

14-15 Individual	1	730639	2	201499	3	22	4	232°	5	1016064
	6	32	7	126	8	1	9	13	10	$\frac{36}{55}$
14-15 Group	1	$\frac{1}{120900}$	2	$\frac{3\sqrt{3}}{2}$	3	4	4	18	5	-15
	6	140	7	$\frac{6}{23}$	8	454	9	$2\sqrt{14}$	10	$\frac{2014}{2015}$

Individual Events

- I1** How many pairs of distinct integers between 1 and 2015 inclusively have their products as multiple of 5?

Multiples of 5 are 5, 10, 15, 20, 25, 30, ..., 2015. Number = 403

Numbers which are not multiples of 5 = $2015 - 403 = 1612$

Let the first number be x , the second number be y .

Number of pairs = No. of ways of choosing any two numbers from 1 to 2015 – no. of ways of choosing such that both x, y are not multiples of 5.

$$= C_2^{2015} - C_2^{1612} = \frac{2015 \times 2014}{2} - \frac{1612 \times 1611}{2} = 403 \times \left(\frac{5 \times 2014}{2} - \frac{4 \times 1611}{2} \right)$$

$$= 403 \times (5 \times 1007 - 2 \times 1611) = 403 \times (5035 - 3222) = 403 \times 1813 = 730639$$

- I2** Given that $(10^{2015})^{-10^2} = \underbrace{0.000 \dots 01}_{n \text{ times}}$. Find the value of n .

$$10^{-201500} = \underbrace{0.000 \dots 01}_{n \text{ times}}$$

$$n = 201500 - 1 = 201499$$

- I3** Let x° be the measure of an interior angle of an n -sided regular polygon, where x is an integer, how many possible values of n are there?

If x° is an integer, then each exterior angle, $360^\circ - x^\circ$, is also an integer.

Using the fact that the sum of exterior angle of a convex polygon is 360° .

Each exterior angle = $\frac{360^\circ}{n}$, which is an integer.

$\therefore n$ must be a positive integral factor of 360.

$n = 1, 2, 3, 4, 5, 6, 8, 9, 10, 12, 15, 18, 20, 24, 30, 36, 40, 45, 60, 72, 90, 120, 180, 360$

However, $n = 1$ and $n = 2$ must be rejected because the least number of sides is 3.

\therefore There are 22 possible value of n .

- I4** As shown in the figure, $\angle EGB = 64^\circ$,

$$\angle A + \angle B + \angle C + \angle D + \angle E + \angle F = ?$$

$$\text{reflex } \angle BGF = \text{reflex } \angle CGE = 180^\circ + 64^\circ = 244^\circ$$

Consider quadrilateral $ABGF$,

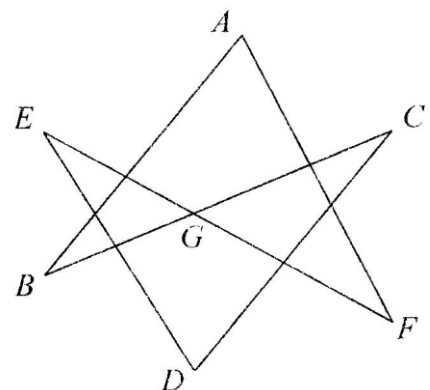
$$\angle A + \angle B + \text{reflex } \angle BGF + \angle F = 360^\circ \quad (\angle \text{ sum of polygon})$$

Consider quadrilateral $CDEG$,

$$\angle C + \angle D + \angle E + \text{reflex } \angle CGE = 360^\circ \quad (\angle \text{ sum of polygon})$$

Add these two equations,

$$\angle A + \angle B + \angle C + \angle D + \angle E + \angle F = 720^\circ - 2(244^\circ) = 232^\circ$$



- I5** It is given that $a_1, a_2, \dots, a_n, \dots$ is a sequence of positive real numbers such that $a_1 = 1$ and a_{n+1}

$$= a_n + \sqrt{a_n} + \frac{1}{4}. \text{ Find the value of } a_{2015}.$$

$$a_2 = 2 + \frac{1}{4} = \frac{9}{4}$$

$$a_3 = \frac{9}{4} + \frac{3}{2} + \frac{1}{4} = \frac{16}{4}$$

$$\text{Claim: } a_n = \frac{(n+1)^2}{4} \text{ for } n \geq 1$$

Pf: By M.I. $n = 1, 2, 3$, proved already.

$$\text{Suppose } a_k = \frac{(k+1)^2}{4} \text{ for some positive integer } k.$$

$$a_{k+1} = a_k + \sqrt{a_k} + \frac{1}{4} = \frac{(k+1)^2}{4} + \frac{k+1}{2} + \frac{1}{4} = \frac{(k+1)^2 + 2(k+1) + 1}{4} = \frac{(k+1+1)^2}{4}$$

By M.I., the statement is true for $n \geq 1$

$$a_{2015} = \frac{2016^2}{4} = 1008^2 = 1016064$$

- I6** As shown in the figure, $ABCD$ is a convex quadrilateral and $AB + BD + CD = 16$. Find the maximum area of $ABCD$.

Let $AB = a$, $BD = b$, $CD = c$, $\angle ABD = \alpha$, $\angle BDC = \beta$

Area of $ABCD$ = area of $\triangle ABD$ + area of $\triangle BCD$

$$= \frac{1}{2}ab\sin\alpha + \frac{1}{2}bc\sin\beta$$

$$\leq \frac{1}{2}ab + \frac{1}{2}bc, \text{ equality holds when } \alpha = 90^\circ, \beta = 90^\circ$$

$$= \frac{1}{2}b(a+c) = \frac{1}{2}b(16-b)$$

$$\leq \frac{1}{2}\left(\frac{b+16-b}{2}\right)^2 \text{ (A.M. } \geq \text{ G.M., equality holds when } b = 8, a + c = 8)$$

$$= 32$$

\therefore The maximum area of $ABCD = 32$

- I7** Let $x, y, z > 1$, $p > 0$, $\log_x p = 18$, $\log_y p = 21$ and $\log_{xyz} p = 9$. Find the value of $\log_z p$.

Reference: 1999 FG1.4, 2001 FG1.4

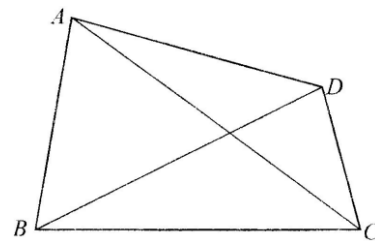
$$\frac{\log p}{\log x} = 18, \frac{\log p}{\log y} = 21, \frac{\log p}{\log xyz} = 9$$

$$\frac{\log x}{\log p} = \frac{1}{18}, \frac{\log y}{\log p} = \frac{1}{21}, \frac{\log x + \log y + \log z}{\log p} = \frac{1}{9}$$

$$\frac{\log x}{\log p} + \frac{\log y}{\log p} + \frac{\log z}{\log p} = \frac{1}{18} + \frac{1}{21} + \frac{\log z}{\log p} = \frac{1}{9}$$

$$\frac{\log z}{\log p} = \frac{1}{126}$$

$$\log_z p = \frac{\log p}{\log z} = 126$$



18 Find the value of $\frac{1}{4029} + \frac{2 \times 2014}{2014^2 + 2015^2} + \frac{4 \times 2014^3}{2014^4 + 2015^4} - \frac{8 \times 2014^7}{2014^8 - 2015^8}$.

