Individual Events

I1	a	0	I2	a	8	I3	a	8	I4	a	2
	b	–1		b	64		b	2		b	1
	c	7		c	15936		c	10		c	$\frac{3}{44}$
	d	18		d	*5312 see the remark		d	1023		d	$\frac{1945}{3872}$

Group Events

G1	a	–1	G2	A	2	G3	R	0	G4	P	*18 see the remark
	b	3		B	28		S	–1		Q	$\frac{63}{512}$
	c	$\frac{1}{9}$		C	300		T	4		R	377
	d	393		D	11		U	$-2\sqrt{3}$		S	5

Individual Event 1

I1.1 若 a 為 $\frac{1}{(x+2)(x+3)}=\frac{1}{(x+1)(x+4)}$ 的實數解的數量，求 a 的值。

If a is the number of real roots of $\frac{1}{(x+2)(x+3)}=\frac{1}{(x+1)(x+4)}$, determine the value of a.

$(x+1)(x+4)=(x+2)(x+3)$ $x^2+5x+4=x^2+5x+6$ $0=2$ 無解，$a=0$	$(x+1)(x+4)=(x+2)(x+3)$ $x^2+5x+4=x^2+5x+6$ $0=2$ No solution, $a=0$

I1.2 若 x 為實數及 b 為 $-|x-a-9|-|10-x|$ 的最大值，求 b 的值。

If x is a real number and b is the maximum value of $-|x-a-9|-|10-x|$, determine the value of b. **(Reference: 2008, HI8, 2010 HG6, 2011 FGS.1, 2012 FG2.3, 2016 FI4.3)**

我們利用三角不等式 $\lvert p\rvert+\lvert q\rvert\ge\lvert p+q\rvert$ $-\lvert x-9\rvert-\lvert 10-x\rvert=-(\lvert x-9\rvert+\lvert 10-x\rvert)$ $\le-\lvert x-9+10-x\rvert=-1$ $\therefore b=-1$	We use the triangle inequality $\lvert p\rvert+\lvert q\rvert\ge\lvert p+q\rvert$ $-\lvert x-9\rvert-\lvert 10-x\rvert=-(\lvert x-9\rvert+\lvert 10-x\rvert)$ $\le-\lvert x-9+10-x\rvert=-1$ $\therefore b=-1$

I1.3 若實數 x 及 y 滿足 $4x^2+4y^2+9xy=-119b$，求 xy 的最大值 c。

If real numbers x and y satisfy $4x^2+4y^2+9xy=-119b$, determine c, the maximum value of xy.

$4x^2+4y^2+9xy=119$

$119\ge 2\sqrt{(2x)^2\cdot(2y)^2}+9xy$ (A.M. ≥ G.M.)

$119\ge 17xy$

$7\ge xy$

$c=7$

I1.4 若正實數 x 滿足方程 $x^2+\frac{1}{x^2}=c$，求 $d=x^3+\frac{1}{x^3}$。

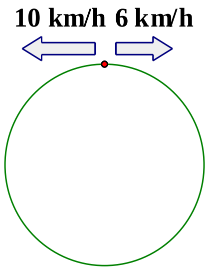
If a positive real number x satisfies $x^2+\frac{1}{x^2}=c$, determine the value of $d=x^3+\frac{1}{x^3}$.

Reference: 1985 FI1.2, 1990 HI12

$x^2+\frac{1}{x^2}=7$ $\Rightarrow\left(x+\frac{1}{x}\right)^2=9$ $\Rightarrow x+\frac{1}{x}=3$ or -3 (捨去，$\because x>0$) $d=x^3+\frac{1}{x^3}=\left(x+\frac{1}{x}\right)\left(x^2-1+\frac{1}{x^2}\right)$ $=3\times(7-1)=18$	$x^2+\frac{1}{x^2}=7$ $\Rightarrow\left(x+\frac{1}{x}\right)^2=9$ $\Rightarrow x+\frac{1}{x}=3$ or -3 (rejected, $\because x>0$) $d=x^3+\frac{1}{x^3}=\left(x+\frac{1}{x}\right)\left(x^2-1+\frac{1}{x^2}\right)$ $=3\times(7-1)=18$

Individual Event 2

I2.1 兩個學生於長 1-km 的圓形跑道的起點開始分別以 10 km/h 及 6 km/h 的速率跑沿相反方向跑步。當他們於起點再相遇時便停止跑步。若 a 為他們開始及停止前相互經過的次數，求 a 的值。



Two students run in opposite directions from a starting point of a 1-km circular track at speeds of 10 km/h and 6 km/h, respectively. They stop running when they meet each other at the starting point again. If a is number of times they cross each other after they start and before they stop, determine the value of a.

