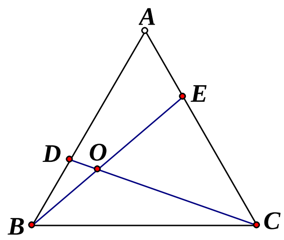
18-19 Individual	**1**	60	**2**	$24\sqrt{3}$	**3**	–4	**4**	1987	**5**	516
	6	516	**7**	$\frac{7\sqrt{5}}{3}$	**8**	8	**9**	32	**10**	4
	11	9	**12**	$5\sqrt{13}$	**13**	9	**14**	3	**15**	7
18-19 Group	**1**	1010	**2**	25	**3**	30	**4**	2	**5**	–1
	6	64	**7**	120	**8**	4	**9**	12	**10**	$25\sqrt{3}+37.5$

Individual Events

I1 在圖一中，ABC 是一個等邊三角形。D 和 E 分別是 AB 和 AC 上的點，使得 $AE = BD$。若 CD 和 BE 相交於 O 及 $\angle COE = y°$，求 y 的值。

In Figure 1, ABC is an equilateral triangle. D and E are points on AB and AC respectively such that $AE = BD$. Id CD and BE intersect at O and $\angle COE = y°$, find the value of y .



Reference: 2000 HG6

$AE = BD$ (已知)
$\angle BAE = \angle CBD = 60°$ (等邊三角形的性質)
$AB = CB$ (等邊三角形的性質)
$\therefore \Delta EAB \cong \Delta DBC$ (S.A.S.)
$\angle ABE = \angle BCD = \theta$ (全等三角形對應邊)
$\angle CBE = 60° - \theta$ (等邊三角形的性質)
$\angle COE = \angle CBE + \angle BCD$ (ΔBCO 的外角)
$= 60° - \theta + \theta = 60°$
$y = 60$

$AE = BD$ (given)
$\angle BAE = \angle CBD = 60°$ (prop. of equilateral Δ)
$AB = CB$ (prop. of equilateral Δ)
$\therefore \Delta EAB \cong \Delta DBC$ (S.A.S.)
$\angle ABE = \angle BCD = \theta$ (cor. sides $\cong \Delta$s)
$\angle CBE = 60° - \theta$ (prop. of equilateral Δ)
$\angle COE = \angle CBE + \angle BCD$ (ext. $\angle$ of ΔBCO)
$= 60° - \theta + \theta = 60°$
$y = 60$

I2 設 O 為極座標系統的極點。若 $P(6, 240°)$ 向右平移 16 單位至 Q 而 ΔOPQ 的面積為 T 平方單位，求 T 的值。

Let O be the pole of the polar coordinate system. If $P(6, 240°)$. If P is translated to the right by 16 units to Q and the area of ΔOPQ is T square units, find the value of T .

Reference: 2016 HI9

P 的直角座標為
$(6 \cos 240°, 6 \sin 240°) = (-3, -3\sqrt{3})$。
Q 的直角座標為 $(13, -3\sqrt{3})$。

$$T = \frac{1}{2}\left\|\begin{matrix} -3 & -3\sqrt{3} \\ 13 & -3\sqrt{3} \end{matrix}\right\| = \frac{1}{2}\left(9\sqrt{3} + 39\sqrt{3}\right) = 24\sqrt{3}$$

The rectangular coordinates of P is
$(6 \cos 240°, 6 \sin 240°) = (-3, -3\sqrt{3})$.
The rectangular coordinates of Q is $(13, -3\sqrt{3})$.

$$T = \frac{1}{2}\left\|\begin{matrix} -3 & -3\sqrt{3} \\ 13 & -3\sqrt{3} \end{matrix}\right\| = \frac{1}{2}\left(9\sqrt{3} + 39\sqrt{3}\right) = 24\sqrt{3}$$

I3 已知 x 及 y 均為實數，若 $y^2 - 4xy + 5x^2 - 8x + 16 = 0$ 及 $F = x - y$，求 F 的值。

Given that x and y are real numbers.
If $y^2 - 4xy + 5x^2 - 8x + 16 = 0$ and $F = x - y$, find the value of F .

Reference: 2015 HG4

$y^2 - 4xy + 4x^2 + x^2 - 8x + 16 = 0$
$(y - 2x)^2 + (x - 4)^2 = 0$
兩個平方之和 $= 0$
$\Rightarrow$ 每一項 $= 0$
$y - 2x = 0$ 及 $x = 4 \Rightarrow y = 8$
$F = x - y = 4 - 8 = -4$

$y^2 - 4xy + 4x^2 + x^2 - 8x + 16 = 0$
$(y - 2x)^2 + (x - 4)^2 = 0$
sum of two squares $= 0$
$\Rightarrow$ Each term $= 0$
$y - 2x = 0$ and $x = 4 \Rightarrow y = 8$
$F = x - y = 4 - 8 = -4$

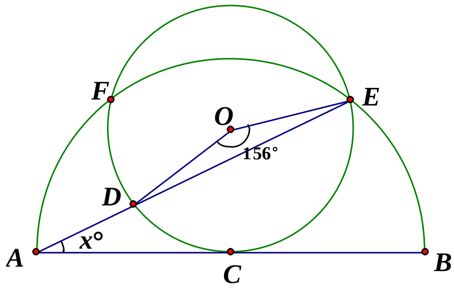
I4 設 n 為正整數。若 $a_n = 1 + 2 + \cdots + 2^n$ 及 $b = a_{10} - a_5 + a_1$，求 b 的值。

Let n be a positive integer. If $a_n = 1 + 2 + \cdots + 2^n$ and $b = a_{10} - a_5 + a_1$, find the value of b.

利用等比級數 n 項之和公式： $a_n = 2^{n+1} - 1$ 由 $n = 1, 2, 3, \cdots$ $b = a_{10} - a_5 + a_1$ $= (2^{11} - 1) - (2^6 - 1) + (1 + 2)$ $= 2048 - 64 + 3 = 1987$	By the sum to n terms of a geometric series formula, $a_n = 2^{n+1} - 1$ for $n = 1, 2, 3, \cdots$ $b = a_{10} - a_5 + a_1$ $= (2^{11} - 1) - (2^6 - 1) + (1 + 2)$ $= 2048 - 64 + 3 = 1987$

I5 在圖二中，AB 為半圓的直徑，C 為半圓的圓心。有一圓形，圓心 O 切 AB 於 C 及交半圓於 E 和 F。若 AE 交此圓形於 D、$\angle DOE = 156°$及$\angle BAE = x°$，求 x 的值。

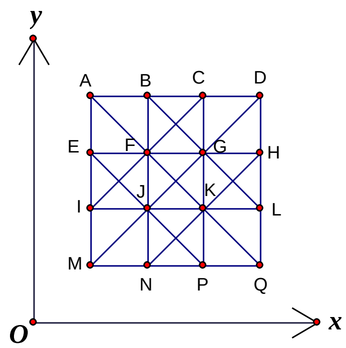
In Figure 2, AB is the diameter of the semi-circle, C is the centre of the semi-circle. A circle with centre at O, touching the semi-circle at C and cutting it at E and F. If AE cuts the circle at D, $\angle DOE = 156°$ and $\angle BAE = x°$, find the value of x.



反角 $\angle DOE = 360° - 156°$ (同頂角) $= 204°$ $\angle DCE = \frac{1}{2}$ 反角 $\angle DOE$ (圓心角兩倍於圓周角) $= 102°$ $\angle ACD = \angle AEC$ (交錯弓形的角) $\angle AEC = x°$ (等腰三角形底角) $\angle BCE = \angle CAE + \angle AEC$ (三角形外角) $= 2x°$ $\angle ACD + \angle DCE + \angle BCE = 180°$ (直綫上的鄰角) $x° + 102° + 2x° = 180°$ $x = 26$	Reflex $\angle DOE = 360° - 156°$ ($\angle$s at a pt.) $= 204°$ $\angle DCE = \frac{1}{2}$ reflex $\angle DOE$ ($\angle$ at centre twice $\angle$ at $\odot^{ce}$) $= 102°$ $\angle ACD = \angle AEC$ ($\angle$ in alt. segment) $\angle AEC = x°$ (base $\angle$s isos. Δ) $\angle BCE = \angle CAE + \angle AEC$ (ext. $\angle$ of Δ) $= 2x°$ $\angle ACD + \angle DCE + \angle BCE = 180°$ (adj. $\angle$s on st. line) $x° + 102° + 2x° = 180°$ $x = 26$

I6 在圖三中，直角座標平面上一個正方形的四個頂點的座標分別為 (1, 1)、(1, 4)、(4, 1)及(4, 4)。若在該正方形中(包括邊界)選擇任何三個座標均為整數的點，問可組成多少個三角形？

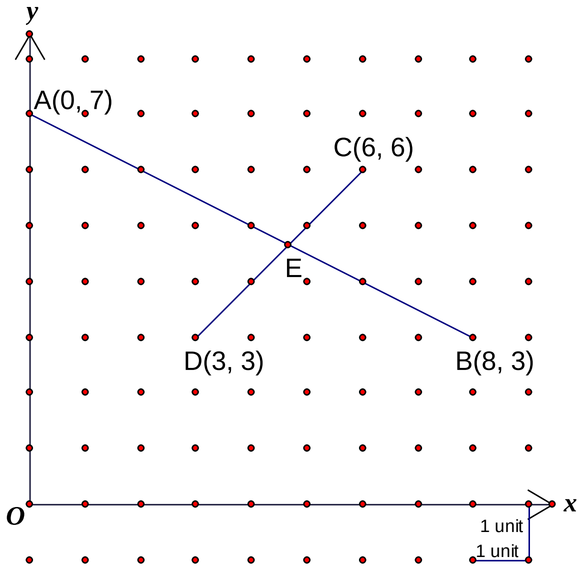
In Figure 3, the vertices of a square in the rectangular coordinate plane are (1, 1), (1, 4), (4, 1) and (4, 4). How many triangles can be formed by selecting any three points in the square (including the boundaries) with integer coordinates?