$$\begin{aligned}
 & \frac{1}{2014+2015} + \frac{2 \times 2014}{2014^2 + 2015^2} + \frac{4 \times 2014^3}{2014^4 + 2015^4} - \frac{8 \times 2014^7}{2014^8 - 2015^8} \\
 = & -\frac{1}{2015-2014} + \frac{1}{2014+2015} + \frac{2 \times 2014}{2014^2 + 2015^2} + \frac{4 \times 2014^3}{2014^4 + 2015^4} - \frac{8 \times 2014^7}{2014^8 - 2015^8} + 1 \\
 = & -\frac{2 \times 2014}{2015^2 - 2014^2} + \frac{2 \times 2014}{2014^2 + 2015^2} + \frac{4 \times 2014^3}{2014^4 + 2015^4} - \frac{8 \times 2014^7}{2014^8 - 2015^8} + 1 \\
 = & -\frac{4 \times 2014^3}{2015^4 - 2014^4} + \frac{4 \times 2014^3}{2014^4 + 2015^4} - \frac{8 \times 2014^7}{2014^8 - 2015^8} + 1 \\
 = & -\frac{8 \times 2014^7}{2015^8 - 2014^8} + \frac{8 \times 2014^7}{2015^8 - 2014^8} + 1 = 1
 \end{aligned}$$

- 19** Let x be a real number. Find the minimum value of $\sqrt{x^2 - 4x + 13} + \sqrt{x^2 - 14x + 130}$.

Reference 2010 FG4.2, 2021 P1Q12

Consider the following problem:

Let $P(2, 3)$ and $Q(7, 9)$ be two points. $R(x, 0)$ is a variable point on x -axis.

To find the minimum sum of distances $PR + RQ$.

Let $y = \text{sum of distances} = \sqrt{(x-2)^2 + 9} + \sqrt{(x-7)^2 + 81}$

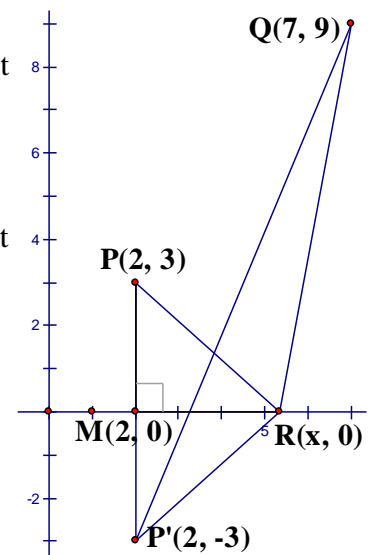
If we reflect $P(2, 3)$ along x -axis to $P'(2, -3)$, $M(2, 0)$ is the foot of perpendicular,

then $\triangle PMR \cong \triangle P'MR$ (S.A.S.)

$y = PR + RQ = P'R + RQ \geq P'Q$ (triangle inequality)

$y \geq \sqrt{(7-2)^2 + (9+3)^2} = 13$

The minimum value of $\sqrt{x^2 - 4x + 13} + \sqrt{x^2 - 14x + 130}$ is 13.



- 110** B, H and I are points on the circle. C is a point outside the circle. BC is tangent to the circle at B . HC and IC cut the circle at D and G respectively. It is given that HDC is the angle bisector of $\angle BCI$, $BC = 12$, $DC = 6$ and $GC = 9$. Find the value of $\frac{\text{area of } \triangle BDH}{\text{area of } DHIG}$.

Reference 2018 HG7

By intersecting chords theorem,

$$CH \times CD = BC^2$$

$$6 \times CH = 12^2$$

$$CH = 24$$

$$DH = 24 - 6 = 18$$

Let $\angle BCD = \theta = \angle GCD$ ($\because HDC$ is the angle bisector)

$$\frac{S_{\triangle BCD}}{S_{\triangle CDG}} = \frac{\frac{1}{2} BC \cdot CD \sin \theta}{\frac{1}{2} GC \cdot CD \sin \theta} = \frac{12}{9} = \frac{4}{3} \dots\dots (1)$$

Consider $\triangle BCD$ and $\triangle BDH$

They have the same height but different bases.

$$\frac{S_{\triangle BDH}}{S_{\triangle BCD}} = \frac{DH}{CD} = \frac{18}{6} = 3 \dots\dots (2)$$

Consider $\triangle CDG$ and $\triangle CIH$

$$\angle DCG = \angle ICH \text{ (common } \angle \text{s)}$$

$$\angle CDG = \angle CIH \text{ (ext. } \angle, \text{ cyclic quad.)}$$

$$\angle CGD = \angle CHI \text{ (ext. } \angle, \text{ cyclic quad.)}$$

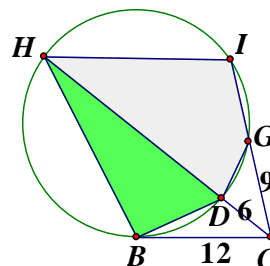
$$\therefore \triangle CDG \sim \triangle CIH \text{ (equiangular)}$$

$$\frac{S_{\triangle CIH}}{S_{\triangle CDG}} = \left(\frac{CH}{CG} \right)^2 = \left(\frac{24}{9} \right)^2 = \frac{64}{9}$$

$$\Rightarrow \frac{S_{DHIG}}{S_{\triangle CDG}} = \frac{64 - 9}{9} = \frac{55}{9}$$

$$\Rightarrow \frac{S_{\triangle CDG}}{S_{DHIG}} = \frac{9}{55} \dots\dots (3)$$

$$(1) \times (2) \times (3): \frac{\text{area of } \triangle BDH}{\text{area of } DHIG} = \frac{S_{\triangle BCD}}{S_{\triangle CDG}} \times \frac{S_{\triangle BDH}}{S_{\triangle BCD}} \times \frac{S_{\triangle CDG}}{S_{DHIG}} = \frac{4}{3} \times 3 \times \frac{9}{55} = \frac{36}{55}$$



Group Events

- G1** Find the value of $\frac{1}{1860 \times 1865} + \frac{1}{1865 \times 1870} + \frac{1}{1870 \times 1875} + \cdots + \frac{1}{2010 \times 2015}$.

Reference: 2010 HI3

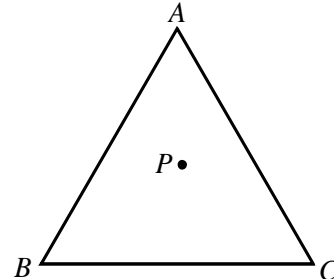
$$\begin{aligned} & \frac{1}{1860 \times 1865} + \frac{1}{1865 \times 1870} + \frac{1}{1870 \times 1875} + \cdots + \frac{1}{2010 \times 2015} \\ &= \frac{1}{25} \cdot \left(\frac{1}{372 \times 373} + \frac{1}{373 \times 374} + \frac{1}{374 \times 375} + \cdots + \frac{1}{402 \times 403} \right) \\ &= \frac{1}{25} \left[\left(\frac{1}{372} - \frac{1}{373} \right) + \left(\frac{1}{373} - \frac{1}{374} \right) + \left(\frac{1}{374} - \frac{1}{375} \right) + \cdots + \left(\frac{1}{402} - \frac{1}{403} \right) \right] \\ &= \frac{1}{25} \left(\frac{1}{372} - \frac{1}{403} \right) \\ &= \frac{31}{3747900} \\ &= \frac{1}{120900} \end{aligned}$$

