and $R = \sqrt{\frac{\log(1+F)}{\log 2}}$, find the value of R.

$F = 1 + 2 + 2^2 + 2^3 + \ldots + 2^{120} = \frac{2^{121}-1}{2-1} = 2^{121} - 1$

$R = \sqrt{\frac{\log(1+F)}{\log 2}} = \sqrt{\frac{\log 2^{121}}{\log 2}} = 11$

SI.4 Let f(x) be a function such that f(n) = (n – 1) f(n – 1) and f(1) ≠ 0.
If $S = \frac{\mathrm{f}(R)}{(R-1)\mathrm{f}(R-3)}$, find the value of S.

$\mathrm{f}(n) = (n-1)\,\mathrm{f}(n-1) = (n-1)(n-2)\,\mathrm{f}(n-2) = \ldots\ldots$

$S = \frac{\mathrm{f}(11)}{(11)\mathrm{f}(11-3)} = \frac{10\times 9\times 8\times \mathrm{f}(8)}{10\times \mathrm{f}(8)} = 9\times 8 = 72$

Remark: Original question:

Let f(x) be a function such that f(n) = (n – 1) f(n – 1). If $S = \frac{\mathrm{f}(R)}{(R-1)\mathrm{f}(R-3)}$, find the value of S.

Note that S is undefined when f(n) = 0 for some integers n.

Individual Event 1

I1.1 Figure 1 has a rectangles, find the value of a.
Reference: 1993HG9
$a = C_2^5 \times C_2^5 = 100$

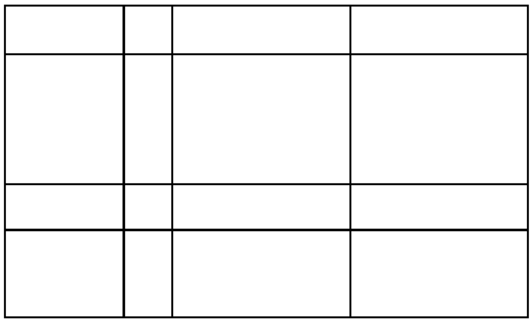


Figure 1

I1.2 Given that 7 divides 111111. If b is the remainder when $\underbrace{111111\ldots111111}_{a-times}$ is divided by 7, find the value of b.

$\underbrace{111111\ldots111111}_{100-times} = \underbrace{111111\ldots11}_{96-times}0000 + 1111$

$= 111111 \times \underbrace{1000001\cdots1000001}_{16\text{ '1's}} \times 10000 + 7\times158 + 5$

$= 7m + 5$, where m is an integer.

$b = 5$

I1.3 If c is the remainder of $\left[(b-2)^{4b^2} + (b-1)^{2b^2} + b^{b^2}\right]$ divided by 3, find the value of c.

$\left[(5-2)^{100} + (4)^{50} + 5^{25}\right] = 3^{100} + 4^{50} + 5^{25}$

$= 3^{100} + (3 + 1)^{50} + (3\times2 - 1)^{25}$

$= 3^{100} + 3m + 1 + 3n - 1$ (by binomial theorem, n, m are integers)

The remainder $c = 0$

I1.4 If $|x + 1| + |y - 1| + |z| = c$, find the value of $d = x^2 + y^2 + z^2$.
Reference: 2005 FI4.1, 2006 FI4.2, 2009 FG1.4, 2011 FI4.3, 2015 HG4, 2015 FI1.1

$|x + 1| + |y - 1| + |z| = 0$

$x = -1$, $y = 1$ and $z = 0$

$d = (-1)^2 + 1^2 + 0^2 = 2$

Individual Event 2

I2.1 Given that functions $f(x) = x^2 + rx + s$ and $g(x) = x^2 - 9x + 6$ have the properties that the sum of roots of $f(x)$ is the product of the roots of $g(x)$, and the product of roots of $f(x)$ is the sum of roots of $g(x)$. If $f(x)$ attains its minimum at $x = a$, find the value of a.

Let α, β be the roots of $f(x)$.

$\alpha + \beta = -r = 6$; $\alpha\beta = s = 9$

$\therefore$ $f(x) = x^2 - 6x + 9 = (x - 3)^2 + 0$

$f(x)$ attains the minimum value at $x = a = 3$

I2.2 The surface area of a cube is b cm². If the length of each side is increased by 3 cm, its volume is increased by $(2b - a)$ cm³, find the value of b.