將這 16 個整數點命名如圖。 其中有 10 條綫段穿過 4 點。 另外有 4 條綫段穿過 3 點。 三角形的數目 $= C_3^{16}$ – 選中三點在同一直綫的數目 $= \frac{16 \times 15 \times 14}{1 \times 2 \times 3} - 10 \times C_3^4 - 4 \times C_3^3$ $= 560 - 40 - 4 = 516$	Label the 16 integral points as shown. There are 10 line segments passing through 4 integral points. There are 4 line segments passing through 3 integral points. Number of triangles $= C_3^{16}$ – number of choices of 3 collinear points $= \frac{16 \times 15 \times 14}{1 \times 2 \times 3} - 10 \times C_3^4 - 4 \times C_3^3$ $= 560 - 40 - 4 = 516$

I7 在圖四中，AB 與 CD 相交於 E。設 AE 的長度為 q 單位，求 q 的值。

In Figure 4, AB and CE intersect at E. Let the length of AE be q units. Find the value of q.

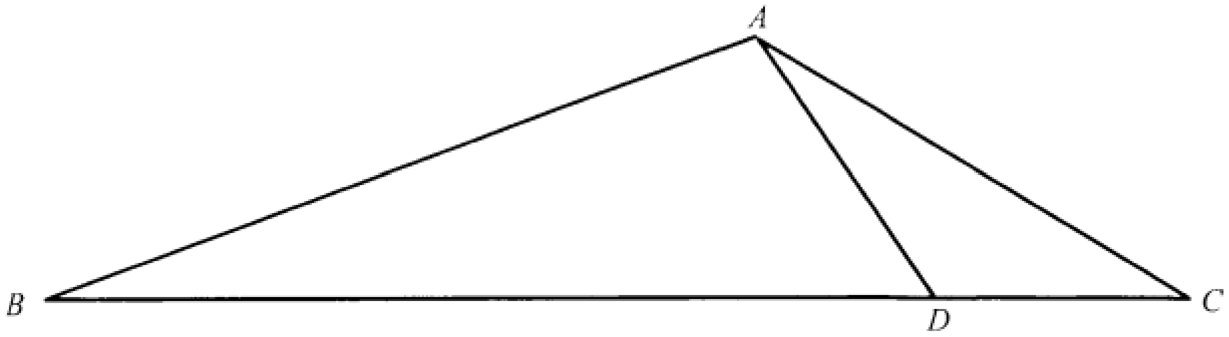


定義一個直角座標系統如圖。 A、B、C 和 D 的座標分別為(0, 7)、(8, 3)、(6, 6)及(3, 3)。 AB 的方程為：$y-7=\frac{7-3}{0-8}\cdot(x-0)$ $y=-\frac{1}{2}x+7 \cdots (1)$ CD 的方程為：$y=x \cdots (2)$ 代 (2) 入 (1): $x=-\frac{1}{2}x+7 \Rightarrow x=\frac{14}{3}=y$ $q=AE=\sqrt{\left(\frac{14}{3}-0\right)^2+\left(\frac{14}{3}-7\right)^2}=\frac{7\sqrt{5}}{3}$	Define a rectangular co-ordinates system as shown. The coordinates of A, B, C and D are (0, 7), (8, 3), (6, 6) and (3, 3) respectively. Equation of AB: $y-7=\frac{7-3}{0-8}\cdot(x-0)$ $y=-\frac{1}{2}x+7 \cdots (1)$ Equation of CD: $y=x \cdots (2)$ Sub. (2) into (1): $x=-\frac{1}{2}x+7 \Rightarrow x=\frac{14}{3}=y$ $q=AE=\sqrt{\left(\frac{14}{3}-0\right)^2+\left(\frac{14}{3}-7\right)^2}=\frac{7\sqrt{5}}{3}$

I8 在圖五中，D 是在 BC 上的一點使得 $\angle ABD = \angle CAD$ 及 $\frac{BD}{AC} = \frac{8}{3}$。若 $\frac{\Delta ABD\text{ 的面積}}{\Delta ADC\text{ 的面積}} = k$，求 k 的值。

In Figure 5, D is a point on BC such that $\angle ABD = \angle CAD$ and $\frac{BD}{AC} = \frac{8}{3}$.

If $\frac{\text{Area of }\Delta ABD}{\text{Area of }\Delta ADC} = k$, find the value of k.



$\Delta ACD \sim \Delta BCA$ (A.A.A.) $\frac{AC}{CD} = \frac{BD+DC}{AC}$ (相似三角形對應邊) $\frac{AC}{CD} = \frac{BD}{AC} + \frac{DC}{AC}$ 設 $t = \frac{AC}{CD}$，則 $\frac{1}{t} = \frac{DC}{AC}$ $t = \frac{8}{3} + \frac{1}{t}$ $3t^2 - 8t - 3 = 0$ $(3t + 1)(t - 3) = 0$ $t = -\frac{1}{3}$ (捨去) 或 $t = 3$ $CD = \frac{1}{3}AC$ $BD = \frac{8}{3}AC$ $k = \frac{\Delta ABD\text{ 的面積}}{\Delta ADC\text{ 的面積}} = \frac{BD}{CD} = 8$	$\Delta ACD \sim \Delta BCA$ (A.A.A.) $\frac{AC}{CD} = \frac{BD+DC}{AC}$ (corr. sides, ~Δs) $\frac{AC}{CD} = \frac{BD}{AC} + \frac{DC}{AC}$ Let $t = \frac{AC}{CD}$, then $\frac{1}{t} = \frac{DC}{AC}$ $t = \frac{8}{3} + \frac{1}{t}$ $3t^2 - 8t - 3 = 0$ $(3t + 1)(t - 3) = 0$ $t = -\frac{1}{3}$ (rejected) or $t = 3$ $CD = \frac{1}{3}AC$ $BD = \frac{8}{3}AC$ $k = \frac{\text{Area of }\Delta ABD}{\text{Area of }\Delta ADC} = \frac{BD}{CD} = 8$

I9 已知 α 及 β 為方程 $x^2 + 32x - 1 = 0$ 的兩個根。

若 $P = (\alpha^2 + 31\alpha - 2)(\beta^2 + 33\beta)$，求 P 的值。

Given that α and β are the two roots of the equation $x^2 + 32x - 1 = 0$.

If $P = (\alpha^2 + 31\alpha - 2)(\beta^2 + 33\beta)$, find the value of P.

Reference: 2013 HG4

$\alpha^2 + 32\alpha - 1 = 0 \Rightarrow \alpha^2 + 31\alpha - 2 = -\alpha - 1$

$\beta^2 + 32\beta - 1 = 0 \Rightarrow \beta^2 + 33\beta = \beta + 1$

$$\begin{aligned}(\alpha^2 + 31\alpha - 2)(\beta^2 + 33\beta) &= (-\alpha - 1)(\beta + 1)\\ &= -(\alpha + 1)(\beta + 1)\\ &= -(\alpha\beta + \alpha + \beta + 1)\\ &= -(-1 - 32 + 1)\\ &= 32\end{aligned}$$

$P = 32$

I10 設 $c=\sqrt[3]{7+5\sqrt{2}}+\sqrt[3]{7-5\sqrt{2}}$。若 $w=c^2$，求 w 的值。

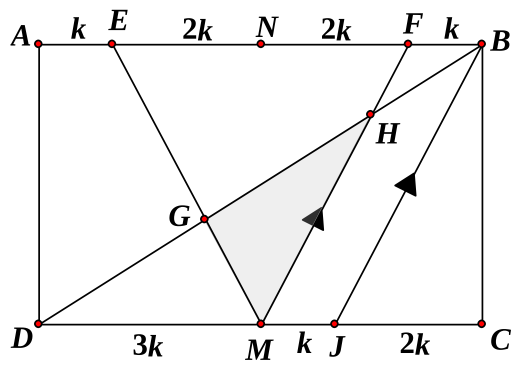
Let $c=\sqrt[3]{7+5\sqrt{2}}+\sqrt[3]{7-5\sqrt{2}}$. If $w = c^2$, find the value of w.