在半小時內，他們分別經過起點 5 次和 3 次。 總跑步距離= $(5 + 3)$km = 8 km 總相遇次數= $a = 8$ 他們相遇在 $\frac{5}{8},\frac{10}{8},\frac{15}{8},\frac{20}{8},\ \frac{25}{8},\frac{30}{8},\frac{35}{8},\frac{40}{8}$。	In half an hour, they will pass the starting point 5 times and 3 times respectively. Total distance travelled = $(5 + 3)$km = 8 km Number of times they meet = $a = 8$ They meet at $\frac{5}{8},\frac{10}{8},\frac{15}{8},\frac{20}{8},\ \frac{25}{8},\frac{30}{8},\frac{35}{8},\frac{40}{8}$.

I2.2 袋中有若干粒紅色及藍色的彈珠，紅色彈珠與藍色彈珠的比例為 3∶1。若加入 a 粒藍色彈珠，紅色彈珠與藍色彈珠的比例則為 2∶1。求彈珠的總數 b。

There is a set of red marbles and blue marbles. When a red marbles are added to the set, the ratio of red marbles to the blue marbles is 3 : 1. When a blue marbles are added, the ratio of red marbles to blue marbles becomes 2 : 1. Determine the total number of marbles, b.

假設原本有 $3k$ 粒紅色彈珠及 k 粒藍色彈珠。 $3k : (k + 8) = 2 : 1$ $3k = 2k + 16$ $k = 16$ 彈珠的總數= $b = 4k = 64$	Let the original number of red marbles and blue marbles be $3k$ and k respectively. $3k : (k + 8) = 2 : 1$ $3k = 2k + 16$ $k = 16$ The total number of marbles = $b = 4k = 64$

I2.3 若 c 為 1 000 000 與一個平方數之最小的相差，其中此平方數為 b 的倍數，求 c 的值。

If c is the smallest difference between 1 000 000 and a square, where the square is a multiple of b, determine the value of c.

假設該平方數為 $64n^2$。 $1000 = 8\times125$ $1\,000\,000 = 64\times125^2$ $1\,000\,000 - 64n^2 = 64\times(125^2 - n^2)$ $64\times(125^2 - 124^2)$ 或 $64\times(126^2 - 125^2)$ $64\times(125 + 124)$ 或 $64\times(126 + 125)$ 最小值= $c = 64\times249 = 15936$	Let the square be $64n^2$. $1000 = 8\times125$ $1\,000\,000 = 64\times125^2$ $1\,000\,000 - 64n^2 = 64\times(125^2 - n^2)$ $64\times(125^2 - 124^2)$ or $64\times(126^2 - 125^2)$ $64\times(125 + 124)$ or $64\times(126 + 125)$ Minimum value = $c = 64\times249 = 15936$

I2.4 於一個月的時間完成建築一個水庫需要 d 個技工或 y 個勞工，當中 $d+y=c$。若挑選 d 個勞工去建築一個同樣的水庫，所需要的時間是挑選 y 個技工的 4 倍，求 d 的值。

The building of a reservoir takes d technicians, or alternatively y labours to complete in a month, where $d+y=c$. If d labours are employed to build the same reservoir, the time taken is 4 times as much as the time taken when y technicians are employed. Determine the value of d.

$d+y=15936$ 每名技工每天工作量$=\dfrac{1}{30d}$. 每名勞工每天工作量$=\dfrac{1}{30y}$. d 名勞工完成工程所需日數$=1\div\dfrac{d}{30y}=\dfrac{30y}{d}$ y 名技工完成工程所需日數$=1\div\dfrac{y}{30d}=\dfrac{30d}{15936-d}$ $\dfrac{30(15936-d)}{d}=\dfrac{4\times 30d}{15936-d}$ $(15936-d)^2=4d^2$ $15936-d=2d$ $d=5312$	$d+y=15936$ Amount of work for one technician per day $=\dfrac{1}{30d}$. Amount of work for one labour per day $=\dfrac{1}{30y}$. Days for d labours to finish the job $=1\div\dfrac{d}{30y}=\dfrac{30y}{d}$ Days for y technicians to finish the job $=1\div\dfrac{y}{30d}=\dfrac{30d}{15936-d}$ $\dfrac{30(15936-d)}{d}=\dfrac{4\times 30d}{15936-d}$ $(15936-d)^2=4d^2$ $15936-d=2d$ $d=5312$

Remark: The Chinese version and the English version have different meaning.