Let the original length of each side be x cm.

Old surface area b cm² = $6x^2$ cm²

Original volume = x^3 cm³

New length of side = $(x + 3)$ cm.

New volume = $(x + 3)^3$ cm³

Increase in volume = $[(x + 3)^3 - x^3]$ cm³ = $(2b - a)$ cm³

$9x^2 + 27x + 27 = 2(6x^2) - 3$

$3x^2 - 27x - 30 = 0$

$x^2 - 9x - 10 = 0$

$(x - 10)(x + 1) = 0$

$x = 10$

$b = 6x^2 = 600$

I2.3 Let $f(1) = 3$, $f(2) = 5$ and $f(n + 2) = f(n + 1) + f(n)$ for positive integers n .

If c is the remainder of $f(b)$ divided by 3, find the value of c.

$f(1) = 3$, $f(2) = 5$, $f(3) = 8$, $f(4) = 13$, $f(5) = 21$, $f(6) = 34$, $f(7) = 55$, $f(8) = 89$,

$\equiv 0$, $\equiv 2$, $\equiv 2$, $\equiv 1$, $\equiv 0$, $\equiv 1$, $\equiv 1$, $\equiv 2 \pmod 3$

$f(9) \equiv 0$, $f(10) \equiv 2$, $f(11) \equiv 2$, $f(12) \equiv 1$, $f(13) \equiv 0$, $f(14) \equiv 1$, $f(15) \equiv 1$, $f(16) \equiv 2 \pmod 3$

$\therefore$ When $f(n)$ is divided by 3, the pattern of the remainders repeats for every 8 integers.

$600 = 8\times75$

$c = 2$

I2.4 In Figure 2, the angles of triangle XYZ satisfy $\angle Z \le \angle Y \le \angle X$ and $c\cdot\angle X = 6\cdot\angle Z$.

If the maximum possible value of $\angle Z$ is $d°$, find the value of d.

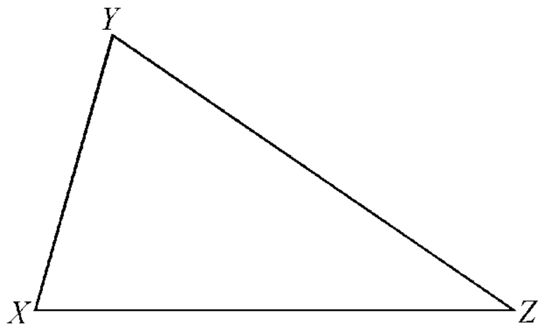


Figure 2

$2\cdot\angle X = 6\cdot\angle Z \Rightarrow \angle X = 3\angle Z$

Let $\angle Z = z°$, $\angle Y = y°$, $\angle X = 3z°$

$z + y + 3z = 180$ ($\angle$ sum of Δ)

$y = 180 - 4z$

$\because$ $\angle Z \le \angle Y \le \angle X$

$\therefore$ $z \le 180 - 4z \le 3z$

$\frac{180}{7} \le z$ and $z \le 36$

d = the maximum possible value of $z = 36$

Individual Event 3

I3.1 If $a=\dfrac{\left(7+4\sqrt{3}\right)^{\frac{1}{2}}-\left(7-4\sqrt{3}\right)^{\frac{1}{2}}}{\sqrt{3}}$, find the value of a.

Reference: 2015 FI4.2

$\left(7+4\sqrt{3}\right)^{\frac{1}{2}}=\sqrt{4+2\sqrt{4}\cdot\sqrt{3}+3}=\sqrt{\left(\sqrt{4}+\sqrt{3}\right)^2}=2+\sqrt{3}$

$\left(7-4\sqrt{3}\right)^{\frac{1}{2}}=\sqrt{4-2\sqrt{4}\cdot\sqrt{3}+3}=\sqrt{\left(\sqrt{4}-\sqrt{3}\right)^2}=2-\sqrt{3}$

$a=\dfrac{2+\sqrt{3}-\left(2-\sqrt{3}\right)}{\sqrt{3}}=2$

Remark: The original question is: ⋯⋯, find the integer value of a. ⋯⋯ 求 a 的整數值。

As the value of a is exact, there is no need to emphasize the integer value of a.

I3.2 Suppose $f(x) = x - a$ and $F(x, y) = y^2 + x$. If $b = F(3, f(4))$, find the value of b.