Reference: 1999 FI3.2, 2005 FI2.2, 2016 FG3.3

設 $\left(a+\sqrt{b}\right)^3=7+5\sqrt{2}$ $a^3+3a^2\sqrt{b}+3ab+b\sqrt{b}=7+5\sqrt{2}$ $b = 2$, $a^3 + 3ab = 7$, $3a^2 + b = 5 \Rightarrow a = 1$ $\left(1+\sqrt{2}\right)^3=7+5\sqrt{2}$ 及 $\left(1-\sqrt{2}\right)^3=7-5\sqrt{2}$ $c=1+\sqrt{2}+\left(1-\sqrt{2}\right)=2$ $w=c^2 = 4$	Let $\left(a+\sqrt{b}\right)^3=7+5\sqrt{2}$ $a^3+3a^2\sqrt{b}+3ab+b\sqrt{b}=7+5\sqrt{2}$ $b = 2$, $a^3 + 3ab = 7$, $3a^2 + b = 5 \Rightarrow a = 1$ $\left(1+\sqrt{2}\right)^3=7+5\sqrt{2}$ and $\left(1-\sqrt{2}\right)^3=7-5\sqrt{2}$ $c=1+\sqrt{2}+\left(1-\sqrt{2}\right)=2$ $w = c^2 = 4$
方法二 $c^3=7+5\sqrt{2}+3\times\sqrt[3]{\left(7+5\sqrt{2}\right)^2\left(7-5\sqrt{2}\right)}$ $+3\times\sqrt[3]{\left(7+5\sqrt{2}\right)\left(7-5\sqrt{2}\right)^2}+7-5\sqrt{2}$ $=14+3\times\sqrt[3]{\left(7+5\sqrt{2}\right)(49-50)}$ $+3\times\sqrt[3]{(49-50)\left(7-5\sqrt{2}\right)}$ $= 14 - 3c$ $c^3 + 3c - 14 = 0$ $(c - 2)(c^2 + 2c + 7) = 0$ $c = 2$ 或 沒有實數解 $w = c^2 = 4$	**Method 2** $c^3=7+5\sqrt{2}+3\times\sqrt[3]{\left(7+5\sqrt{2}\right)^2\left(7-5\sqrt{2}\right)}$ $+3\times\sqrt[3]{\left(7+5\sqrt{2}\right)\left(7-5\sqrt{2}\right)^2}+7-5\sqrt{2}$ $=14+3\times\sqrt[3]{\left(7+5\sqrt{2}\right)(49-50)}$ $+3\times\sqrt[3]{(49-50)\left(7-5\sqrt{2}\right)}$ $= 14 - 3c$ $c^3 + 3c - 14 = 0$ $(c - 2)(c^2 + 2c + 7) = 0$ $c = 2$ or no real solution $w = c^2 = 4$

I11 在圖六中，$ABCD$ 為一個長方形。M 和 N 分別是 DC 和 AB 的中點且 $AE : EN = BF : FN = 1 : 2$。$DB$ 分別交 EM 和 FM 於 G 及 H。若長方形 $ABCD$ 及三角形 GHM 的面積分別是 96 和 S，求 S 的值。

In Figure 6, $ABCD$ is rectangle M and N are the mid-points of DC and AB respectively and $AE : EN = BF : FN = 1 : 2$. DB intersects EM and FM at G and H respectively. If the areas of the rectangle $ABCD$ and the triangle GHM are 96 and S respectively, find the value of S.



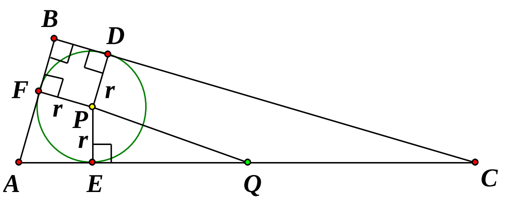
圖六 Figure 6

Reference 1998 HG5, 2016 HI14, 2018 FG3.1

設 $AE = BF = k$, $EN = NF = 2k$, $DM = MC = 3k$	Let $AE = BF = k$, $EN = NF = 2k$, $DM = MC = 3k$
$\Delta BHF \sim \Delta DHM$ (A.A.A.)	$\Delta BHF \sim \Delta DHM$ (A.A.A.)
$\Delta BGE \sim \Delta DGM$ (A.A.A.)	$\Delta BGE \sim \Delta DGM$ (A.A.A.)
$\frac{DH}{HB} = \frac{DM}{BF} = \frac{3k}{k} = 3$ (相似三角形對應邊)	$\frac{DH}{HB} = \frac{DM}{BF} = \frac{3k}{k} = 3$ (corr. side ~Δs)
$\frac{DG}{GB} = \frac{DM}{BE} = \frac{3k}{5k} = \frac{3}{5}$ (相似三角形對應邊)	$\frac{DG}{GB} = \frac{DM}{BE} = \frac{3k}{5k} = \frac{3}{5}$ (corr. side ~Δs)
$BH = \frac{1}{4}DB$, $DG = \frac{3}{8}DB$	$BH = \frac{1}{4}DB$, $DG = \frac{3}{8}DB$
$GH = DB - DG - BH = \left(1 - \frac{3}{8} - \frac{1}{4}\right)DB = \frac{3}{8}DB$	$GH = DB - DG - BH = \left(1 - \frac{3}{8} - \frac{1}{4}\right)DB = \frac{3}{8}DB$
$DG : GH = \frac{3}{8}DB : \frac{3}{8}DB = 1 : 1$ …… (1)	$DG : GH = \frac{3}{8}DB : \frac{3}{8}DB = 1 : 1$ …… (1)
過 B 作 $BJ // MF$，交 CD 於 J。	Draw $BJ // MF$, cutting CD at J.
$\frac{DM}{MJ} = \frac{DH}{HB} = 3$ (等比定理)	$\frac{DM}{MJ} = \frac{DH}{HB} = 3$ (theorem of equal ratios)
$MJ = k$, $JC = 2k$	$MJ = k$, $JC = 2k$
$\Delta BCD \cong \Delta DAB$ (S.S.S.)	$\Delta BCD \cong \Delta DAB$ (S.S.S.)
$S_{\Delta BCD} = S_{\Delta DAB} = \frac{1}{2} \times 96 = 48$	$S_{\Delta BCD} = S_{\Delta DAB} = \frac{1}{2} \times 96 = 48$
$\frac{S_{\Delta BDJ}}{S_{\Delta BCJ}} = \frac{DJ}{CJ} = \frac{4k}{2k} = 2 \Rightarrow S_{\Delta BDJ} = \frac{2}{3} \times 48 = 32$	$\frac{S_{\Delta BDJ}}{S_{\Delta BCJ}} = \frac{DJ}{CJ} = \frac{4k}{2k} = 2 \Rightarrow S_{\Delta BDJ} = \frac{2}{3} \times 48 = 32$
$\Delta DMH \sim \Delta DJB$ (A.A.A.)	$\Delta DMH \sim \Delta DJB$ (A.A.A.)
$\frac{S_{\Delta DMH}}{S_{\Delta DJB}} = \left(\frac{DM}{DJ}\right)^2 = \left(\frac{3}{4}\right)^2 \Rightarrow S_{\Delta DMH} = \frac{9}{16} \times 32 = 18$	$\frac{S_{\Delta DMH}}{S_{\Delta DJB}} = \left(\frac{DM}{DJ}\right)^2 = \left(\frac{3}{4}\right)^2 \Rightarrow S_{\Delta DMH} = \frac{9}{16} \times 32 = 18$
由(1), $S_{\Delta GHM} = S_{\Delta GDM} = \frac{1}{2} \times S_{\Delta DMH} = 9$	By (1), $S_{\Delta GHM} = S_{\Delta GDM} = \frac{1}{2} \times S_{\Delta DMH} = 9$

I12 在三角形 ABC 中，$AB = 14$、$BC = 48$ 及 $AC = 50$。將 P 及 Q 分別記為ΔABC 的內心及外心。設 PQ 的長度為 d 單位。求 d 的值。

In triangle ABC, $AB = 14$, $BC = 48$ and $AC = 50$. Denote the in-centre and circumcentre of ΔABC by P and Q respectively. Let the length of PQ be d units. Find the value of d.



$AB^2 + BC^2 = 14^2 + 48^2 = 196+2304 = 2500 = AC^2$ $\angle ABC = 90°$　(畢氏定理的逆定理) AC 是外接圓 ABC 的直徑(半圓上的圓周角的定理) $Q = AC$ 的中點　(外接圓的圓心) $AQ = 25 \cdots (1)$ 假設內切圓分別切 BC、AC 及 AB 於 D、E 及 F。 $PD\perp BC$，$PE\perp AC$，$PF\perp AB$ (切綫⊥半徑) $PDBF$ 是一個長方形　(它有 3 隻直角) 設內切圓的半徑為 r。 $PD = PE = PF = r$ $PDBF$ 是一個正方形　($PD = PF$) $BF = BD = r$ $AF = 14 - r$, $CD = 48 - r$ $AE = 14 - r$, $CE = 48 - r$　(由外點引切綫) $AE + EC = AC$ $14 - r + 48 - r = 50$ $r = 6$, $AE = 14 - 6 = 8$ $EQ = AQ - AE = 25 - 8 = 17$ 在ΔPEQ 中，$PE^2 + EQ^2 = PQ^2$ (畢氏定理) $6^2 + 17^2 = PQ^2$ $d = \sqrt{325} = 5\sqrt{13}$	$AB^2 + BC^2 = 14^2 + 48^2 = 196+2304 = 2500 = AC^2$ $\angle ABC = 90°$　(converse, Pyth. thm.) AC is the diameter of the circumcircle ABC (converse, $\angle$ in semi-circle) Q = mid-point of AC　(centre of circumcircle) $AQ = 25 \cdots (1)$ Suppose the in-circle touches BC, AC and AB at D, E and F respectively. $PD\perp BC$, $PE\perp AC$, $PF\perp AB$ (tangent ⊥ radius) $PDBF$ is a rectangle　(it has 3 ⊥ ∠s) Let r be the radius of the inscribed circle. $PD = PE = PF = r$ $PDBF$ is a square　($PD = PF$) $BF = BD = r$ $AF = 14 - r$, $CD = 48 - r$ $AE = 14 - r$, $CE = 48 - r$　(tangent from ext. pt.) $AE + EC = AC$ $14 - r + 48 - r = 50$ $r = 6$, $AE = 14 - 6 = 8$ $EQ = AQ - AE = 25 - 8 = 17$ In ΔPEQ, $PE^2 + EQ^2 = PQ^2$ (Pythagoras' thm.) $6^2 + 17^2 = PQ^2$ $d = \sqrt{325} = 5\sqrt{13}$

I13 已知正整數 a、b 及 c 滿足下列條件：

(i) $a > b > c$，

(ii) $(a - b)(b - c)(a - c) = 84$，

(iii) $abc < 100$。

設 M 為 a 的最大值。求 M 的值。

Given that a, b and c are positive integers satisfying the following conditions:

(i) $a > b > c$,

(ii) $(a - b)(b - c)(a - c) = 84$,

(iii) $abc < 100$.