Original version: …所需要的時間較挑選 y 個技工的多4倍… the time taken is 4 times as much as …

New version: …所需要的時間是挑選 y 個技工的4倍… the time taken is 4 times as much as …

Individual Event 3

I3.1 若 $\{x_0, y_0, z_0\}$ 為以下方程組的解，求 $a = x_0 + y_0 + z_0$ 的值。

If $\{x_0, y_0, z_0\}$ is a solution to the set of simultaneous equations below, determine the value of $a = x_0 + y_0 + z_0$.

$$\begin{cases} 2x-2y+z=-15 \\ x+2y+2z=18 \\ 2x-y+2z=-5 \end{cases}$$

$$\left[\begin{array}{ccc|c} 2 & -2 & 1 & -15 \\ 1 & 2 & 2 & 18 \\ 2 & -1 & 2 & -5 \end{array}\right] \begin{array}{c} R_2 \to R_1 \\ R_3 - R_1 \\ 2R_2 - R_3 \end{array} \sim \left[\begin{array}{ccc|c} 1 & 2 & 2 & 18 \\ 0 & 1 & 1 & 10 \\ 0 & 5 & 2 & 41 \end{array}\right] \begin{array}{c} R_1 - 2R_2 \\ \\ 5R_2 - R_3 \end{array} \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & -2 \\ 0 & 1 & 1 & 10 \\ 0 & 0 & 3 & 9 \end{array}\right] \begin{array}{c} \\ R_2 - \frac{1}{3}R_3 \\ \frac{1}{3}R_3 \end{array} \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

$x_0 = -2, y_0 = 7, z_0 = 3$

$a = -2 + 7 + 3 = 8$

I3.2 求 $b = \dfrac{\sqrt{6+2\sqrt{a}}+\sqrt{6-2\sqrt{a}}}{2}$ 的值。

Determine the value of $b = \dfrac{\sqrt{6+2\sqrt{a}}+\sqrt{6-2\sqrt{a}}}{2}$.

Reference: 2011 HI7, 2013 HI1, 2013 FI3.1, 2015 FG3.1, 2016 FG4.3

$$b = \frac{\sqrt{6+2\sqrt{8}}+\sqrt{6-2\sqrt{8}}}{2}$$

$$= \frac{\sqrt{\left(\sqrt{4}\right)^2+2\sqrt{4}\cdot\sqrt{2}+\left(\sqrt{2}\right)^2}+\sqrt{\left(\sqrt{4}\right)^2-2\sqrt{4}\cdot\sqrt{2}+\left(\sqrt{2}\right)^2}}{2}$$

$$= \frac{\sqrt{\left(\sqrt{4}+\sqrt{2}\right)^2}+\sqrt{\left(\sqrt{4}-\sqrt{2}\right)^2}}{2}$$

$$= \frac{\sqrt{4}+\sqrt{2}+\sqrt{4}-\sqrt{2}}{2} = 2$$

I3.3 若 x 是正整數且 $\log_{10} b^x > 3$，求 x 的最小值 c。

If x is a positive integer and $\log_{10} b^x > 3$, determine c, the minimum value of x.

$\log_{10} 2^x > 3 = \log_{10} 1000$

$2^9 = 512 < 1000 < 1024 = 2^{10}$

$c = 10$

I3.4 若 $f(x) = 2^0 + 2^1 + 2^2 + \cdots + 2^{x-2} + 2^{x-1}$，求 $d = f(c)$ 的值。

If $f(x) = 2^0 + 2^1 + 2^2 + \cdots + 2^{x-2} + 2^{x-1}$, determine the value of $d = f(c)$.

Reference: 2009 FI1.3, 2015 FI1.4

$d = f(10) = 2^0 + 2^1 + 2^2 + \cdots + 2^8 + 2^9$

$= 2^{10} - 1$

$= 1023$

Individual Event 4

I4.1 若 a 為正整數，求 a 的最大值使得 $ax^2 - (a-3)x + (a-2) = 0$ 有實根。

If a is a positive integer, determine the greatest value of a such that $ax^2 - (a-3)x + (a-2) = 0$ has real root(s).

$\Delta = (a-3)^2 - 4a(a-2) \ge 0$ $a^2 - 6a + 9 - 4a^2 + 8a \ge 0$ $3a^2 - 2a - 9 \le 0$ Let $3a^2 - 2a - 9 = 0$ $a = \frac{2 \pm \sqrt{2^2 + 4(3)(9)}}{2 \cdot 3}$ $a = \frac{1 \pm \sqrt{28}}{3}$ $a \approx \frac{1 \pm 5.3}{3} = 2.1$ or -1.4 $-1.4 \le a \le 2.1$ a 的最大整數值$= 2$。	$\Delta = (a-3)^2 - 4a(a-2) \ge 0$ $a^2 - 6a + 9 - 4a^2 + 8a \ge 0$ $3a^2 - 2a - 9 \le 0$ Let $3a^2 - 2a - 9 = 0$ $a = \frac{2 \pm \sqrt{2^2 + 4(3)(9)}}{2 \cdot 3}$ $a = \frac{1 \pm \sqrt{28}}{3}$ $a \approx \frac{1 \pm 5.3}{3} = 2.1$ 或 -1.4 $-1.4 \le a \le 2.1$ The largest integral value of $a = 2$.