Reference: 1985 FI3.3, 1990 HI3, 2015 FI4.3

$f(x) = x - 2$

$f(4) = 4 - 2 = 2$

$b = F(3, f(4))$

$= F(3, 2)$

$= 2^2 + 3 = 7$

I3.3 The remainder when 392 is divided by a 2-digit positive integer is b.
If c is the number of such 2-digit positive integers, find the value of c.

$392 - 7 = 385 = 5\times77 = 5\times7\times11$

Possible 2-digit positive integer = 11, 35, 55 or 77

$c = 4$

I3.4 If x is a real number and d is the maximum value of the function $y=\dfrac{3x^2+3x+c}{x^2+x+1}$,
find the value of d.

$(x^2 + x + 1)y = 3x^2 + 3x + 4$

$(3 - y)x^2 + (3 - y) + (4 - y) = 0$ ⋯⋯ (*), this is a quadratic equation in x.

For any real value of x, $y=\dfrac{3x^2+3x+c}{x^2+x+1}$ is a well-defined function

$\therefore$ (*) must have real roots in x.

$\Delta = (3 - y)^2 - 4(3 - y)(4 - y) \ge 0$

$(3 - y)(3 - y - 16 + 4y) \ge 0$

$(y - 3)(3y - 13) \le 0$

$3\le y\le\dfrac{13}{3}$

d = the maximum value of $y=\dfrac{13}{3}$

Method 2

$y=\dfrac{3x^2+3x+4}{x^2+x+1}=3+\dfrac{1}{x^2+x+1}=3+\dfrac{1}{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}}$

$\left(x+\dfrac{1}{2}\right)^2+\dfrac{3}{4}\ge\dfrac{3}{4}$

$y\le 3+\dfrac{4}{3}=\dfrac{13}{3}$

d = the maximum value of $y=\dfrac{13}{3}$

Individual Event 4

I4.1 Let f(x) be a real value function that satisfies f(xy) = f(x) f(y) for all real numbers x and y and f(0) ≠ 0. Find the value of a = f(1).

Reference: 2015 FI1.3

f(0×0) = f(0) f(0)

f(0) – [f(0)]^2 = 0

f(0)[1 – f(0)] = 0

∵ f(0) ≠ 0

∴ f(0) = 1

f(0) = f(1×0) = f(1) f(0)

1 = f(1) ⇒ a = f(1) = 1

I4.2 Let $F(n)$ be a function with $F(1) = F(2) = F(3) = a$ and $F(n+1) = \frac{F(n)\cdot F(n-1)+1}{F(n-2)}$ for positive integer $n \geq 3$, find the value of $b = F(6)$.

$F(4) = F(3+1) = \frac{F(3)\cdot F(3-1)+1}{F(3-2)} = \frac{F(3)\cdot F(2)+1}{F(1)} = \frac{1\cdot 1+1}{1} = 2$

$F(5) = F(4+1) = \frac{F(4)\cdot F(4-1)+1}{F(4-2)} = \frac{F(4)\cdot F(3)+1}{F(2)} = \frac{2\cdot 1+1}{1} = 3$

$F(6) = \frac{F(5)\cdot F(4)+1}{F(3)} = \frac{3\cdot 2+1}{1} = 7$

I4.3 If $b-6$, $b-5$, $b-4$ are three roots of the equation $x^4 + rx^2 + sx + t = 0$, find the value of $c = r + t$.

Reference: 2015 FI2.4

The three roots are 1, 2 and 3. Let the fourth root be α.

$\alpha + 1 + 2 + 3 = 0 \Rightarrow \alpha = -6$

$r = -6\times1 - 6\times2 - 6\times3 + 1\times2 + 1\times3 + 2\times3 = -25$

$t = -6\times1\times2\times3 = -36$

$c = r + t = -25 - 36 = -61$

I4.4 Suppose that (x_0, y_0) is a solution of the system: $\begin{cases} xy = 6 \\ x^2y + xy^2 + x + y + c = 2 \end{cases}$.

Find the value of $d = x_0^2 + y_0^2$.

Reference: 1993 HG8, 2010 FI1.3

From (2): $xy(x + y) + x + y - 61 = 2$

$6(x + y) + (x + y) - 63 = 0$

$x + y = 9$

$d = x^2 + y^2 = (x + y)^2 - 2xy = 9^2 - 2\times6 = 69$

Sample Group Event (2009 Final Group Event 1)

SG.1 Given some triangles with side lengths a cm, 2 cm and b cm, where a and b are integers and $a \le 2 \le b$. If there are q non-congruent classes of triangles satisfying the above conditions, find the value of q.