- G2** Given an equilateral triangle ABC with each side of length 3 and P is an interior point of the triangle. Let PX , PY and PZ be the feet of perpendiculars from P to AB , BC and CA respectively, find the value of $PX + PY + PZ$. (**Reference 1992 HG8, 2005 HG9, 2021 P1Q6**)

Let the distance from P to AB , BC , CA be h_1 , h_2 , h_3 respectively.

$$\frac{1}{2} \cdot 3h_1 + \frac{1}{2} \cdot 3h_2 + \frac{1}{2} \cdot 3h_3 = \text{area of } \triangle ABC = \frac{1}{2} \cdot 3^2 \sin 60^\circ = \frac{9\sqrt{3}}{4}$$

$$PX + PY + PZ = h_1 + h_2 + h_3 = \frac{3\sqrt{3}}{2}$$



- G3** The coordinates of P are $(\sqrt{3} + 1, \sqrt{3} + 1)$. P is rotated 60° anticlockwise about the origin to Q . Q is then reflected along the y -axis to R . Find the value of PR^2 . (**Reference: 2007 HI10**)

Let the inclination of OP be θ .

$$\tan \theta = \frac{\sqrt{3} + 1}{\sqrt{3} + 1} = 1 \Rightarrow \theta = 45^\circ$$

$$\text{Inclination of } OQ = 45^\circ + 60^\circ = 105^\circ$$

$$\text{Angle between } OQ \text{ and positive } y\text{-axis} = 105^\circ - 90^\circ = 15^\circ$$

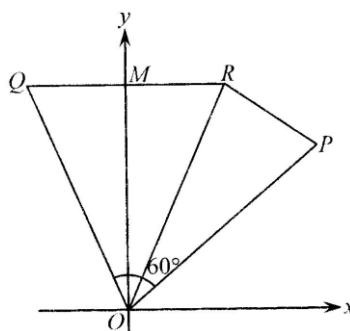
$$\therefore \text{Inclination of } OR = 90^\circ - 15^\circ = 75^\circ$$

$$\angle POR = 75^\circ - 45^\circ = 30^\circ$$

$$OP = OR = (\sqrt{3} + 1)\sqrt{1^2 + 1^2} = \sqrt{6} + \sqrt{2}$$

Apply cosine rule on $\triangle POR$

$$\begin{aligned} PR^2 &= (\sqrt{6} + \sqrt{2})^2 + (\sqrt{6} + \sqrt{2})^2 - 2(\sqrt{6} + \sqrt{2})(\sqrt{6} + \sqrt{2})\cos 30^\circ \\ &= (6 + 2 + 2\sqrt{12}) \left(2 - 2 \times \frac{\sqrt{3}}{2} \right) = (8 + 4\sqrt{3})(2 - \sqrt{3}) \\ &= 4 \end{aligned}$$



- G4** Given that $a^2 + \frac{b^2}{2} + 9 \leq ab - 3b$, where a and b are real numbers. Find the value of ab .

Reference: 2005 FI4.1, 2006 FI4.2, 2009 FG1.4, 2011 FI4.3, 2013 FI1.4, 2015 FI1.1

$$4a^2 + 2b^2 + 36 \leq 4ab - 12b$$

$$(4a^2 - 4ab + b^2) + (b^2 + 12b + 36) \leq 0$$

$$(2a - b)^2 + (b + 6)^2 \leq 0$$

$$\Rightarrow 2a - b = 0 \text{ and } b + 6 = 0$$

$$\Rightarrow b = -6 \text{ and } a = -3$$

$$ab = 18$$

- G5** Given that the equation $x^2 + 15x + 58 = 2\sqrt{x^2 + 15x + 66}$ has two real roots. Find the sum of the roots.

$$\text{Let } y = x^2 + 15x$$

$$(y + 58)^2 = 4(y + 66)$$

$$y^2 + 116y + 3364 = 4y + 264$$

$$y^2 + 112y + 3100 = 0$$

$$(y + 62)(y + 50) = 0$$

$$x^2 + 15x + 62 = 0 \text{ or } x^2 + 15x + 50 = 0$$

$$\Delta = 225 - 248 < 0 \text{ or } \Delta = 225 - 200 > 0$$

\therefore The first equation has no real roots and the second equation has two real roots

\therefore Sum of the two real roots = -15

- G6** Given that the sum of two interior angles of a triangle is n° , and the largest interior angle is 30° greater than the smallest one. Find the largest possible value of n .

Let the 3 angles of the triangle be x° , y° and $x^\circ - 30^\circ$, where $x \geq y \geq x - 30 \dots\dots (1)$

$$x + y + x - 30 = 180 \quad (\angle \text{ sum of } \Delta)$$

$$\Rightarrow y = 210 - 2x \dots\dots (2)$$

$$\text{Sub. (2) into (1): } x \geq 210 - 2x \geq x - 30$$

$$x \geq 70 \text{ and } 80 \geq x$$

$$\therefore 80 \geq x \geq 70 \dots\dots (3)$$

$$n = x + y = x + 210 - 2x \text{ by (2)}$$

$$\therefore x = 210 - n \dots\dots (4)$$

$$\text{Sub. (4) into (3): } 130 \leq n \leq 140$$

\therefore The largest possible value of $n = 140$

- G7** Four circles with radii 1 unit, 2 units, 3 units and r units are touching one another as shown in the figure. Find the value of r .

Let the centre of the smallest circle be O and the radius be r . Let

the centres of the circles with radii 2, 3, 1 be A , B and C respectively. $AB = 3 + 2 = 5$, $AC = 2 + 1 = 3$, $BC = 3 + 1 = 4$

$$AC^2 + BC^2 = 3^2 + 4^2 = 25 = AB^2$$

ΔABC is a $\perp \angle \Delta$ with $\angle C = 90^\circ$ (converse, Pythagoras' theorem)

$$OA = r + 2, OB = r + 3, OC = r + 1$$

Let the feet of \perp drawn from O to BC and AC respectively.

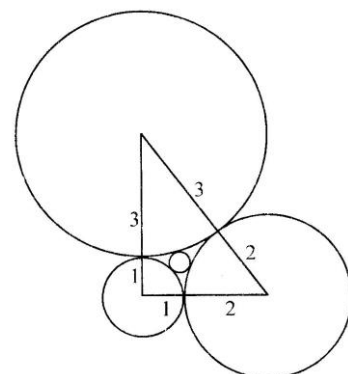
Let $CQ = x$, $CP = y$; then $AQ = 3 - x$, $BP = 4 - y$.

$$\text{In } \Delta OCQ, x^2 + y^2 = (r + 1)^2 \dots\dots (1) \text{ (Pythagoras' theorem)}$$

$$\text{In } \Delta OAQ, (3 - x)^2 + y^2 = (r + 2)^2 \dots\dots (2) \text{ (Pythagoras' theorem)}$$

$$\text{In } \Delta OBP, x^2 + (4 - y)^2 = (r + 3)^2 \dots\dots (3) \text{ (Pythagoras' theorem)}$$

$$(2) - (1): 9 - 6x = 2r + 3 \Rightarrow x = 1 - \frac{1}{3}r \dots\dots (4)$$



$$ME = OP + PI = \sqrt{\frac{10a}{3}} + \sqrt{\frac{24a}{5}} = \frac{11\sqrt{2}}{\sqrt{15}} \cdot \sqrt{a} = \frac{11\sqrt{2}}{\sqrt{15}} \cdot \frac{\sqrt{30}}{11} = 2$$

$$DI = DE - IE = 6 - \frac{30}{121} = \frac{696}{121}$$

$$PI = \sqrt{\frac{24a}{5}} = \sqrt{\frac{24}{5} \times \frac{30}{121}} = \frac{12}{11}$$

In $\triangle DPI$, $DI^2 + PI^2 = DP^2$ (Pythagoras' theorem)

$$\Rightarrow \left(\frac{696}{121}\right)^2 + \left(\frac{12}{11}\right)^2 = DP^2$$

$$DP^2 = \frac{501840}{121^2} \Rightarrow DP = \frac{\sqrt{501840}}{121}$$

Lemma 2 Given a circle C with centre at O and radius r .