I4.2 若 x 及 y 為實數且 $1 < y < x$ 及 $\log_x y + 3\log_y x = \frac{13}{a}$，求 $b = \frac{x + y^4}{x^2 + y^2}$ 的值。

If x and y are real numbers with $1 < y < x$ and $\log_x y + 3\log_y x = \frac{13}{a}$,

determine the value of $b = \frac{x + y^4}{x^2 + y^2}$.

設 $t = \log_x y$, 則 $\log_y x = \frac{1}{t}$ 原式變成： $t + \frac{3}{t} = \frac{13}{2}$ $2t^2 - 13t + 6 = 0$ $(2t-1)(t-6) = 0$ $t = \frac{1}{2}$ 或 $t = 6$ $\because 1 < y < x \therefore \log_x y < 1$ $\log_x y = \frac{1}{2}$ $y = \sqrt{x}$ $b = \frac{x + y^4}{x^2 + y^2} = \frac{x + x^2}{x^2 + x} = 1$	Let $t = \log_x y$, then $\log_y x = \frac{1}{t}$ The equation becomes: $t + \frac{3}{t} = \frac{13}{2}$ $2t^2 - 13t + 6 = 0$ $(2t-1)(t-6) = 0$ $t = \frac{1}{2}$ or $t = 6$ $\because 1 < y < x \therefore \log_x y < 1$ $\log_x y = \frac{1}{2}$ $y = \sqrt{x}$ $b = \frac{x + y^4}{x^2 + y^2} = \frac{x + x^2}{x^2 + x} = 1$

I4.3 一個袋中有紅球 $b+2$ 個，白球 $b+3$ 個及藍球 $b+4$ 個，從袋中隨機抽出 3 個並不重新放進袋中。求三個抽出的球都是相同顏色的概率 c 的值。

A bag contains $b+2$ red balls, $b+3$ white balls and $b+4$ blue balls. Three balls are randomly drawn from the bag without replacement. Determine the value of the probability, c, that the 3 balls are of the same colours.

紅球 $=3$、白球 $=4$、藍球 $=5$ P(同色) $=\dfrac{3\times2\times1+4\times3\times2+5\times4\times3}{12\times11\times10}$ $=\dfrac{3}{44}$	Red balls = 3, White balls = 4, Blue balls = 5 P(same colour) $=\dfrac{3\times2\times1+4\times3\times2+5\times4\times3}{12\times11\times10}$ $=\dfrac{3}{44}$

I4.4 若 $\cos 2\theta = c$，求 $d=\sin^4\theta+\cos^4\theta$ 的值。

If $\cos 2\theta = c$, determine the value of $d=\sin^4\theta+\cos^4\theta$.

$d=(\cos^2\theta+\sin^2\theta)^2-2\sin^2\theta\cos^2\theta$

$d=1-0.5\sin^2 2\theta$

$d=1-\frac{1}{2}\left(1-\cos^2 2\theta\right)$

$d=\frac{1}{2}\left(1+\frac{3^2}{44^2}\right)$

$d=\frac{1936+9}{2\cdot1936}$

$d=\frac{1945}{3872}$

Group Event 1

G1.1 若實數 x、y 及 z 滿足 $x+\frac{1}{y}=-1$，$y+\frac{1}{z}=-2$ 及 $z+\frac{1}{x}=-5$。求 $a=\frac{1}{xyz}$ 的值。

If real numbers x, y and z satisfy $x+\frac{1}{y}=-1$, $y+\frac{1}{z}=-2$ and $z+\frac{1}{x}=-5$. Determine the value of $a=\frac{1}{xyz}$. **(Reference: 2008 FG2.4, 2010 FG2.2)**

(1)×(2)×(3) – (1) – (2) – (3):

$$xyz+x+y+z+\frac{1}{x}+\frac{1}{y}+\frac{1}{z}+\frac{1}{xyz}-\left(x+y+z+\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)=-10+8\Rightarrow xyz+\frac{1}{xyz}=-2$$

$(xyz+1)^2=0$

$a=\frac{1}{xyz}=-1$

G1.2 若 $\left|x-|2x-1|\right|=\frac{1}{2}$ 為實數方程，求實根數量 b 的值。

If $\left|x-|2x-1|\right|=\frac{1}{2}$ is a real equation, determine the value of b, the number of real solutions of the equation.