When $a = 1$, possible $b = 2$

When $a = 2$, possible $b = 2$ or 3

$\therefore q = 3$

SG.2 Given that the equation $|x| - \frac{4}{x} = \frac{3|x|}{x}$ has k distinct real root(s), find the value of k.

When $x > 0$: $x^2 - 4 = 3x \Rightarrow x^2 - 3x - 4 = 0 \Rightarrow (x + 1)(x - 4) = 0 \Rightarrow x = 4$

When $x < 0$: $-x^2 - 4 = -3x \Rightarrow x^2 - 3x + 4 = 0$; $D = 9 - 16 < 0 \Rightarrow$ no real roots.

$k = 1$ (There is only one real root.)

SG.3 Given that x and y are non-zero real numbers satisfying the equations $\frac{\sqrt{x}}{\sqrt{y}} - \frac{\sqrt{y}}{\sqrt{x}} = \frac{7}{12}$ and $x - y = 7$. If $w = x + y$, find the value of w.

The first equation is equivalent to $\frac{x-y}{\sqrt{xy}} = \frac{7}{12} \Rightarrow \sqrt{xy} = 12 \Rightarrow xy = 144$

Sub. $y = \frac{144}{x}$ into $x - y = 7$: $x - \frac{144}{x} = 7 \Rightarrow x^2 - 7x - 144 = 0 \Rightarrow (x + 9)(x - 16) = 0$

$x = -9$ or 16; when $x = -9$, $y = -16$ (rejected $\because \sqrt{x}$ is undefined); when $x = 16$; $y = 9$

$w = 16 + 9 = 25$

Method 2 The first equation is equivalent to $\frac{x-y}{\sqrt{xy}} = \frac{7}{12} \Rightarrow \sqrt{xy} = 12 \Rightarrow xy = 144$ ⋯⋯ (1)

$\because x - y = 7$ and $x + y = w$

$\therefore x = \frac{w+7}{2}, y = \frac{w-7}{2}$

Sub. these equations into (1): $\left(\frac{w+7}{2}\right)\left(\frac{w-7}{2}\right) = 144$

$w^2 - 49 = 576 \Rightarrow w = \pm 25$

$\because$ From the given equation $\frac{\sqrt{x}}{\sqrt{y}} - \frac{\sqrt{y}}{\sqrt{x}} = \frac{7}{12}$, we know that both $x > 0$ and $y > 0$

$\therefore w = x + y = 25$ only

SG.4 Given that x and y are real numbers and $\left|x - \frac{1}{2}\right| + \sqrt{y^2 - 1} = 0$. Let $p = |x| + |y|$, find the value of p.

Reference: 2006 FI4.2 … $y^2 + 4y + 4 + \sqrt{x + y + k} = 0$. If $r = |xy|$, …

Both $\left|x - \frac{1}{2}\right|$ and $\sqrt{y^2 - 1}$ are non-negative numbers.

The sum of two non-negative numbers = 0 means each of them is zero

$x = \frac{1}{2}$, $y = \pm 1$; $p = \frac{1}{2} + 1 = \frac{3}{2}$

Group Event 1

G1.1 Find the units digit of $(2^{13} + 1)(2^{14} + 1)(2^{15} + 1)(2^{16} + 1)$.

$2^{10} = 1024$, $2^{11} = 2048$, $2^{12} = 4096$, $2^{13} = 8192$, $2^{14} = 16384$, $2^{15} = 32768$, $2^{16} = 65536$

$(2^{13} + 1)(2^{14} + 1)(2^{15} + 1)(2^{16} + 1) = 8193\times16385\times32769\times65537 \equiv 3\times5\times9\times7 \equiv 5 \pmod{10}$

Units digit = 5

G1.2 Find the integral part of $16 \div (0.40 + 0.41 + 0.42 + \ldots + 0.59)$.

$0.40 + 0.41 + 0.42 + \ldots + 0.59 = \frac{20}{2}\cdot(0.40 + 0.59) = 9.9$

$1 = 9.9 \div 9.9 < 16 \div 9.9 < 18 \div 9 = 2$

Integral part = 1

G1.3 Choose three digits from 1, 2, 4, 6, 7 to construct three-digit numbers. Of these three-digit numbers, how many of them are divisible by 3?