Let M be the maximum value of a. Find the value of M.

84 的正因子包括 1、2、3、4、6、7、12、14、21、28、42 及 84。 $(a - b) + (b - c) = a - c$ $(a - b, b - c, a - c)$的可能值$= (3, 4, 7)$或$(4, 3, 7)$ $(a, b, c) = (a, a - 3, a - 7)$或$(a, a - 4, a - 7)$ 為了使得 a 為最大，b 和 c 必須盡量小 $\therefore (a, b, c) = (a, a - 4, a - 7)$ $9\times5\times2 = 90$，$10\times6\times3 = 180$ $M = 9$	Positive factors of 84 are 1, 2, 3, 4, 6, 7, 12, 14, 21, 28, 42 and 84. $(a - b) + (b - c) = a - c$ Possible $(a - b, b - c, a - c) = (3, 4, 7)$ or $(4, 3, 7)$ $(a, b, c) = (a, a - 3, a - 7)$ or $(a, a - 4, a - 7)$ For largest a, b and c must be as small as possible $\therefore (a, b, c) = (a, a - 4, a - 7)$ $9\times5\times2 = 90$, $10\times6\times3 = 180$ $M = 9$

I14 已知 $3\sin x + 2\sin y = 4$。設 N 為 $3\cos x + 2\cos y$ 的最大值。求 N 的值。

Given that $3\sin x + 2\sin y = 4$. Let N be the maximum value of $3\cos x + 2\cos y$.
Find the value of N.

The following method is provided by Ms. Wong Ka Man from St. Mark's College.

$(3\cos x + 2\cos y)^2$
$= 9\cos^2 x + 12\cos x\cos y + 4\cos^2 y$
$= 9(1 - \sin^2 x) + 12(\cos x\cos y + \sin x\sin y) + 4(1 - \sin^2 y) - 12\sin x\sin y$
$= 13 + 12\cos(x - y) - (3\sin x + 2\sin y)^2$
$= 13 + 12\cos(x - y) - 4^2 = 12\cos(x - y) - 3$
$\le 12 - 3 = 9$
$\therefore 3\cos x + 2\cos y \le 3$
$N = 3$

I15 已知 x、y 及 z 為正實數且滿足 $\begin{cases} x^2 + xy + y^2 = 7 \\ y^2 + yz + z^2 = 21 \\ x^2 + xz + z^2 = 28 \end{cases}$。若 $a = x + y + z$，求 a 的值。

Given that x, y and z are positive real numbers satisfying $\begin{cases} x^2 + xy + y^2 = 7 & \cdots(1) \\ y^2 + yz + z^2 = 21 & \cdots(2) \\ x^2 + xz + z^2 = 28 & \cdots(3) \end{cases}$.

If $a = x + y + z$, find the value of a.

<table>
<tr>
<td>
$\begin{cases} (x-y)(x^2+xy+y^2) = 7(x-y) \\ (y-z)(y^2+yz+z^2) = 21(y-z) \\ (z-x)(x^2+xz+z^2) = 28(z-x) \end{cases}$
$\begin{cases} x^3 - y^3 = 7x - 7y \\ y^3 - z^3 = 21y - 21z \\ z^3 - x^2 = 28z - 28x \end{cases}$
將以上三題方程相加: $0 = -21x + 14y + 7z$
$z = 3x - 2y \cdots (4)$
$(1) + (2) - (3)$：$2y^2 + (x + z)y - xz = 0 \cdots (5)$
代(4)入(5)：$2y^2 + (x + 3x - 2y)y - x(3x - 2y) = 0$
$2y^2 + (4x - 2y)y - (3x^2 - 2xy) = 0$
$3x^2 - 6xy = 0$
$x = 0$ (x 為正實數，捨去) 或 $x = 2y \cdots (6)$
代(6)入(1)：$4y^2 + 2y^2 + y^2 = 7$
$y = 1$ 或 -1 (y 為正實數，捨去)
$x = 2$
$z = 3x - 2y = 4$
$a = x + y + z = 7$
</td>
<td>
$\begin{cases} (x-y)(x^2+xy+y^2) = 7(x-y) \\ (y-z)(y^2+yz+z^2) = 21(y-z) \\ (z-x)(x^2+xz+z^2) = 28(z-x) \end{cases}$
$\begin{cases} x^3 - y^3 = 7x - 7y \\ y^3 - z^3 = 21y - 21z \\ z^3 - x^2 = 28z - 28x \end{cases}$
Add up these equations: $0 = -21x + 14y + 7z$
$z = 3x - 2y \cdots (4)$
$(1) + (2) - (3)$: $2y^2 + (x + z)y - xz = 0 \cdots (5)$
Sub. (4) into (5):
$2y^2 + (x + 3x - 2y)y - x(3x - 2y) = 0$
$2y^2 + (4x - 2y)y - (3x^2 - 2xy) = 0$
$3x^2 - 6xy = 0$
$x = 0$ (x is real positive, rejected) or $x = 2y \cdots (6)$
Sub. (6) into (1): $4y^2 + 2y^2 + y^2 = 7$
$y = 1$ or -1 (y is real positive, rejected)
$x = 2$
$z = 3x - 2y = 4$
$a = x + y + z = 7$
</td>
</tr>
</table>

Group Events

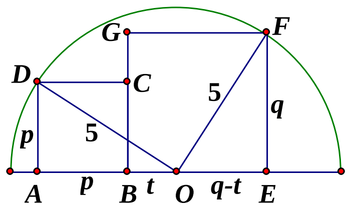
G1 對所有正實數 x，定義 $\mathrm{f}(x)=\log_{2019} x^{2020}$。若 $D=\mathrm{f}\left(\sqrt{3}\right)+\mathrm{f}\left(\sqrt{673}\right)$，求 D 的值。

For all positive value real numbers x, define $\mathrm{f}(x)=\log_{2019} x^{2020}$. If $D=\mathrm{f}\left(\sqrt{3}\right)+\mathrm{f}\left(\sqrt{673}\right)$, find the value of D.

$$D=\log_{2019}\left(\sqrt{3}\right)^{2020}+\log_{2019}\left(\sqrt{673}\right)^{2020}$$
$$=\log_{2019}\left(\sqrt{3}\times\sqrt{673}\right)^{2020}$$
$$=\log_{2019}(2019)^{1010}$$
$$=1010$$

G2 圖一所示，$ABCD$ 和 $BEFG$ 是兩個緊貼的正方形，躺臥在一個以 O 為圓心，半徑為 5 cm 的半圓上。其中 A、B 和 E 在半圓的直徑，D 和 F 在半圓的弧上。設 $ABCD$ 與 $BEFG$ 的面積之和為 S cm^2，求 S 的值。

Figure 1 shows two adjacent squares $ABCD$ and $BEFG$ lying on a semi-circle with centre O and radius 5 cm. A, B and E lie on the diameter of the semi-circle, D and F lie on the semi-circular arc. Let the sum of areas of $ABCD$ and $BEFG$ be S cm^2, find the value of S.

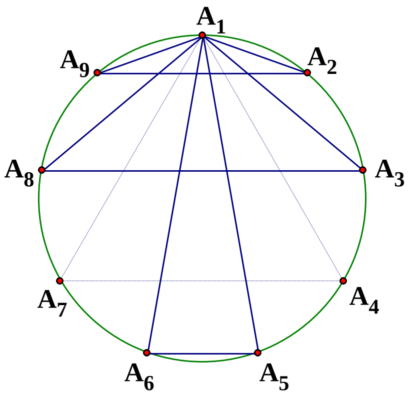


圖一 Figure 1

$OD=OE=5$ cm。設 $AD=p$、$EF=q$。 不妨假設 $q>p$。 設 $OB=t$，則 $OE=q-t$. $AD\perp AB$，$FE\perp BE$ $AD^2+AO^2=OE^2+EF^2=OF^2$ (畢氏定理) $p^2+(p+t)^2=(q-t)^2+q^2=5^2$ $p^2+p^2+2pt+t^2=q^2-2qt+t^2+q^2$ $2p^2+2pt=2q^2-2qt$ $p^2+pt=q^2-qt$ $(q+p)t=q^2-p^2$ $t=q-p$ $OA=p+t=p+q-p=q$ $AD^2+AO^2=OD^2$ (畢氏定理) $p^2+q^2=5^2$ $S=p^2+q^2=25$	$OD=OE=5$ cm. Let $AD=p$, $EF=q$. Without loss of generality, assume $q>p$. Let $OB=t$, then $OE=q-t$. $AD\perp AB$, $FE\perp BE$ $AD^2+AO^2=OE^2+EF^2=OF^2$ (Pythagoras' theorem) $p^2+(p+t)^2=(q-t)^2+q^2=5^2$ $p^2+p^2+2pt+t^2=q^2-2qt+t^2+q^2$ $2p^2+2pt=2q^2-2qt$ $p^2+pt=q^2-qt$ $(q+p)t=q^2-p^2$ $t=q-p$ $OA=p+t=p+q-p=q$ $AD^2+AO^2=OD^2$ (Pythagoras' theorem) $p^2+q^2=5^2$ $S=p^2+q^2=25$