Reference: 2012 FG4.3, 2005 FG4.2, 2009 HG9, 2012 FG4.2

$x-\lvert 2x-1\rvert=\frac{1}{2}$ 或 $x-\lvert 2x-1\rvert=-\frac{1}{2}$	$x-\lvert 2x-1\rvert=\frac{1}{2}$ or $x-\lvert 2x-1\rvert=-\frac{1}{2}$
$x-\frac{1}{2}=\lvert 2x-1\rvert$ 或 $x+\frac{1}{2}=\lvert 2x-1\rvert$	$x-\frac{1}{2}=\lvert 2x-1\rvert$ or $x+\frac{1}{2}=\lvert 2x-1\rvert$
$2x-1=\pm\left(x-\frac{1}{2}\right)$ 或 $2x-1=\pm\left(x+\frac{1}{2}\right)$	$2x-1=\pm\left(x-\frac{1}{2}\right)$ or $2x-1=\pm\left(x+\frac{1}{2}\right)$
$x=\frac{1}{2}$ 或 $x=\frac{3}{2}$, $\frac{1}{6}$	$x=\frac{1}{2}$ or $x=\frac{3}{2}$, $\frac{1}{6}$
b =實根數量= 3	b = number of real solution = 3

G1.3 若實數 x 及 y 滿足 $xy>0$ 及 $x+y=3$，求 $\left(1-\frac{1}{x}\right)\left(1-\frac{1}{y}\right)$ 的最大值 c。

If real numbers x and y satisfy $xy > 0$ and $x + y = 3$,

find c, the maximum value of $\left(1-\frac{1}{x}\right)\left(1-\frac{1}{y}\right)$.

$\because\ xy>0,\ x+y=3$ $\therefore\ x>0,\ y>0$ $0<xy\le\left(\frac{x+y}{2}\right)^2=\frac{9}{4}$ (A.M. ≥ G.M.) $0<xy\le 1$ 或 $1\le xy\le\frac{9}{4}$ $\frac{1}{xy}\ge 1$ 或 $\frac{4}{9}\le\frac{1}{xy}\le 1\Rightarrow\frac{4}{9}\le\frac{1}{xy}$ $-\frac{4}{9}\ge-\frac{1}{xy}$ $\left(1-\frac{1}{x}\right)\left(1-\frac{1}{y}\right)=\frac{xy-(x+y)+1}{xy}=1-\frac{2}{xy}\le\frac{1}{9}=c$	$\because\ xy>0,\ x+y=3$ $\therefore\ x>0,\ y>0$ $0<xy\le\left(\frac{x+y}{2}\right)^2=\frac{9}{4}$ (A.M. ≥ G.M.) $0<xy\le 1$ or $1\le xy\le\frac{9}{4}$ $\frac{1}{xy}\ge 1$ or $\frac{4}{9}\le\frac{1}{xy}\le 1\Rightarrow\frac{4}{9}\le\frac{1}{xy}$ $-\frac{4}{9}\ge-\frac{1}{xy}$ $\left(1-\frac{1}{x}\right)\left(1-\frac{1}{y}\right)=\frac{xy-(x+y)+1}{xy}=1-\frac{2}{xy}\le\frac{1}{9}=c$

G1.4 若實數 x 滿足 $x-\frac{1}{x}=3$，求 $d=x^5-\frac{1}{x^5}$ 的值。

If a real number x satisfies $x-\frac{1}{x}=3$, determine the value of $d=x^5-\frac{1}{x^5}$.

$$\left(x-\frac{1}{x}\right)^2=3^2=9$$

$$x^2+\frac{1}{x^2}=11$$

$$\left(x^2+\frac{1}{x^2}\right)^2=11^2=121$$

$$x^4+\frac{1}{x^4}=119$$

$$d=x^5-\frac{1}{x^5}=\left(x-\frac{1}{x}\right)\left(x^4+x^2+1+\frac{1}{x^2}+\frac{1}{x^4}\right)$$

$d = 3\times(119 + 11 + 1)$

$d = 393$

Group Event 2

G2.1 在六進制中，若 A 為 $123456_6 \div 13_6$ 的餘數，求 A 的值。

In base-6 system, if $123456_6 \div 13_6$ has remainder A, determine the value of A.

$123456_6 = 6^4 + 2\times6^3 + 3\times6^2 + 4\times6 + 5 = 1296 + 2\times216 + 108 + 24 + 5 = 1865_{10}$

$13_6 = 9_{10}$

$1865 \div 9$, $A = 2$

G2.2 立方體的任意兩個頂點可相連成一線段。若 B 為最多所能夠相連成的直線的數量，求 B 的值。

Any two vertices in a cube can form a line segment. If B is the greatest number of line segments thus formed, determine the value of B.