126, 246, 147, 267 are divisible by 3.

The permutations of the digits of 126, 246, 147, 267 are also divisible by 3.

Total number of such integers = $3!\times4 = 24$

G1.4 Using numbers: 1, 2, 3, 4, 5, 6 to form a six-digit number: *ABCDEF* such that *A* is divisible by 1, *AB* is divisible by 2, *ABC* is divisible by 3, *ABCD* is divisible by 4, *ABCDE* is divisible by 5, *ABCDEF* is divisible by 6. Find the greatest value of *A*.

Reference: http://www2.hkedcity.net/citizen_files/aa/gi/fh7878/public_html/Number_Theory/1234567890.pdf

$\overline{ABCDE}$ is divisible by 5 $\Rightarrow E = 5$

$(A, C) = (1, 3)$ or $(3, 1)$

$\because$ $\overline{AB}$ is divisible by 2, $\overline{ABCD}$ is divisible by 4, $\overline{ABCDEF}$ is divisible by 6

$\therefore$ B, D, F are even.

$\overline{ABC}$ is divisible by 3 $\Rightarrow 1 + B + 3$ is divisible by 3 $\Rightarrow B = 2$

$\Rightarrow (D, F) = (4, 6)$ or $(6, 4)$

$\overline{ABCD}$ is divisible by 4 $\Rightarrow$ $\overline{CD}$ is divisible by 4

When $C = 1$, $D = 6$ … (1)

When $C = 3$, $D = 6$ … (2)

$\Rightarrow F = 4$

$\therefore$ $\overline{ABCDEF} = \overline{A2C654}$

Greatest value of $A = 3$

Group Event 2

G2.1 If $4^3 + 4^r + 4^4$ is a perfect square and r is a positive integer, find the minimum value of r.

$4^3 + 4^r + 4^4 = 2^2(4^2 + 4^{r-1} + 4^3) = 2^2(80 + 4^{r-1})$

The least perfect square just bigger than 80 is $81 = 9^2$.

$4^{r-1} = 1 \Rightarrow r = 1$

$\therefore$ The minimum value of r is 1.

G2.2 Three boys B_1, B_2, B_3 and three girls G_1, G_2, G_3 are to be seated in a row according to the following rules:

1) A boy will not sit next to another boy and a girl will not sit next to another girl,
2) Boy B_1 must sit next to girl G_1

If s is the number of different such seating arrangements, find the value of s.

First, arrange the three boys in a line, there are 3! permutations.

$B_1B_2B_3$, $B_1B_3B_2$, $B_2B_1B_3$, $B_2B_3B_1$, $B_3B_1B_2$, $B_3B_2B_1$.

If B_1 sits in the middle, there are two different cases. For instance, B_2 B_1 B_3. Then the possible seating arrangements are:

B_2 G_2 B_1 G_1 B_3 G_3, B_2 G_3 B_1 G_1 B_3 G_2, G_2 B_2 G_1 B_1 G_3 B_3 or G_3 B_2 G_1 B_1 G_2 B_3,
G_3 B_2 G_2 B_1 G_1 B_3, G_2 B_2 G_3 B_1 G_1 B_3, B_2 G_1 B_1 G_3 B_3 G_2 or B_2 G_1 B_1 G_2 B_3 G_3.

If B_1 sits in the left end or right end, there are four different cases. For instance, B_2 B_3 B_1, then the possible seating arrangements are:

B_2 G_2 B_3 G_3 B_1 G_1, B_2 G_3 B_3 G_2 B_1 G_1, G_2 B_2 G_3 B_3 G_1 B_1, G_3 B_2 G_2 B_3 G_1 B_1,
B_2 G_2 B_3 G_1 B_1 G_3, B_2 G_3 B_3 G_1 B_1 G_2.

$\therefore$ Total number of seating arrangements $= 2\times8 + 4\times6 = 40$

Method 2 Label the 6 positions as

1	2	3	4	5	6

B_1 and G_1 sit next to each other, their positions can be 12, 23, 34, 45 or 56, altogether 5 ways.

B_1 and G_1 can interchange positions to G_1B_1, 2 different ways.

For the other positions, the two other boys and the two other girls can sit in $2\times2 = 4$ ways.