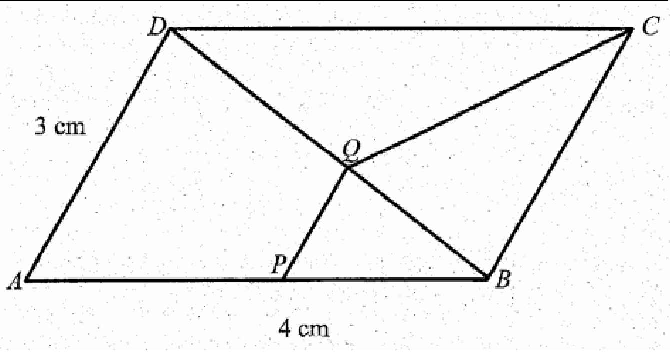
G3 若從一個正九邊形的 9 個頂點中選 3 點，共可組成多少個等腰三角形？

If three vertices are chosen from the nine vertices of a regular nonagon, how many possible isosceles triangles are there ?



將 9 個頂點依次序命名為 A_1、A_2、⋯、A_9。 其中有 4 個等腰三角形通過 $A_iA_1A_j$ 及 $A_1A_i = A_1A_j$。當中 $A_4A_1A_7$ 是一個等邊三角形。 若果不計算等邊三角形，所有等腰三角形的總數為 $3\times9 = 27$。 若果包括了所有等邊三角形，所有等腰三角形的總數為 $27 + 3 = 30$。	Label the 9 vertices as A_1, A_2, ⋯ , A_9 in order. There are 4 isosceles triangles in the form $A_iA_1A_j$ such that $A_1A_i = A_1A_j$. Amongst these 4 isosceles triangles, $A_4A_1A_7$ is an equilateral triangle. If we do not count these equilateral triangles, the total number of isosceles triangles are $3\times9 = 27$ If we include these equilateral triangles, the total number of isosceles triangles = $27 + 3 = 30$

G4 在圖二中，$ABCD$ 為一個平行四邊形，其中 $AB = 4$ cm、$AD = 3$ cm 及 $\sin A = \frac{2}{3}$。P 和 Q 分別是 AB 和 BD 上的點使得 $PQ // AD$，且四邊形 $PBCQ$ 的面積為 3 cm^2。設 AP 的長度為 q cm，求 q 的值。



圖二 Figure 2

In Figure 2, $ABCD$ is a parallelogram, where $AB = 4$ cm, $AD = 3$ cm and $\sin A = \frac{2}{3}$. P and Q are points on AB and BD respectively such that $PQ // AD$, and the area of the quadrilateral $PBCQ$ is 3 cm^2 . Let the length of AP be q cm, find the value of q .

設 S 表示面積。 $S_{\Delta ABD} = S_{\Delta CDB} = \frac{1}{2}AB\cdot AD\sin A = \frac{1}{2}\cdot3\cdot4\cdot\frac{2}{3} = 4$ $\Delta BPQ \sim \Delta BAD$ (A.A.A.) 設 $BQ : QD = k : (1 - k)$ $\frac{S_{\Delta BPQ}}{S_{\Delta BAD}} = k^2 \Rightarrow S_{\Delta BPQ} = 4k^2$ ΔBCQ 及 ΔBCD 有相同高度 $\frac{S_{\Delta BCQ}}{S_{\Delta BCD}} = k \Rightarrow S_{\Delta BCQ} = 4k$ $S_{PBCQ} = 3 \Rightarrow 4k^2 + 4k = 3$ $(2k + 3)(2k - 1) = 0$ $k = -1.5$ (捨去) 或 0.5 $BP : PA = BQ : QD = 0.5 : (1 - 0.5) = 1 : 1$ $\Rightarrow q = 2$	Let S denote the area. $S_{\Delta ABD} = S_{\Delta CDB} = \frac{1}{2}AB\cdot AD\sin A = \frac{1}{2}\cdot3\cdot4\cdot\frac{2}{3} = 4$ $\Delta BPQ \sim \Delta BAD$ (A.A.A.) Let $BQ : QD = k : (1 - k)$ $\frac{S_{\Delta BPQ}}{S_{\Delta BAD}} = k^2 \Rightarrow S_{\Delta BPQ} = 4k^2$ ΔBCQ and ΔBCD have the same height $\frac{S_{\Delta BCQ}}{S_{\Delta BCD}} = k \Rightarrow S_{\Delta BCQ} = 4k$ $S_{PBCQ} = 3 \Rightarrow 4k^2 + 4k = 3$ $(2k + 3)(2k - 1) = 0$ $k = -1.5$ (rejected) or 0.5 $BP : PA = BQ : QD = 0.5 : (1 - 0.5) = 1 : 1$ $\Rightarrow q = 2$

G5 已知 $\mathrm{f}(x)-2\mathrm{f}\left(\frac{1}{x}\right)=x$，其中 $x\neq 0$。設 y 為滿足方程 $\mathrm{f}(x)=1$ 的 x 的最大值。求 y 的值。

Given that $\mathrm{f}(x)-2\mathrm{f}\left(\frac{1}{x}\right)=x$, where $x\neq 0$. Let y be the maximum value of x that satisfies the equation $\mathrm{f}(x)=1$. Find the value of y. **Reference: 2018 HG4**

$\mathrm{f}(x)-2\mathrm{f}\left(\frac{1}{x}\right)=x \quad \cdots (1)$ $\mathrm{f}\left(\frac{1}{x}\right)-2\mathrm{f}(x)=\frac{1}{x} \quad \cdots (2)$ $(1)+2(2)$: $-3\mathrm{f}(x)=x+\frac{2}{x}$ $\Rightarrow \mathrm{f}(x)=-\frac{1}{3}\left(x+\frac{2}{x}\right)$ $\mathrm{f}(x)=1 \Rightarrow -\frac{1}{3}\left(x+\frac{2}{x}\right)=1$ $x^2+2=-3x$ $x^2+3x+2=0$ $x=-1$ 或 -2 $y=-1$	$\mathrm{f}(x)-2\mathrm{f}\left(\frac{1}{x}\right)=x \quad \cdots (1)$ $\mathrm{f}\left(\frac{1}{x}\right)-2\mathrm{f}(x)=\frac{1}{x} \quad \cdots (2)$ $(1)+2(2)$: $-3\mathrm{f}(x)=x+\frac{2}{x}$ $\Rightarrow \mathrm{f}(x)=-\frac{1}{3}\left(x+\frac{2}{x}\right)$ $\mathrm{f}(x)=1 \Rightarrow -\frac{1}{3}\left(x+\frac{2}{x}\right)=1$ $x^2+2=-3x$ $x^2+3x+2=0$ $x=-1$ or -2 $y=-1$

G6 設 a_k 為多項式 $(2x-2)^3(2x+2)^3(2x+1)^3$ 中 x^k 的係數。

若 $Q=a_2+a_4+a_6+a_8$，求 Q 的值。

Let a_k be the coefficient of x^k in the polynomial $(2x-2)^3(2x+2)^3(2x+1)^3$.

If $Q=a_2+a_4+a_6+a_8$, find the value of Q.

$(2x-2)^3(2x+2)^3(2x+1)^3=64(x^2-1)^3(2x+1)^3=64(x^6-3x^4+3x^2-1)(8x^3+12x^2+6x+1)$

$a_2=64(3-12)=64\times(-9)$

$a_4=64(-3+3\times 12)=64\times 33$

$a_6=64(1-3\times 12)=64\times(-35)$

$a_8=64\times 12$

$Q=a_2+a_4+a_6+a_8=64\times(-9)+64\times 33+64\times(-35)+64\times 12=64\times(-9+33-35+12)=64$

G7 設 $\mathrm{f}(x)=-6x^2+4x\cos\theta+\sin\theta$，其中 $0^\circ\leq\theta\leq 360^\circ$。已知對所有實數 x，$\mathrm{f}(x)\leq 0$。若 θ 的最大值與最小值之差為 d°，求 d 的值。

Let $\mathrm{f}(x)=-6x^2+4x\cos\theta+\sin\theta$, where $0^\circ\leq\theta\leq 360^\circ$. If is given that $\mathrm{f}(x)\leq 0$ for all real numbers x. If d° is the difference between the greatest and the least values of θ, find the value of d.