立方體有 8 個頂點。 從中選取兩點形成一線段。 $B = C^8_2 = 28$	There are 8 vertices in a cube. Select any two vertices to form a line segment. $B = C^8_2 = 28$

G2.3 若實數 x、y 及 z 滿足 $(x + y + z) = 30$ 及 $C = x^2 + y^2 + z^2$，求 C 的最小值。

If real numbers x, y and z satisfy $(x + y + z) = 30$ and $C = x^2 + y^2 + z^2$, determine the least value of C.

考慮 $t^2 - 2xt + x^2 = (t - x)^2 \cdots (1)$ $t^2 - 2yt + y^2 = (t - y)^2 \cdots (2)$ $t^2 - 2zt + z^2 = (t - z)^2 \cdots (3)$ $(1) + (2) + (3)$: L.H.S. $= 3t^2 - 2(x + y + z)t + (x^2 + y^2 + z^2)$ 此函數必為非負 $\Delta = 4(x + y + z)^2 - 4(3)(x^2 + y^2 + z^2) \le 0$ $(1^2 + 1^2 + 1^2)(x^2 + y^2 + z^2) \ge (x + y + z)^2$ $3C \ge 30^2$ $C \ge 300$ C 的最小值 $= 300$	Consider $t^2 - 2xt + x^2 = (t - x)^2 \cdots (1)$ $t^2 - 2yt + y^2 = (t - y)^2 \cdots (2)$ $t^2 - 2zt + z^2 = (t - z)^2 \cdots (3)$ $(1) + (2) + (3)$: L.H.S. $= 3t^2 - 2(x + y + z)t + (x^2 + y^2 + z^2)$ The function is always non-negative $\Delta = 4(x + y + z)^2 - 4(3)(x^2 + y^2 + z^2) \le 0$ $(1^2 + 1^2 + 1^2)(x^2 + y^2 + z^2) \ge (x + y + z)^2$ $3C \ge 30^2$ $C \ge 300$ The minimum value of $C = 300$

G2.4 已知 $D = (x - 1)^3 + 3$。當 $-3 \le x \le 3$，求 D 的最大值。

Given that $D = (x - 1)^3 + 3$. Determine the greatest value of D for $-3 \le x \le 3$.

$-4 \le x - 1 \le 2$ $-64 \le (x - 1)^3 \le 8$ $-61 \le (x - 1)^3 + 3 \le 11$ D 的最大值 $= 11$	$-4 \le x - 1 \le 2$ $-64 \le (x - 1)^3 \le 8$ $-61 \le (x - 1)^3 + 3 \le 11$ The greatest value of $D = 11$

Group Event 3

G3.1 設 a、b 及 c 為整數且 $1 < a < b < c$。若 $(ab-1)(bc-1)(ac-1)$可被 abc 整除，求 $ab+bc+ac-1$ 除以 abc 所得之餘數 R 的值。

Let a, b and c be integers with $1 < a < b < c$. If $(ab-1)(bc-1)(ac-1)$ is divisible by abc, determine the value of the remainder R when $ab+bc+ac-1$ is divided by abc.

$(ab-1)(bc-1)(ac-1)$ $= (abc)^2 - abc(a+b+c) + (ab+bc+ca) - 1$ 它可被abc 整除。 $\therefore ab+bc+ca-1 = mabc$，$m$為整數。 餘數$R = 0$	$(ab-1)(bc-1)(ac-1)$ $= (abc)^2 - abc(a+b+c) + (ab+bc+ca) - 1$ It is divisible by abc. $\therefore ab+bc+ca-1 = mabc$, m is an integer The remainder $R = 0$

G3.2 若 $0 < x < 1$，求 $S=\left(\frac{\sqrt{1+x}}{\sqrt{1+x}-\sqrt{1-x}}+\frac{1-x}{\sqrt{1-x^2}+x-1}\right)\cdot\left(\sqrt{\frac{1}{x^2}-1}-\frac{1}{x}\right)$ 的值。

If $0 < x < 1$, determine the value of $S=\left(\frac{\sqrt{1+x}}{\sqrt{1+x}-\sqrt{1-x}}+\frac{1-x}{\sqrt{1-x^2}+x-1}\right)\cdot\left(\sqrt{\frac{1}{x^2}-1}-\frac{1}{x}\right)$.