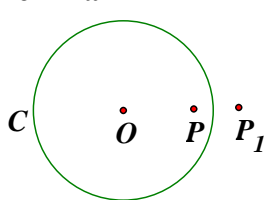


Figure 1

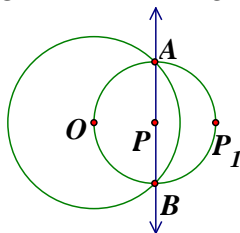


Figure 2

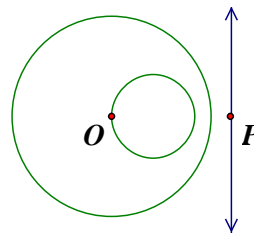


Figure 3

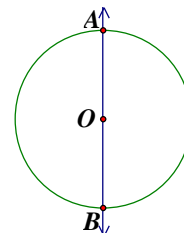


Figure 4

P and P_1 are points such that O, P, P_1 are collinear.

If $OP \times OP_1 = r^2$, then P_1 is the point of inversion of P respect to the circle C . (Figure 1)

P is also the point of inversion of P_1 . O is called the centre of inversion.

If P lies on the circumference of the circle, then $OP = r$, $OP_1 = r$, P and P_1 coincide.

If $OP < r$, then $OP_1 > r$; if $OP > r$, then $OP_1 < r$; if $OP = 0$, then $OP_1 = \infty$; $OP = \infty$, $OP_1 = 0$.

If $OP < r$ and APB is a chord, then the inversion of APB is the arc AP_1B ; the inversion of the straight line AB is the circle AP_1B which has a common chord AB . (Figure 2)

If $OP > r$, the inversion of a line (outside the given circle) is another smaller circle inside the given circle passing through the centre O . (Figure 3)

If $OP = 0$, the inversion of a line through the centre is itself, the line AB . (Figure 4)

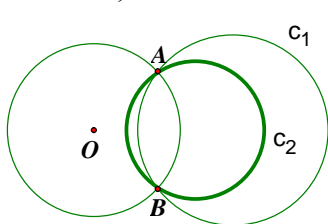


Figure 5

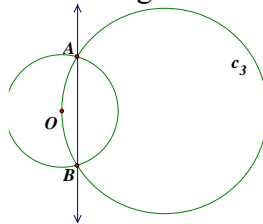


Figure 6

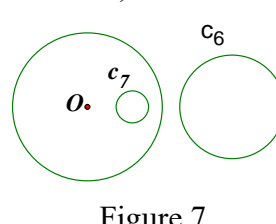


Figure 7

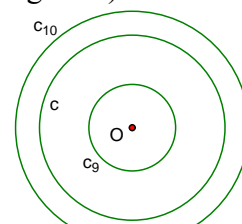


Figure 8

Given another circle C_1 which intersects the original circle at A and B , but does not pass through O . Then the inversion of C_1 with respect to the given circle is another circle C_2 passing through A and B but does not pass through O . (Figure 5)

Given another circle C_3 which intersects the original circle at A and B , and passes through O . Then the inversion of C_3 with respect to the given circle is the straight line through A and B . (Figure 6)

Given a circle C_6 outside but does not intersect the original circle. The inversion of C_6 respect to the given circle is another circle C_7 inside but does not pass through O . Conversely, the inversion of C_7 is C_6 . (Figure 7)

Given a concentric circle C_9 with the common centre O inside the given circle C . Then the inversion of C_9 is another concentric circle C_{10} outside C . Conversely, the inversion of C_{10} is C_9 . (Figure 8)