設 $a=-6$，$b=4\cos\theta$，$c=\sin\theta$ $\mathrm{f}(x)$ 的最大值 $=\frac{4ac-b^2}{4a}\leq 0$ $\frac{4(-6)\sin\theta-(4\cos\theta)^2}{4(-6)}\leq 0$ $24\sin\theta+16\cos^2\theta\leq 0$ $3\sin\theta+2(1-\sin^2\theta)\leq 0$ $2\sin^2\theta-3\sin\theta-2\geq 0$ $(2\sin\theta+1)(\sin\theta-2)\geq 0$ $\sin\theta\leq -0.5$ 或 $\sin\theta\geq 2$ (捨去) $210^\circ\leq\theta\leq 330^\circ\Rightarrow d=330-210=120$	Let $a=-6$, $b=4\cos\theta$, $c=\sin\theta$ Maximum value of $\mathrm{f}(x)=\frac{4ac-b^2}{4a}\leq 0$ $\frac{4(-6)\sin\theta-(4\cos\theta)^2}{4(-6)}\leq 0$ $24\sin\theta+16\cos^2\theta\leq 0$ $3\sin\theta+2(1-\sin^2\theta)\leq 0$ $2\sin^2\theta-3\sin\theta-2\geq 0$ $(2\sin\theta+1)(\sin\theta-2)\geq 0$ $\sin\theta\leq -0.5$ or $\sin\theta\geq 2$ (rejected) $210^\circ\leq\theta\leq 330^\circ\Rightarrow d=330-210=120$

G8 設 $\{a_n\}$ 為一個正實數序列使當 $n>1$ 時，$a_n=a_{n-1}a_{n+1}-1$。
已知 2018 在序列中及 $a_2=2019$。若 a_1 的所有可取的數目為 s，求 s 的值。
Let $\{a_n\}$ be a sequence of positive real numbers such that $a_n=a_{n-1}a_{n+1}-1$ for $n>1$.
It is given that 2018 is in the sequence and $a_2=2019$. If the number of all possible values of a_1 is s, find the value of s.

$a_{n+1}=\frac{1+a_n}{a_{n-1}}=\frac{1+\frac{1+a_{n-1}}{a_{n-2}}}{a_{n-1}}=\frac{a_{n-2}+a_{n-1}+1}{a_{n-1}a_{n-2}}$ $=\frac{a_{n-2}+\frac{1+a_{n-2}}{a_{n-3}}+1}{\frac{1+a_{n-2}}{a_{n-3}}\cdot a_{n-2}}=\frac{a_{n-2}a_{n-3}+1+a_{n-2}+a_{n-3}}{(1+a_{n-2})\cdot a_{n-2}}$ $=\frac{(1+a_{n-2})(1+a_{n-3})}{(1+a_{n-2})\cdot a_{n-2}}=\frac{1+a_{n-3}}{a_{n-2}}=\frac{1+a_{n-3}}{\frac{1+a_{n-3}}{a_{n-4}}}$ $=a_{n-4}$ 對於 $n\geq 5$ $\therefore$ $a_1=a_6=a_{11}=\cdots=a_{5n+1}$ $a_2=a_7=a_{12}=\cdots=a_{5n+2}=2019$ $\because$ $a_k=2018$ 對某些正整數 $k\neq 5n+2$。 $\therefore$ a_n 的數值由 a_2 及 a_k 決定。 a_1 有 4 種不同的值，$s=4$	$a_{n+1}=\frac{1+a_n}{a_{n-1}}=\frac{1+\frac{1+a_{n-1}}{a_{n-2}}}{a_{n-1}}=\frac{a_{n-2}+a_{n-1}+1}{a_{n-1}a_{n-2}}$ $=\frac{a_{n-2}+\frac{1+a_{n-2}}{a_{n-3}}+1}{\frac{1+a_{n-2}}{a_{n-3}}\cdot a_{n-2}}=\frac{a_{n-2}a_{n-3}+1+a_{n-2}+a_{n-3}}{(1+a_{n-2})\cdot a_{n-2}}$ $=\frac{(1+a_{n-2})(1+a_{n-3})}{(1+a_{n-2})\cdot a_{n-2}}=\frac{1+a_{n-3}}{a_{n-2}}=\frac{1+a_{n-3}}{\frac{1+a_{n-3}}{a_{n-4}}}$ $=a_{n-4}$ for $n\geq 5$ $\therefore$ $a_1=a_6=a_{11}=\cdots=a_{5n+1}$ $a_2=a_7=a_{12}=\cdots=a_{5n+2}=2019$ $\because$ $a_k=2018$ for some positive integer $k\neq 5n+2$. $\therefore$ a_n is uniquely determined by a_2 and a_k. a_1 can have 4 different values, $s=4$

G9 有多少對正整數 x、y 可滿足 $xy=6\left(x+y+\sqrt{x^2+y^2}\right)$？
How many pairs of positive integers x, y are there satisfying $xy=6\left(x+y+\sqrt{x^2+y^2}\right)$?

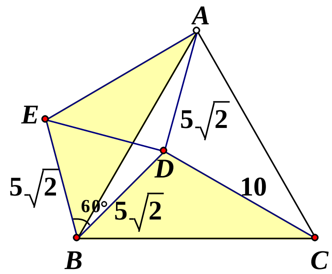
$(xy-6x-6y)^2=36(x^2+y^2)$ $x^2y^2-12x^2y-12xy^2+72xy=0$ $xy-12x-12y+72=0$ $xy-12x-12y+144=72$ $(x-12)(y-12)=72$ $(x-12, y-12)=(1, 72), (2, 36), (3, 24), (4, 18), (6, 12), (8, 9), (9, 8), (12, 6), (18, 4), (24, 3), (36, 2), (72, 1)$。 一共有 12 對正整數。 註：當 $(x-12, y-12)=(-8,-9)$ 或 $(-9,-8)$ $(x, y)=(4, 3)$ 或 $(3, 4)$。這兩組答案未能滿足原方程 $\therefore$ 捨去。	$(xy-6x-6y)^2=36(x^2+y^2)$ $x^2y^2-12x^2y-12xy^2+72xy=0$ $xy-12x-12y+72=0$ $xy-12x-12y+144=72$ $(x-12)(y-12)=72$ $(x-12, y-12)=(1, 72), (2, 36), (3, 24), (4, 18), (6, 12), (8, 9), (9, 8), (12, 6), (18, 4), (24, 3), (36, 2), (72, 1)$. There are 12 pairs of positive integers. Remark: When $(x-12, y-12)=(-8,-9)$ or $(-9,-8)$ $(x, y)=(4, 3)$ or $(3, 4)$. These two solutions do not satisfy the original equation $\therefore$ rejected.

G10 D 是等邊三角形 ABC 內的一點使得 $AD = BD = 5\sqrt{2}$ 及 $CD = 10$。設 ΔABC 的面積為 S，求 S 的值。

D is a point inside the equilateral triangle ABC such that $AD = BD = 5\sqrt{2}$ and $CD = 10$.

Let the area of ΔABC be S, find the value of S.

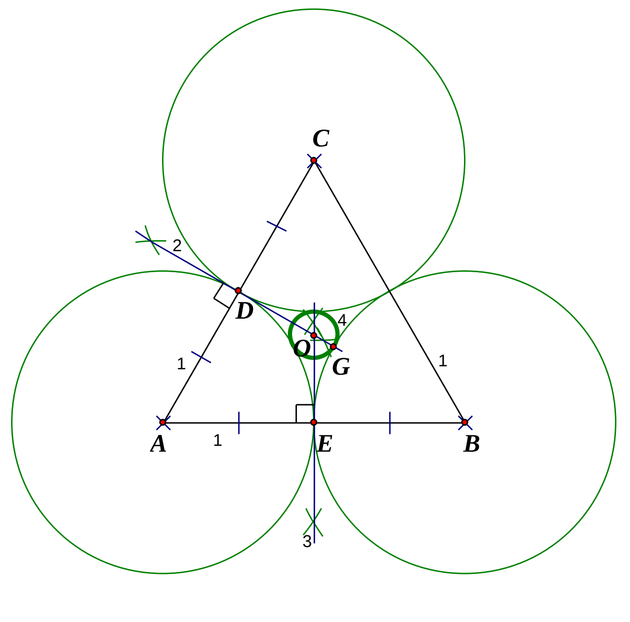
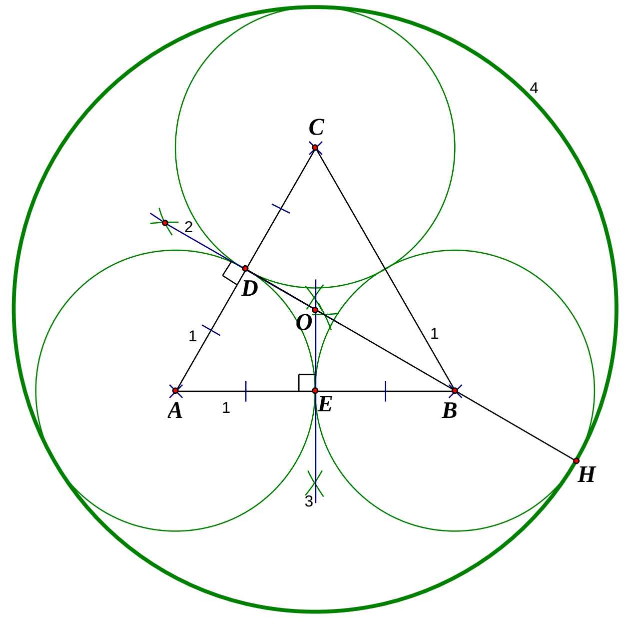
Reference: 2014 HI3