For instance, if B_1G_1 sit in the 2-3 positions,

1	B_1	G_1	4	5	6

then B_2, B_3, G_2, G_3 can sit in the following 4 ways:

G_2 B_1 G_1 B_2 G_3 B_3, G_3 B_1 G_1 B_2 G_1 B_3, G_2 B_1 G_1 B_3 G_3 B_2, G_3 B_1 G_1 B_3 G_2 B_2.

The total number of sitting arrangements $= 5\times2\times4 = 40$ ways

G2.3 Let $f(x)=\dfrac{x+a}{x^2+\frac{1}{2}}$, where x is a real number and the maximum value of f(x) is $\dfrac{1}{2}$ and the minimum value of f(x) is –1. If t = f(0), find the value of t.

Let $y=\dfrac{x+a}{x^2+\frac{1}{2}}=\dfrac{2x+2a}{2x^2+1}\Rightarrow 2yx^2+y=2x+2a\Rightarrow(2y)x^2-2x+(y-2a)=0$

For real values of x, $\Delta=(-2)^2-4(2y)(y-2a)\geq 0$

$1-(2y^2-4ay)\geq 0\Rightarrow 2y^2-4ay-1\leq 0$ …… (*)

Given that $-1\leq y\leq\dfrac{1}{2}\Rightarrow(y+1)(2y-1)\leq 0\Rightarrow 2y^2+y-1\leq 0$ …… (**)

(*) is equivalent to (**) $\therefore a=-\dfrac{1}{4}$

$f(x)=\dfrac{x-\frac{1}{4}}{x^2+\frac{1}{2}}$

$t=f(0)=\dfrac{-\frac{1}{4}}{\frac{1}{2}}$

$=-\dfrac{1}{2}$

G2.4 In Figure 3, ABC is an isosceles triangle with $\angle ABC=u°$, $AB=BC=a$ and $AC=b$. If the quadratic equation $ax^2-\sqrt{2}\cdot bx+a=0$ has two real roots, whose absolute difference is $\sqrt{2}$, find the value of u.

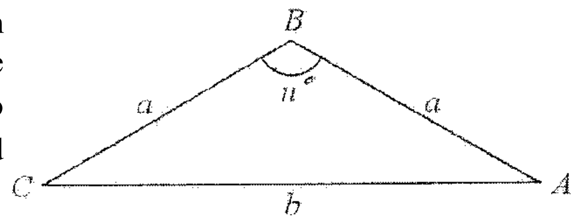


Figure 3

Let the roots be α, β.

$|\alpha-\beta|=\sqrt{2}$

$(\alpha-\beta)^2=2$

$(\alpha+\beta)^2-4\alpha\beta=2$

$\left(\dfrac{\sqrt{2}b}{a}\right)^2-4=2$

$b^2=3a^2$

$\cos u°=\dfrac{a^2+a^2-b^2}{2a^2}$

$=\dfrac{2a^2-3a^2}{2a^2}$

$=-\dfrac{1}{2}$

$u=120$

Group Event 3

G3.1 If m and n are positive integers with $m^2 - n^2 = 43$, find the value of $m^3 - n^3$.

$(m + n)(m - n) = 43$, which is a prime number.

$\begin{cases} m+n=43 \\ m-n=1 \end{cases} \Rightarrow m = 22,\ n = 21$

$m^3 - n^3 = (m - n)(m^2 + mn + n^2) = 1\times[(m + n)^2 - mn] = 43^2 - 22\times21 = 1849 - 462 = 1387$

G3.2 Let $x_1, x_2, \cdots, x_{10}$ be non-zero integers satisfying $-1 \le x_i \le 2$ for $i = 1, 2, \cdots, 10$.

If $x_1 + x_2 + \cdots + x_{10} = 11$, find the maximum possible value for $x_1^2 + x_2^2 + \cdots + x_{10}^2$.

In order to maximize $x_1^2 + x_2^2 + \cdots + x_{10}^2$, the number of "2" appeared in $x_1, x_2, \cdots, x_{10}$ must be as many as possible and the remaining numbers should be "–1".

Let the number of "2" be n and the number of "–1" be $10 - n$.

$2n - 1\times(10 - n) = 11$

$\Rightarrow n = 7$

Maximum $= 2^2 + 2^2 + 2^2 + 2^2 + 2^2 + 2^2 + 2^2 + 1 + 1 + 1 = 31$

G3.3 If $f(n) = a^n + b^n$, where n is a positive integer and $f(3) = [f(1)]^3 + f(1)$, find the value of $a\cdot b$.