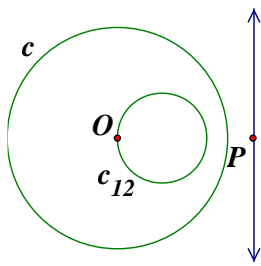


Figure 9

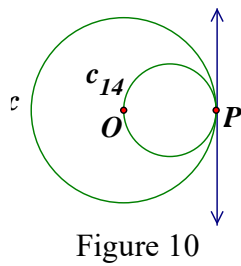


Figure 10

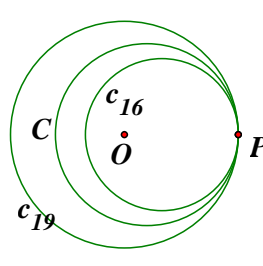


Figure 11

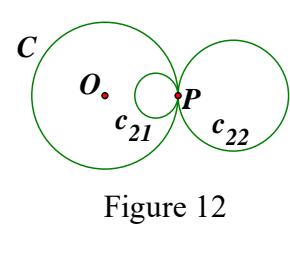


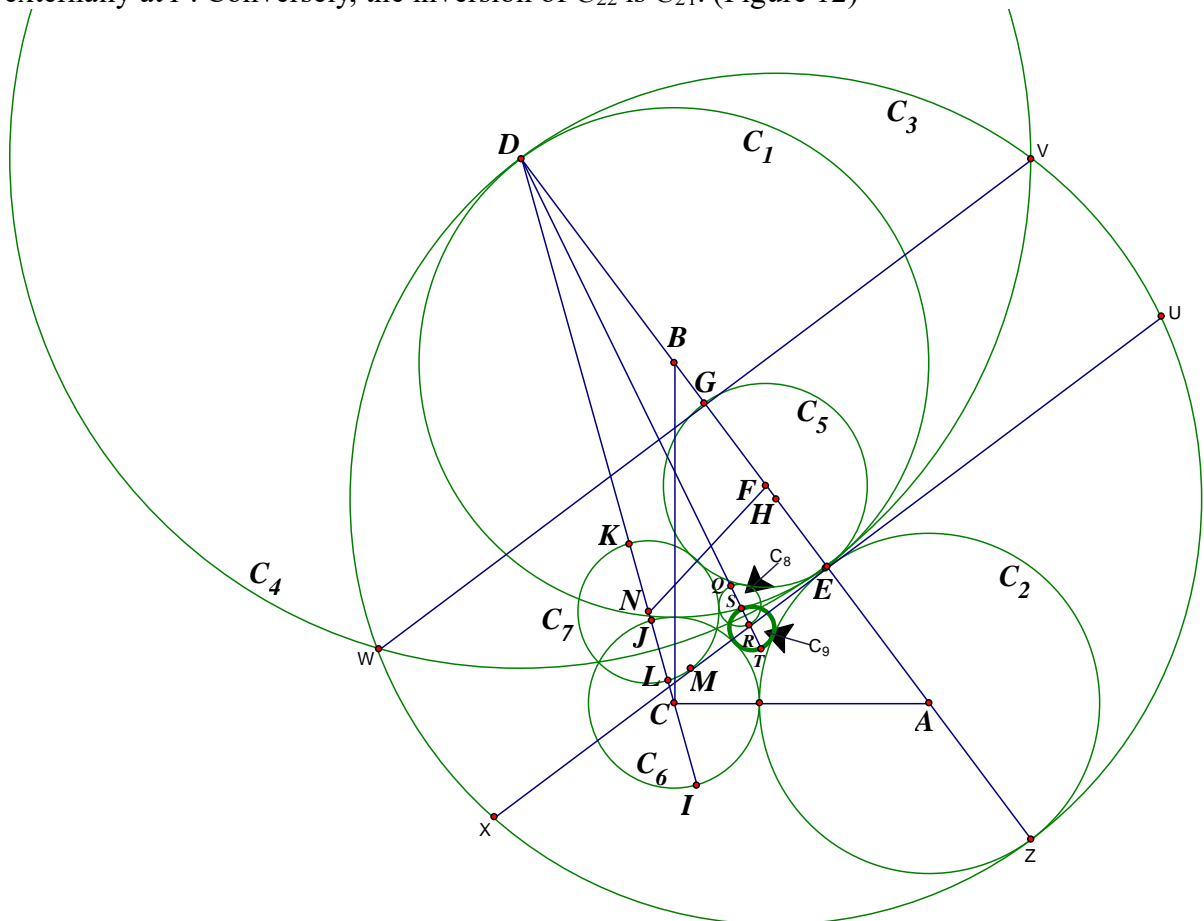
Figure 12

Given a circle C_{12} inside the given circle C but does not intersect the original circle, and passes through O . Then the inversion of C_{12} with respect to the given circle is the straight line outside C . (Figure 9)

Given a circle C_{14} inside the given circle C passes through O and touches C internally at P . Then the inversion of C_{14} with respect to the given circle is the tangent at P . Conversely, the inversion of the tangent at P is C_{14} . (Figure 10)

Given a circle C_{16} inside the given circle C which encloses O but touches C internally at P . Then the inversion of C_{16} with respect to the given circle is a circle C_{19} encloses C and touches C at P . Conversely, the inversion of C_{19} is C_{16} . (Figure 11)

Given a circle C_{21} inside the given circle C which does not enclose O but touches C internally at P . Then the inversion of C_{21} with respect to the given circle is a circle C_{22} which touches C externally at P . Conversely, the inversion of C_{22} is C_{21} . (Figure 12)



Suppose the circle C_1 with centre at B and the circle C_2 with centre at A touch each other at E . Draw a common tangent XEU . Let EZ and ED be the diameters of these two circles.

Let H be the mid-point of DZ . Use H as centre HD as radius to draw a circle C_3 .

Use D as centre, DE as radius to draw a circle C_4 . C_4 and C_3 intersect at W and V .

Join VW . VW intersects DZ at G . Let F be the mid-point of EG .

Use F as centre, FE as radius to draw a circle C_5 .

$BE = 3$, $AE = 2$ (given), $DE = 6$, $EZ = 4$, $DZ = 6 + 4 = 10$.

$$HD = HZ = HW = 5$$

Let the diameter of C_5 be x , i.e. $GE = x$, $DG = 6 - x$.

$$HG = HD - DG = 5 - (6 - x) = x - 1$$

In $\triangle DGW$, $WG^2 = 6^2 - (6 - x)^2 = 12x + x^2 \dots\dots (1)$ (Pythagoras' theorem)

In $\triangle HGW$, $WG^2 = 5^2 - (x - 1)^2 = 24 + 2x - x^2 \dots\dots (2)$ (Pythagoras' theorem)

$$(1) = (2): 24 + 2x - x^2 = 12x + x^2 \Rightarrow x = 2.4 \Rightarrow \text{The radius of } C_5 \text{ is } 1.2 = \frac{6}{5}$$

$DG \times DZ = (6 - 2.4) \times 10 = 36 = DE^2 \Rightarrow G$ is the point of inversion of Z w.r.t. C_4 .

Clearly E is the point of inversion of E w.r.t. C_4 .

\Rightarrow The inversion of C_2 w.r.t. C_4 is C_5 .

The inversion of C_1 w.r.t. C_4 is the tangent UEX (see Figure 10).

Let the circle, with centre at C and radius 1 be C_6 .

Join DC . DC cuts C_6 at J . DC is produced to cut C_6 again I . Then IJ = diameter of C_6 = 2.

In $\triangle ABC$, let $\angle ABC = \theta$, $\cos \theta = \frac{4}{5}$, $\angle CBD = 180^\circ - \theta$ (adj. \angle s on st. line)

$$\text{In } \triangle BCD, CD^2 = 3^2 + 4^2 - 2 \times 3 \times 4 \cos(180^\circ - \theta) = 25 + 24 \times \frac{4}{5} = \frac{221}{5} \Rightarrow CD = \sqrt{\frac{221}{5}}$$

$$DJ = DC - CJ = \sqrt{\frac{221}{5}} - 1; DI = DC + CI = \sqrt{\frac{221}{5}} + 1$$

Invert C_6 w.r.t. C_4 to C_7 centre at N . Suppose DI intersects C_7 at K and L in the figure.

$$DI \times DK = 6^2 \text{ and } DJ \times DL = 6^2 \Rightarrow \left(\sqrt{\frac{221}{5}} + 1 \right) DK = 36 \text{ and } \left(\sqrt{\frac{221}{5}} - 1 \right) DL = 36$$

$$\Rightarrow DK = \frac{36}{\sqrt{\frac{221}{5}} + 1} = \frac{5}{6} \left(\sqrt{\frac{221}{5}} - 1 \right) \text{ and } DL = \frac{36}{\sqrt{\frac{221}{5}} - 1} = \frac{5}{6} \left(\sqrt{\frac{221}{5}} + 1 \right)$$

$$LK = DL - DK = \frac{5}{6} \left(\sqrt{\frac{221}{5}} + 1 \right) - \frac{5}{6} \left(\sqrt{\frac{221}{5}} - 1 \right) = \frac{5}{3} \Rightarrow \text{The radius of } C_7 \text{ is } \frac{5}{6}.$$

Now construct a smaller circle C_8 centre P , touches C_5 and C_7 externally and also touches XU . P is not shown in the figure.

By the result of **Lemma 1**, the radius of C_8 is $\frac{30}{121}$; $ME = 2$ and $DP = \frac{\sqrt{501840}}{121}$

DP cuts C_8 at Q , DP is produced further to cut C_8 again at R .

$$DQ = DP - PQ = \frac{\sqrt{501840}}{121} - \frac{30}{121}; DR = DP + PR = \frac{\sqrt{501840}}{121} + \frac{30}{121}$$

Now invert C_8 w.r.t C_4 to give C_9 . This circle will touch C_1 , C_2 and C_6 externally.

DR intersects C_9 at S , produce DR further to meet C_9 again at T . Then

$$DS \times DR = 6^2 \text{ and } DT \times DQ = 6^2 \Rightarrow \left(\frac{\sqrt{501840} + 30}{121} \right) DS = 36 \text{ and } \left(\frac{\sqrt{501840} - 30}{121} \right) DT = 36$$

$$\Rightarrow DS = \frac{36 \times 121}{\sqrt{501840} + 30} = \frac{\sqrt{501840} - 30}{115}; DT = \frac{36 \times 121}{\sqrt{501840} - 30} = \frac{\sqrt{501840} + 30}{115}$$

$$ST = DT - DS = \frac{\sqrt{501840} + 30}{115} - \frac{\sqrt{501840} - 30}{115} = \frac{12}{23} = \text{diameter of } C_9$$

\therefore The radius of C_9 is $\frac{6}{23}$.