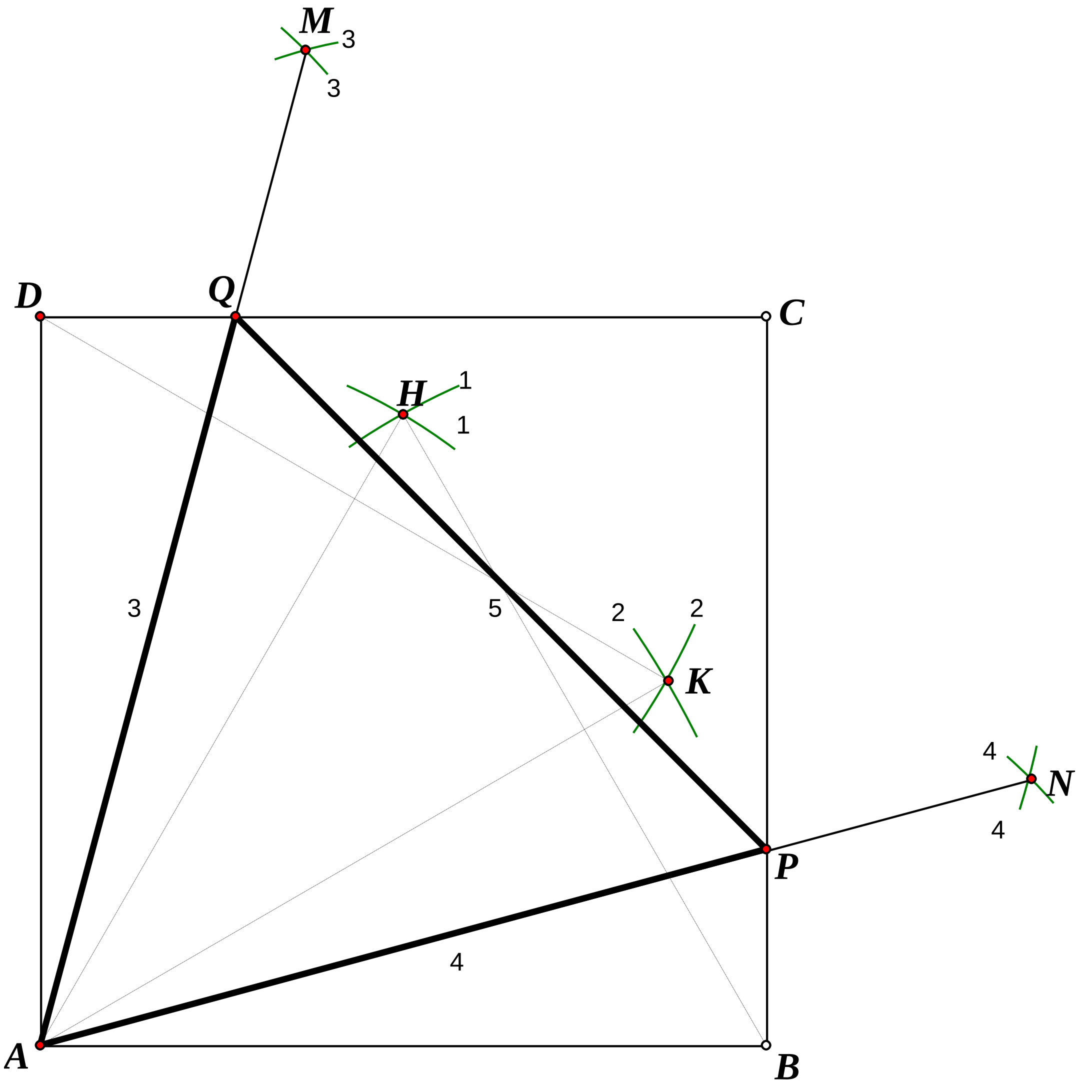
如圖所示，將 BD 繞 B 反時針方向轉 60°，得 BE。	As shown in the figure, rotate BD about B anti-clockwise through 60° to BE.
由作圖所得，$BD = BE = 5\sqrt{2}$ 及 $\angle DBE = 60°$	By construction, $BD=BE=5\sqrt{2}$ and $\angle DBE= 60°$
$\angle BDE = \angle BED$ (等腰三角形底角)	$\angle BDE = \angle BED$ (base ∠s isos. Δ)
$= \frac{180^\circ - 60^\circ}{2} = 60°$ (三角形內角和)	$= \frac{180^\circ - 60^\circ}{2} = 60°$ (∠ sum of Δ)
ΔBDE 是一個等邊三角形。	ΔBDE is an equilateral triangle.
$DE = BD = 5\sqrt{2}$ (等邊三角形性質)	$DE = BD = 5\sqrt{2}$ (prop. of equil. Δ)
$AB = AC$ (等邊三角形性質)	$AB = AC$ (prop. of equil. Δ)
$\angle ABC = 60°$ (等邊三角形性質)	$\angle ABC = 60°$ (prop. of equil. Δ)
$\angle ABE = \angle DBE - \angle ABD = 60° - \angle ABD$	$\angle ABE = \angle DBE - \angle ABD = 60° - \angle ABD$
$= \angle CBD$	$= \angle CBD$
$\Delta ABE \cong \Delta CBD$ (S.A.S.)	$\Delta ABE \cong \Delta CBD$ (S.A.S.)
$AE = CD = 10$ (全等三角形對應邊)	$AE = CD = 10$ (corr. sides, ≅ Δs)
$DE^2 + DA^2 = (5\sqrt{2})^2 + (5\sqrt{2})^2 = 100 = AE^2$	$DE^2 + DA^2 = (5\sqrt{2})^2 + (5\sqrt{2})^2 = 100 = AE^2$
$\angle ADE = 90°$ (畢氏定理逆定理)	$\angle ADE = 90°$ (converse, Pyth. thm.)
$\angle ADB = \angle ADE + \angle BDE = 90° + 60° = 150°$	$\angle ADB = \angle ADE + \angle BDE = 90° + 60° = 150°$
設 $AB = x$。於ΔABD 中應用餘弦公式：	Let $AB = x$. Apply cosine formula on ΔABD:
$x^2 = AD^2 + BD^2 - 2AD \cdot BD \cos \angle ADB$	$x^2 = AD^2 + BD^2 - 2AD \cdot BD \cos \angle ADB$
$x^2 = (5\sqrt{2})^2 + (5\sqrt{2})^2 - 2(5\sqrt{2})^2 \cos 150^\circ$	$x^2 = (5\sqrt{2})^2 + (5\sqrt{2})^2 - 2(5\sqrt{2})^2 \cos 150^\circ$
$x^2 = 100 - 100\left(-\frac{\sqrt{3}}{2}\right) = 100 + 50\sqrt{3}$	$x^2 = 100 - 100\left(-\frac{\sqrt{3}}{2}\right) = 100 + 50\sqrt{3}$
$S = \Delta ABC$ 的面積 $= \frac{1}{2} \cdot AB \cdot BC \sin 60^\circ$	$S =$ area of $\Delta ABC = \frac{1}{2} \cdot AB \cdot BC \sin 60^\circ$
$= \frac{1}{2} \cdot (100 + 50\sqrt{3}) \cdot \frac{\sqrt{3}}{2}$	$= \frac{1}{2} \cdot (100 + 50\sqrt{3}) \cdot \frac{\sqrt{3}}{2}$
$= \frac{25}{2} \cdot (2\sqrt{3} + 3) = 25\sqrt{3} + 37.5$	$= \frac{25}{2} \cdot (2\sqrt{3} + 3) = 25\sqrt{3} + 37.5$

Geometrical Construction

1. 圖一所示為三個半徑相等且兩兩相切的圓。試作一圓使得它與圖中每一圓相切於一點。
Figure 1 shows three circles with equal radius which are pairwise tangents to each other. Construct a circle which will touch each circle in the figure at a point.

	
作圖步驟： (1) 連接 AB、AC 及 BC。 (2) 作 AC 的垂直平分綫。D 為 AC 的中點。此中垂綫交以 B 為圓心的圓形於 G。($OG < AD$) (3) 作 AB 的垂直平分綫。E 為 AB 的中點。兩中垂綫相交於 O。 (4) 作圓 $\odot(O, OG)$。 此圓滿足所求。	Steps: (1) Join AB, AC and BC. (2) Draw the perpendicular bisectors of AC. D is the mid-point of AC. It intersects the circle with centre B at G. ($OG < AD$) (3) Draw the perpendicular bisectors of AB. E is the mid-point of AB. The 2 $\perp$ bisectors intersect at O. (4) Draw a circle $\odot(O, OG)$. This is the required circle.
方法二： 於步驟(2)中，中垂綫交以 B 為圓心的圓形於 H。 ($OH > AD$) (4) 作圓 $\odot(O, OH)$。 此圓亦滿足所求。	**Method 2** In step (2), the perpendicular bisector intersects the circle with centre B at H. ($OH > AD$) (4) Draw a circle $\odot(O, OH)$. This is another solution.