Reference: 2016 FI3.3

$$S=\left(\frac{\sqrt{1+x}}{\sqrt{1+x}-\sqrt{1-x}}+\frac{1-x}{\sqrt{1-x^2}+x-1}\right)\times\left(\sqrt{\frac{1}{x^2}-1}-\frac{1}{x}\right)$$
$$=\left\{\frac{\sqrt{1+x}\cdot\left(\sqrt{1+x}+\sqrt{1-x}\right)}{(1+x)-(1-x)}+\frac{(1-x)\cdot\left[\sqrt{1-x^2}-(x-1)\right]}{\left(1-x^2\right)-(x-1)^2}\right\}\times\left(\sqrt{\frac{1-x^2}{x^2}}-\frac{1}{x}\right)$$
$$=\left\{\frac{1+x+\sqrt{1-x^2}}{2x}+\frac{(1-x)\cdot\left[\sqrt{1-x^2}+(1-x)\right]}{\left(1-x^2\right)-\left(1-2x+x^2\right)}\right\}\times\left(\frac{\sqrt{1-x^2}}{x}-\frac{1}{x}\right)$$
$$=\left\{\frac{1+x+\sqrt{1-x^2}}{2x}+\frac{(1-x)\cdot\left[\sqrt{1-x^2}+(1-x)\right]}{2x(1-x)}\right\}\times\left(\frac{\sqrt{1-x^2}-1}{x}\right)$$
$$=\left[\frac{1+x+\sqrt{1-x^2}}{2x}+\frac{\sqrt{1-x^2}+(1-x)}{2x}\right]\times\left(\frac{\sqrt{1-x^2}-1}{x}\right)$$
$$=\left(\frac{2+2\sqrt{1-x^2}}{2x}\right)\times\left(\frac{\sqrt{1-x^2}-1}{x}\right)=\left(\frac{1+\sqrt{1-x^2}}{x}\right)\times\left(\frac{\sqrt{1-x^2}-1}{x}\right)$$
$$=\frac{\left(1-x^2\right)-1}{x^2}=-1$$

Remark: You may substitute $x = 0.5$ directly to find the value of c.

G3.3 求方程 $x^4 + (x-4)^4 = 544$ 的實根之和 T 的值。

Determine the value of T, the sum of real roots of $x^4 + (x-4)^4 = 544$.

Reference: 2014 FG4.4

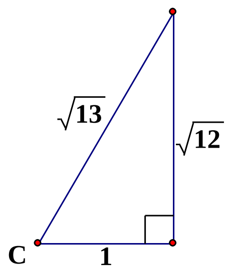
設 $t = 2-x$，則 $x = t+2$，$x-4 = t-2$ 方程變成：$(t+2)^4 + (t-2)^4 = 544$ $2[t^4 + 6(2)^2t^2 + 2^4] = 544$ $t^4 + 24t^2 - 256 = 0$ $(t^2+32)(t^2-8) = 0$ $t^2 = -32$ (捨去) or $t^2 = 8$ $x = 2\pm 2\sqrt{2}$ T=實根之和 $= 4$	Let $t = 2-x$, then $x = t+2$, $x-4 = t-2$ The equation becomes: $(t+2)^4 + (t-2)^4 = 544$ $2[t^4 + 6(2)^2t^2 + 2^4] = 544$ $t^4 + 24t^2 - 256 = 0$ $(t^2+32)(t^2-8) = 0$ $t^2 = -32$ (rejected) or $t^2 = 8$ $x = 2\pm 2\sqrt{2}$ T = Sum of real roots = 4

G3.4 在三角形 ABC 中，$BC = a$，$\angle ABC = \frac{\pi}{3}$ 及面積為 $\sqrt{3}a^2$。求 $U = \tan(\angle ACB)$的值。

In triangle ABC, $BC = a$, $\angle ABC = \frac{\pi}{3}$ and its area is $\sqrt{3}a^2$.

Determine the value of $U = \tan(\angle ACB)$.

設 $AB = c$ $\frac{1}{2}ac\sin\frac{\pi}{3} = \sqrt{3}a^2$ $c = 4a$ $BC^2 = a^2 + (4a)^2 - 2a\cdot 4a\cos\frac{\pi}{3}$ $BC = \sqrt{13}a$ $\cos\angle ACB = \frac{a^2 + \left(\sqrt{13}a\right)^2 - (4a)^2}{2\cdot a\cdot\sqrt{13}a} = -\frac{1}{\sqrt{13}}$ $U = \tan\angle ACB = -\frac{\sqrt{13-1}}{1} = -2\sqrt{3}$	Let $AB = c$ $\frac{1}{2}ac\sin\frac{\pi}{3} = \sqrt{3}a^2$ $c = 4a$ $BC^2 = a^2 + (4a)^2 - 2a\cdot 4a\cos\frac{\pi}{3}$ $BC = \sqrt{13}a$ $\cos\angle ACB = \frac{a^2 + \left(\sqrt{13}a\right)^2 - (4a)^2}{2\cdot a\cdot\sqrt{13}a} = -\frac{1}{\sqrt{13}}$ $U = \tan\angle ACB = -\frac{\sqrt{13-1}}{1} = -2\sqrt{3}$