- G8** Given that a, b, x and y are non-zero integers, where $ax + by = 4$, $ax^2 + by^2 = 22$, $ax^3 + by^3 = 46$ and $ax^4 + by^4 = 178$. Find the value of $ax^5 + by^5$.

$$ax + by = 4 \dots (1), ax^2 + by^2 = 22 \dots (2), ax^3 + by^3 = 46 \dots (3), ax^4 + by^4 = 178 \dots (4).$$

$$\text{Let } ax^5 + by^5 = m \dots (5)$$

$$(x + y)(2): (x + y)(ax^2 + by^2) = 22(x + y)$$

$$ax^3 + by^3 + xy(ax + by) = 22(x + y)$$

$$\text{Sub. (1) and (3): } 46 + 4xy = 22(x + y) \Rightarrow 23 + 2xy = 11(x + y) \dots (6)$$

$$(x + y)(3): (x + y)(ax^3 + by^3) = 46(x + y)$$

$$ax^4 + by^4 + xy(ax^2 + by^2) = 46(x + y)$$

$$\text{Sub. (2) and (4): } 178 + 22xy = 46(x + y) \Rightarrow 89 + 11xy = 23(x + y) \dots (7)$$

$$11(7) - 23(6): 450 + 75xy = 0 \Rightarrow xy = -6 \dots (8)$$

$$11(6) - 2(7): 75(x + y) = 75 \Rightarrow x + y = 1 \dots (9)$$

$$(x + y)(4): (x + y)(ax^4 + by^4) = 178(x + y)$$

$$ax^5 + by^5 + xy(ax^3 + by^3) = 178(x + y)$$

$$\text{Sub. (3) and (5): } m + 46xy = 178(x + y)$$

$$\text{Sub. (8) and (9): } m + 46(-6) = 178 \times 1$$

$$\Rightarrow m = 454$$

- G9** Given that, in the figure, ABC is an equilateral triangle with $AF = 2$, $FG = 10$, $GC = 1$ and $DE = 5$. Find the value of HI . (**Reference: 2017 HG9**)

$$AF + FG + GC = 2 + 10 + 1 = 13$$

$$\therefore AB = BC = CA = 13 \text{ (property of equilateral triangle)}$$

$$\text{Let } AD = x, \text{ then } BE = 13 - 5 - x = 8 - x$$

$$\text{Let } HI = y, BH = z, \text{ then } IC = 13 - y - z$$

By intersecting chords theorem,

$$AD \times AE = AF \times AG$$

$$x(x + 5) = 2 \times 12$$

$$x^2 + 5x - 24 = 0$$

$$(x - 3)(x + 8) = 0$$

$$x = 3 \text{ or } -8 \text{ (rejected)}$$

$$BE = 8 - x = 5$$

$$BH \times BI = BE \times BD$$

$$z(z + y) = 5 \times 10 = 50 \dots (1)$$

$$CI \times CH = CG \times CF$$

$$(13 - y - z)(13 - z) = 11$$

$$169 - 13(y + 2z) + z(y + z) = 11 \dots (2)$$

$$\text{Sub. (1) into (2): } 169 - 13(y + 2z) + 50 = 11$$

$$y + 2z = 16$$

$$y = 16 - 2z \dots (3)$$

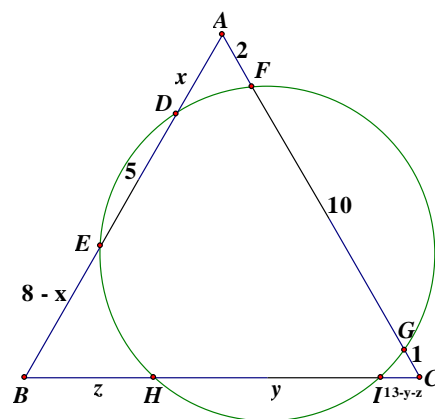
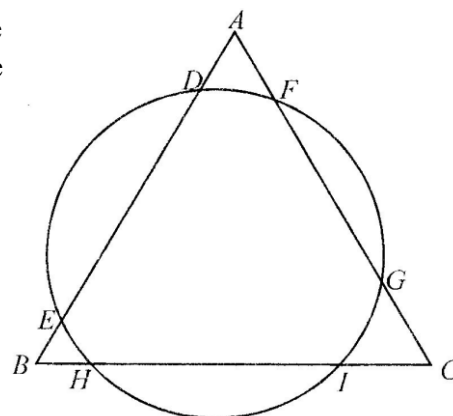
$$\text{Sub. (3) into (1): } z(z + 16 - 2z) = 50$$

$$z^2 - 16z + 50 = 0$$

$$z = 8 + \sqrt{14} \text{ or } 8 - \sqrt{14}$$

$$\text{From (3), } 2z \leq 16 \Rightarrow z \leq 8 \Rightarrow 8 + \sqrt{14} \text{ is rejected } \Rightarrow z = 8 - \sqrt{14} \text{ only}$$

$$HI = y = 16 - 2z = 16 - 2(8 - \sqrt{14}) = 2\sqrt{14}$$



G10 Let a_n and b_n be the x -intercepts of the quadratic function $y = n(n-1)x^2 - (2n-1)x + 1$, where n is an integer greater than 1. Find the value of $a_2b_2 + a_3b_3 + \dots + a_{2015}b_{2015}$.

Reference: 2005 HI6

The quadratic function can be written as $y = (nx - 1)[(n-1)x - 1]$

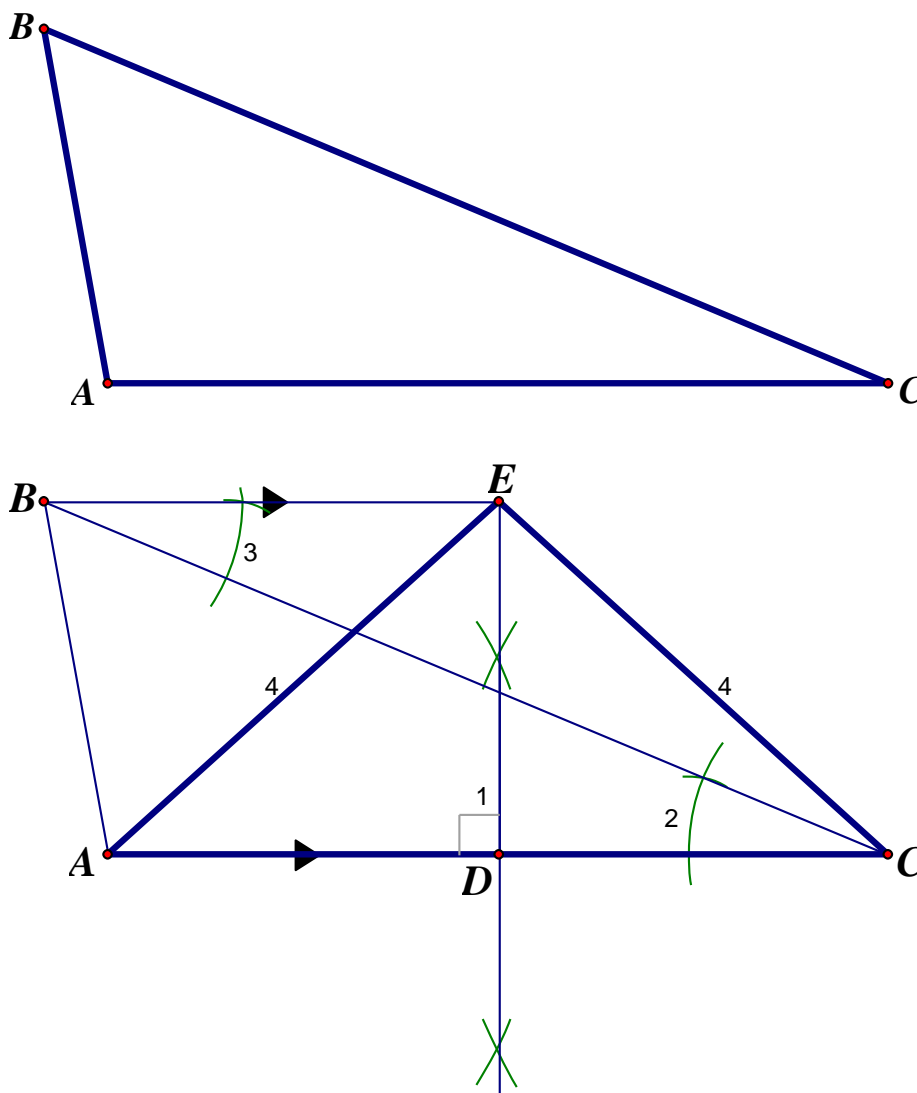
\therefore The x -intercepts are $\frac{1}{n}$ and $\frac{1}{n-1}$.