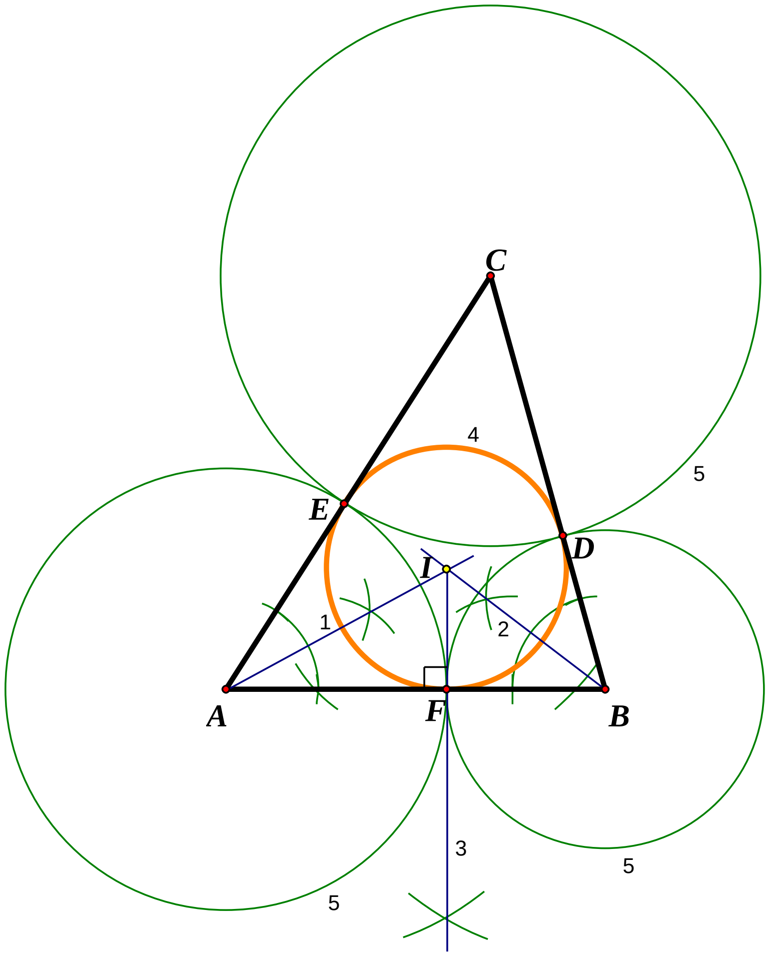
2. 圖二所示為一個邊長為 1 單位的正方形 $ABCD$。試作一個三角形 APQ，其中 P、Q 分別位於綫段 BC、CD 上且 $\angle PAB = \angle QAD = 15°$。寫出 APQ 是哪一類三角形。
Figure 2 shows a square $ABCD$ with side 1 unit. Construct a triangle APQ, in which P, Q lie on the line segments BC and CD respectively, and $\angle PAB = \angle QAD = 15°$. Write down the type of triangle that APQ is.



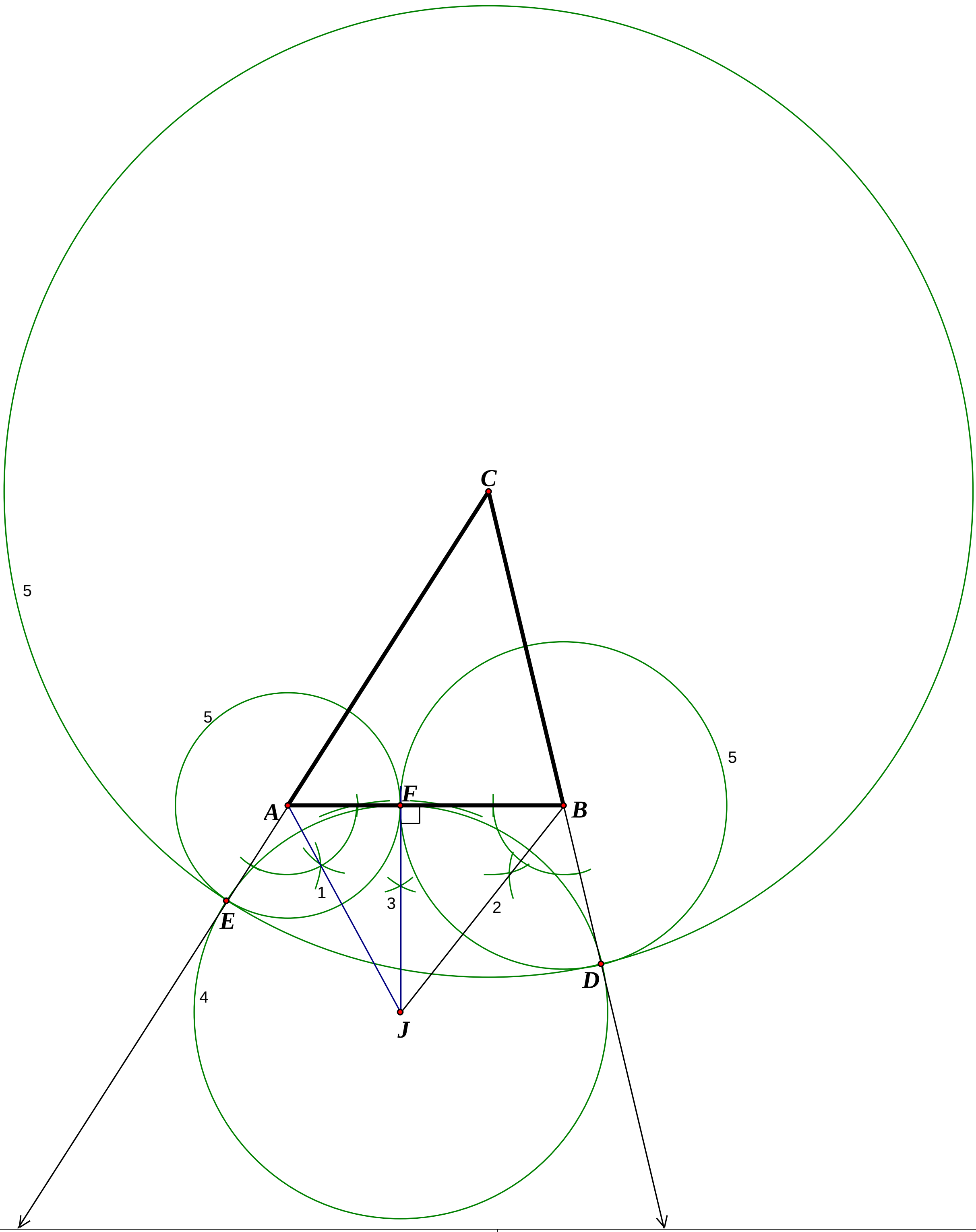
作圖步驟：	Steps:
(1) 作等邊三角形 AHB。 $\angle BAH = 60°$，$\angle DAH = 30°$。 (2) 作等邊三角形 AKD。 $\angle DAK = 60°$，$\angle BAK = 30°$。 (3) 作$\angle DAH$ 的角平分綫 AM，交 CD 於 Q。 $\angle DAQ = 15°$。 (4) 作$\angle BAK$ 的角平分綫 AN，交 CB 於 P。 $\angle BAP = 15°$。 (5) 連接 PQ。 ΔAPQ 是一個等邊三角形。	(1) Construct an equilateral triangle AHB. $\angle BAH = 60°$, $\angle DAH = 30°$. (2) Construct an equilateral triangle AKD. $\angle DAK = 60°$, $\angle BAK = 30°$. (3) Construct the angle bisector AM of $\angle DAH$, cutting CD at Q. $\angle DAQ = 15°$. (4) Construct the angle bisector AN of $\angle BAK$, cutting CB at P. $\angle BAP = 15°$. (5) Join PQ. ΔAPQ is an equilateral triangle.

3. 圖三所示為一個三角形 ABC。試以 A、B 及 C 為圓心分別構作三個圓，使得它們兩兩相切。

Figure 3 shows a triangle ABC. Use A, B and C as centres to construct three circles respectively that are pairwise tangent to each other. **Reference: 2009 HSC1, 2012HC2, 2014 HC1**



作圖步驟：	Steps:
(1) 作$\angle A$ 的角平分綫。	(1) Construct the angle bisector of $\angle A$.
(2) 作$\angle B$ 的角平分綫。 兩條角平分綫相交於內切圓心 I。	(2) Construct the angle bisector of $\angle B$. The two $\angle$ bisectors intersect at the incentre I.
(3) 作綫段 $IF \perp AB$。	(3) Construct a line $IF \perp AB$.
(4) 作內切圓$\odot(I, IF)$，分別切 BC 和 AC 於 D 和 E。	(4) Construct the incircle $\odot(I, IF)$, touching BC and AC at D and E respectively.
由切綫性質，$AE = AF$、$BD = BF$、$CD = CE$。	By tangent property, $AE = AF$, $BD = BF$, $CD = CE$.
(5) 作三圓$\odot(A, AE)$、$\odot(B, BD)$、$\odot(C, CE)$。	(5) Draw 3 circles $\odot(A, AE)$, $\odot(B, BD)$, $\odot(C, CE)$.



方法二作圖步驟： (1) 作$\angle A$ 的外角平分綫。 (2) 作$\angle B$ 的外角平分綫。 兩條角平分綫相交於旁切圓心 J。 (3) 作綫段 $JF \perp AB$。 (4) 作旁切圓$\odot(J, JF)$，分別切 CB 和 CA 的延綫於 D 和 E。 由切綫性質，$AE = AF$、$BD = BF$、$CD = CE$。 (5) 作三圓$\odot(A, AE)$、$\odot(B, BD)$、$\odot(C, CE)$。	**Method 2** Steps: (1) Construct the exterior angle bisector of $\angle A$. (2) Construct the exterior angle bisector of $\angle B$. The two $\angle$ bisectors intersect at the excentre J. (3) Construct a line $JF \perp AB$. (4) Construct the excircle $\odot(J, JF)$, touching CB produced and CA produced at D and E respectively. By tangent property, $AE = AF$, $BD = BF$, $CD = CE$. (5) Draw 3 circles $\odot(A, AE)$, $\odot(B, BD)$, $\odot(C, CE)$.