$f(1) = a + b$

$f(3) = (a + b)^3 + a + b = a^3 + b^3$

$a^2 + 2ab + b^2 + 1 = a^2 - ab + b^2$

$3ab = -1$

$\Rightarrow ab = -\dfrac{1}{3}$

G3.4 In Figure 4, AD, BC and CD are tangents to the circle with centre at O and diameter $AB = 12$. If $AD = 4$, find the value of BC.

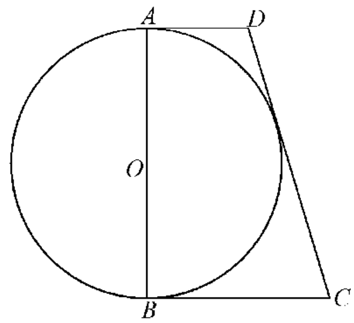


Figure 4

Suppose CD touches the circle at E. Let $BC = x$.

$DE = 4$ and $CE = x$ (tangent from ext. point)

From D, draw a line segment $DF \,//\, AB$, cutting BC at F.

$\angle DAB = \angle ABC = 90°$ (tangent $\perp$ radius)

$\angle DFC = 90°$ (corr. $\angle$s $AB \,//\, DC$)

$\therefore$ $ABFD$ is a rectangle.

$DF = 12$, $BF = 4$ (opp. sides of rectangle)

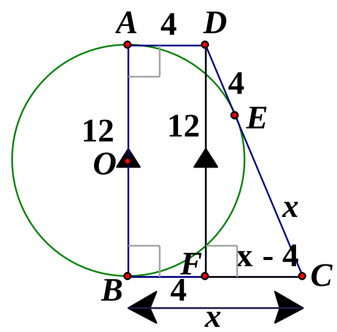
$CF = x - 4$

In ΔCDF, $(x - 4)^2 + 12^2 = (x + 4)^2$ (Pythagoras' theorem)

$x^2 - 8x + 16 + 144 = x^2 + 8x + 16$

$144 = 16x$

$BC = x = 9$



Group Event 4

G4.1 In P be the product of 3,659,893,456,789,325,678 and 342,973,489,379,256 , find the number of digits of P.

Reference: 2015 FG1.3, 2023 FG4.4

$3,659,893,456,789,325,678 = 3.7\times10^{18}$ (correct to 2 sig. fig.)

$342,973,489,379,256 = 3.4\times10^{14}$ (correct to 2 sig. fig.)

$P \approx 3.7\times10^{18}\times3.4\times10^{14} = 12.58\times10^{32} = 1.258\times10^{33}$

The number of digits is 34.

G4.2 If $\frac{1}{4}+4\left(\frac{1}{2013}+\frac{1}{x}\right)=\frac{7}{4}$, find the value of $1872+48\times\left(\frac{2013x}{x+2013}\right)$.

$4\cdot\frac{x+2013}{2013x}=\frac{3}{2}$

$\frac{2013x}{x+2013}=\frac{8}{3}$

$1872+48\times\left(\frac{2013x}{x+2013}\right)=1872+48\times\frac{8}{3}=1872+128=2000$

G4.3 The remainders of an integer when divided by 10, 9, 8, $\cdots$, 2 are 9, 8, 7, $\cdots$, 1 respectively. Find the smallest such an integer.

Reference: 1985 FG7.2, 1990 HI13

Let the integer be N.

$N + 1$ is divisible by 10, 9, 8, 7, 6, 5, 4, 3, 2.

The L.C.M. of 2, 3, 4, 5, 6, 7, 8, 9, 10 is 2520.

$\therefore N = 2520k - 1$, where k is an integer.

The least positive integral of $N = 2520 - 1 = 2519$

G4.4 In Figure 5, A, B, C, D, E represent different digits.
Find the value of $A + B + C + D + E$.

$$\begin{array}{r} ABCDE \\ \times\quad\quad 9 \\ \hline 1AAA0E \end{array}$$

Figure 5

$9E \equiv E \pmod{10} \Rightarrow E = 0$ or 5

Consider the multiplication of ten thousands digit

$9A$ + carry digit $= 10 + A \Rightarrow A = 1$ or 2

Possible products are 122205, 111105, 122200, 111100.

Of these 4 numbers, only 111105 is divisible by 9.

$\overline{ABCDE} = 111105\div9 = 12345$

$A + B + C + D + E = 1 + 2 + 3 + 4 + 5 = 15$