$$a_n b_n = \frac{1}{n(n-1)} = \frac{1}{n-1} - \frac{1}{n} \quad \text{for } n > 1$$

$$\begin{aligned} a_2b_2 + a_3b_3 + \dots + a_{2015}b_{2015} &= \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \dots + \left(\frac{1}{2014} - \frac{1}{2015}\right) \\ &= 1 - \frac{1}{2015} = \frac{2014}{2015} \end{aligned}$$

Geometrical Construction

1. Construct an isosceles triangle which has the same base and height to the following triangle.



Steps

- (1) Construct the perpendicular bisector of AC , D is the mid point of AC .
- (2) Copy $\angle ACB$.
- (3) Draw $\angle CBE$ so that it is equal to $\angle ACB$, then $BE \parallel AC$ (alt. \angle s eq.)
 BE and the \perp bisector in step 1 intersect at E .
- (4) Join AE , CE .

Then $\triangle AEC$ is the required isosceles triangle with $AE = CE$. Construction steps completed.

Proof: $\triangle ABC$ and $\triangle AEC$ are two triangles with the same base and the same height

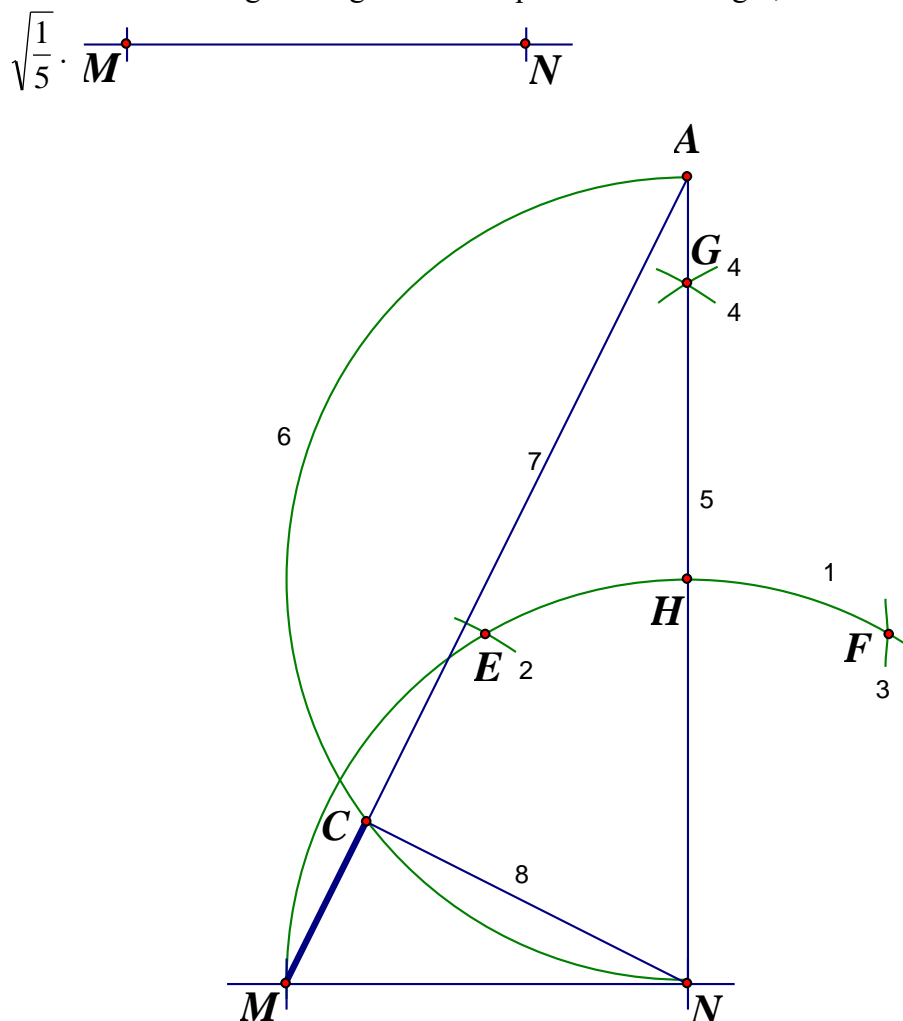
$\therefore \triangle ABC$ and $\triangle AEC$ have the same areas

$DE = DE$	(common sides)
$\angle ADE = 90^\circ = \angle CDE$	(property of perpendicular bisector)
$AD = DC$	(property of perpendicular bisector)
$\triangle ADE \cong \triangle CDE$	(S.A.S.)
$AE = CE$	(corr. sides $\cong \Delta$ s)

Then $\triangle AEC$ is the required isosceles triangle with $AE = CE$.

The proof is completed.

2. Given the following line segment MN represent a unit length, construct a line segment of length



Steps

- (1) Use N as centre, MN as radius to draw an arc.
- (2) Use M as centre, MN as radius to draw an arc, cutting the arc in step 1 at E .
- (3) Use E as centre, MN as radius to draw an arc, cutting the arc in step 1 at F .
- (4) Use E as centre, MN as radius to draw an arc. Use F as centre, MN as radius to draw an arc. The two arcs intersect at G . $\triangle EFG$ is an equilateral triangle.
- (5) Join NG and produce it longer. NG intersects the arc in step 1 at H .
- (6) Use H as centre, HN as radius to draw a semi-circle, cutting NG produced at A .
- (7) Join AM , cutting the semicircle in step (6) at C .
- (8) Join NC .

Then MC is the required length.