Group Event 4

G4.1 製作某玩具，需要先倒模，後上色。甲先生每日可以為 3 件玩具倒模，或為 15 件玩具上色；乙先生每日則可以為 5 件玩具倒模，或為 15 件玩具上色。各人每日只能倒模或上色，而不能同做兩事。若甲先生和乙先生合作，求最小多少日 P 才可以製作 120 件玩具。

To make a specific toy, it must be first moulded and then painted. Mr. A can mould 3 pieces of toys or paint 15 pieces of toys in one day, whereas Mr. B can mould 5 pieces or paint 15 pieces of toys in one day. Each of them can either mould or paint toys in one day, but not both. If Mr. A and Mr. B work together, determine the least number of days P to make 120 toys.

A倒模的速度比B慢，而A上色的速度和B一樣。因此，為了要使得製作120件玩具的日數最短，B所有時間皆被指派去倒模。假設A花了x日倒模，y日上色。 $3x + 5(x + y) = 120 \Rightarrow 8x + 5y = 120$ …… (1) $15y = 120 \Rightarrow y = 8$ …… (2) 代 (2) 入 (1): $8x + 40 = 120$ $x = 10$ 最小日數$P = 18$	The speed of making moulds for A is slower than B while the speed of painting for A is the same as B. So, in order to minimize the number of days to make 120 toys, all time of B is allocated for moulding. Let A uses x days for moulding and y days for painting. $3x + 5(x + y) = 120 \Rightarrow 8x + 5y = 120$ …… (1) $15y = 120 \Rightarrow y = 8$ …… (2) Sub. (2) into (1): $8x + 40 = 120$ $x = 10$ The least number of days $P = 18$

Remark: The following sentence is missing in the Chinese version:
各人每日只能倒模或上色，而不能同做兩事。

G4.2 在一個射鴨子遊戲中一男孩射了 10 發子彈，該男孩每發子彈射中鴨子的概率為 0.5。求他於最後一發子彈射中第六隻鴨子的概率 Q。

In a duck shooting game, a boy fires 10 shots. The probability of him shooting down a duck with a shot is 0.5. Determine the probability Q of him shooting down the 6th duck at the last shot.

P(最後一發子彈射中第六隻鴨子) = P(頭9發射中5隻，第10發射中1隻) $= \frac{C_5^9}{2^9} \cdot \frac{1}{2} = \frac{63}{512}$	P(shoot down the 6th duck at the last shot) = P(1-9 shots 5 ducks, last shot 6th duck) $= \frac{C_5^9}{2^9} \cdot \frac{1}{2} = \frac{63}{512}$

G4.3 如圖 1，求按箭咀方向由 A 往 B 的路線總數 R。

As shown in Figure 1 below, determine the number of ways R getting from point A to B with the direction indicated by the arrows.

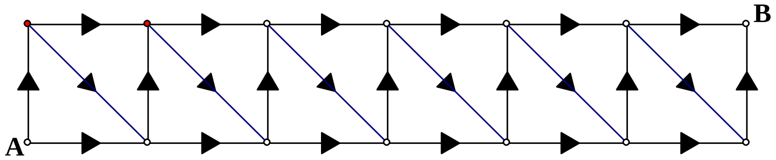
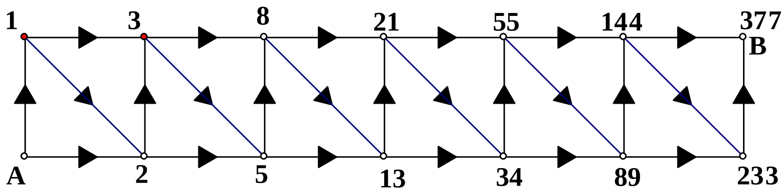


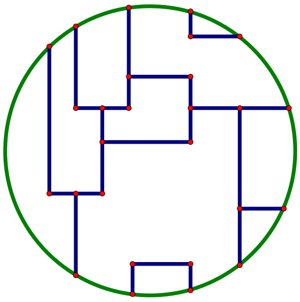
Figure 1/圖 1

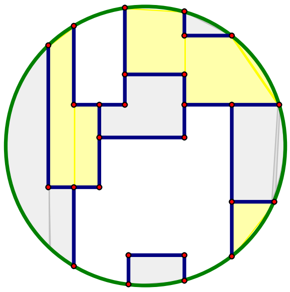


$R = 377$

G4.4 如果用 3 款顏料替下圖中所有區域著色，並且相鄰的區域不可用相同顏料。求同一款顏料最多可用作上色的區域數目 S。

To shade all the regions inside the following circular map using 3 colours, for which adjacent regions must not be in the same colour. Determine the maximum number S of regions being shaded by the same colour.





$S = 5$