Proof: $\angle MNE = 60^\circ$

$$\angle FNE = 60^\circ$$

$$\angle MNG = 60^\circ + 30^\circ = 90^\circ$$

$$\angle ACN = 90^\circ$$

$$AN = 2, AM = \sqrt{1^2 + 2^2} = \sqrt{5}$$

$$\triangle CMN \sim \triangle NMA$$

$$\frac{MC}{MN} = \frac{MN}{AM}$$

$$MC = \sqrt{\frac{1}{5}}$$

($\triangle MNE$ is an equilateral triangle)

($\triangle FNE$ is an equilateral triangle)

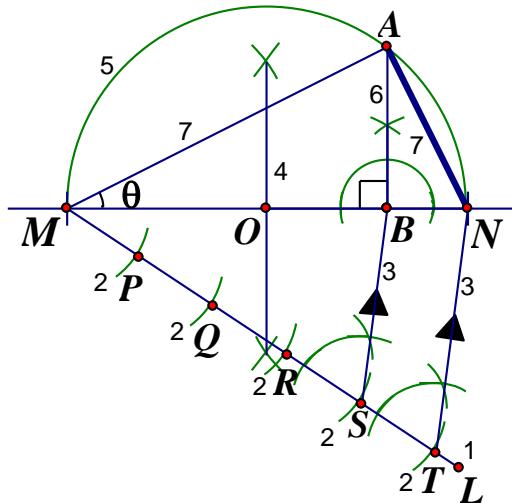
(NG is the \angle bisector of $\angle ENF$)

(\angle in semi-circle)

(Pythagoras' theorem)

(equiangular)

(cor. sides, $\sim \Delta$'s)

Method 2**Steps**

- (1) Draw a line segment ML in any direction which is not parallel to MN .
- (2) Use any radius to mark points P, Q, R, S, T on ML so that $MP = PQ = QR = RS = ST$.
- (3) Join TN , draw a line segment SB parallel to TN , cutting MN at B .
- (4) Construct the perpendicular bisector of MN , O is the mid-point of MN .
- (5) Use O as centre, OM as radius to draw a semi-circle with MN as diameter.
- (6) Through B , draw a line segment $AB \perp MN$, cutting the semi-circle in step 5 at A .
- (7) Join AM and AN .

Then $AN = \sqrt{\frac{1}{5}}$. The construction is completed.

Proof: By steps (2) and (3), $MS : ST = 4 : 1$ and $SB \parallel TN$

$$\therefore MB : BN = MS : ST = 4 : 1 \quad (\text{Theorem of equal ratios})$$

$$MB = \frac{4}{5}, BN = \frac{1}{5}$$

$$\angle MAN = 90^\circ \quad (\angle \text{ in semi-circle})$$

$$\text{Let } \angle AMN = \theta$$

$$\angle AMN = 180^\circ - 90^\circ - \theta = 90^\circ - \theta \quad (\angle \text{ sum of } \triangle AMN)$$

$$\angle BAN = \theta \quad (\text{ext } \angle \text{ of } \triangle ABN)$$

$$\frac{AB}{MB} = \frac{BN}{AB} = \tan \theta$$

$$AB^2 = MB \times BN = \frac{4}{5} \times \frac{1}{5} = \frac{4}{25}$$

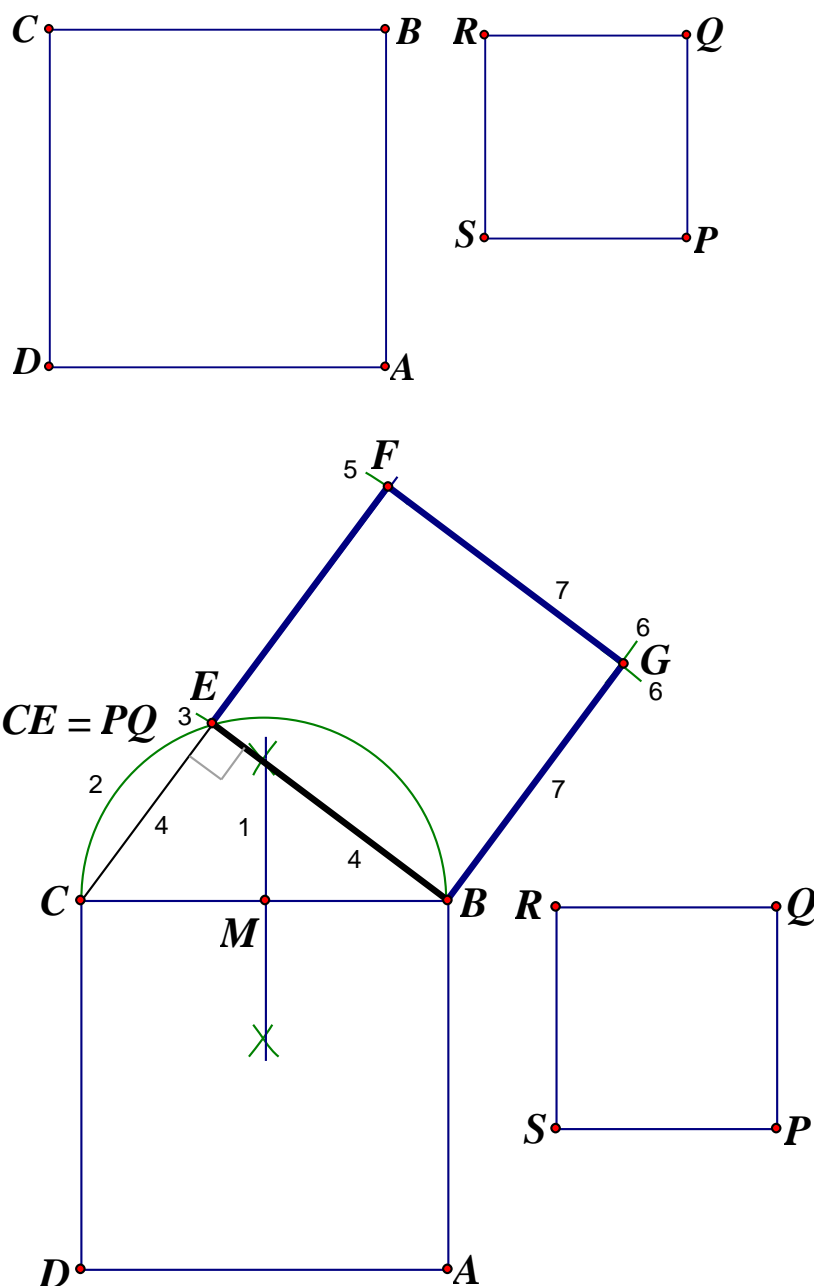
$$AB^2 + BN^2 = AN^2 \quad (\text{Pythagoras' theorem on } \triangle ABN)$$

$$AN^2 = \frac{4}{25} + \frac{1}{25} = \frac{1}{5}$$

$$AN = \sqrt{\frac{1}{5}}$$

The proof is completed

3. Construct a square whose area is equal to the difference between the areas of the following two squares $ABCD$ and $PQRS$. (Reference: 2018 HC1)



Steps

- (1) Draw the perpendicular bisector of BC , M is the mid-point of BC .
 - (2) Use M as centre, MB as radius to draw a semi-circle outside the square $ABCD$.
 - (3) Use C as centre, PQ as radius to draw an arc, cutting the semicircle in (2) at E .
 - (4) Join CE and produce it longer. Join BE .
 - (5) Use E as centre, BE as radius to draw an arc, cutting CE produced at F .
 - (6) Use B as centre, BE as radius to draw an arc. Use F as centre, FE as radius to draw an arc. The two arcs intersect at G .
 - (7) Join FG and BG .
- $\angle BEC = 90^\circ$ (\angle in semi-circle)
 $BE^2 = BC^2 - CE^2 = BC^2 - PQ^2$ (Pythagoras' theorem)
 $BEFG$ is the required square.