22-23 Individual	**1**	–13	**2**	7	**3**	52	**4**	52	**5**	9
	6	$16\sqrt{2}$	**7**	$\frac{4}{5}$	**8**	4	**9**	289	**10**	315
	11	14	**12**	$\frac{1}{3}$	**13**	2697	**14**	100	**15**	$\frac{2}{7}$
22-23 Group	**1**	27	**2**	–1	**3**	$\frac{67}{25}$	**4**	86400	**5**	$\frac{19}{12}$
	6	8	**7**	401	**8**	172	**9**	–5	**10**	64

Individual Events

I1 已知 a 和 b 均為實數。若 $a^2 + b^2 - 8a + 34b + 305 = 0$，求 $a + b$ 的值。

Given that a and b are real numbers. If $a^2 + b^2 - 8a + 34b + 305 = 0$, find the value of $a + b$.

Reference: 2021 P1Q15

$a^2 - 8a + 16 + b^2 + 34b + 289 = 0$

$(a - 4)^2 + (b + 17)^2 = 0$

Sum of two squares = 0 ⇔ each term = 0

$a = 4$ and $b = -17$

$a + b = 4 - 17 = -13$

I2 若 x 及 y 均為正整數且滿足 $x + 8xy + y = 28$，求 $x + 2y$ 的最大可能值。

If x and y are positive integers satisfying $x + 8xy + y = 28$,
find the largest possible value of $x + 2y$.

Reference: 2022 P2Q3

$x(1 + 8y) + y = 28$

$8x(1 + 8y) + 1 + 8y = 225$

$(8x + 1)(8y + 1) = 225$

$8x + 1$	$8y + 1$	x	y	$x + 2y$
1	225	no positive solution		
3	75	no integral solution		
5	45	no integral solution		
9	25	1	3	7
15	15	no integral solution		
25	9	3	1	5

The largest possible value of $x + 2y = 7$.

I3 設 m 為一個整數常數，其中 $4 < m < 40$。若方程 $x^2 - 2(2m - 3)x + 4m^2 - 14m + 8 = 0$ 有兩個整數根，求 x 的最大可能值。

Let m be an integral constant, where $4 < m < 40$. If the equation
$x^2 - 2(2m - 3)x + 4m^2 - 14m + 8 = 0$ has two integral roots, find the largest possible value of x.

Reference: 2011 FI3.1

$\Delta = 4(2m - 3)^2 - 4(4m^2 - 14m + 8) = P^2$, where P is an integer.

$\left(\frac{P}{2}\right)^2 = 4m^2 - 12m + 9 - 4m^2 + 14m - 8 = 2m + 1$, $x = -\frac{b}{2} \pm \sqrt{\frac{\Delta}{4}} = (2m-3) \pm \sqrt{2m+1}$

The possible square numbers are 9, 16, 25, 36, 49, 64, 81, ⋯

$2m + 1$	m	quad. equation	x
9	4	rejected	
25	12	$x^2 - 42x + 416 = 0$	16, 26
49	24	$x^2 - 90x + 1976 = 0$	38, 52
81	40	rejected	

The largest possible value of x is 52.

I4 設 a 為一正實數。若 $a^2+\frac{1}{a^2}=14$，求 $a^3+\frac{1}{a^3}$ 的值。

Let a be a positive real number. If $a^2+\frac{1}{a^2}=14$, find the value of $a^3+\frac{1}{a^3}$.

Reference: 2017 FI1.4

$a^2+\frac{1}{a^2}=14$

$\left(a+\frac{1}{a}\right)^2=16$

$a+\frac{1}{a}=4$ or -4 (rejected, $\because a>0$)

$a^3+\frac{1}{a^3}=\left(a+\frac{1}{a}\right)\left(a^2-1+\frac{1}{a^2}\right)$

$= 4\times(14-1) = 52$

I5 若干正整數之和是 60。最大正整數為 15 及其中有一個正整數是 12。除卻這正整數 12，其餘正整數恰好組成一個等差數列。求最小的正整數。

The sum of certain number of positive integers is 60. The largest positive integer is 15 and one of the positive integers is 12. Apart from this positive integer 12, the remaining positive integers form an arithmetic sequence. Find the smallest positive integer.

Let the smallest positive integer be ℓ and the number of terms be n. then $n < 15$.

$\frac{n}{2}(15+\ell)+12=60$

$n(15+\ell) = 96$

Possible integral values of n are 1, 2, 3, 4, 6, 8 and 12.

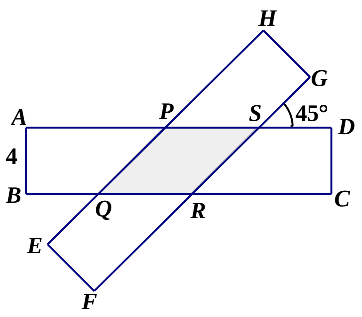
The corresponding values of ℓ are 81, 33, 17, 9, –1, –3 and –7.

$\because 1 \le \ell \le 15$

$\therefore \ell = 9$ only

I6 在圖一中，把長方形 $ABCD$ 繞它的中心逆時針轉 45° 得長方形 $EFGH$。若 $AB = 4$，求陰影部分 $PQRS$ 的面積。

In Figure 1, the rectangle $ABCD$ is rotated about its centre 45° anticlockwise to obtain the rectangle $EFGH$. If $AB = 4$, find the area of the shaded region $PQRS$.



圖一 Figure 1

Reference: 2011 FG2.1

Let $PQ = x$. Then $x \cos 45° = 4$

$x=4\sqrt{2}$

$\because EF = AB = 4$ and $\angle FRQ = 45°$

$\therefore QR=4\sqrt{2}$

$\therefore PQRS$ is a rhombus with length of side $=4\sqrt{2}$ and $\angle PQR = 45°$

Area of $PQRS=\left(4\sqrt{2}\right)^2\sin 45^\circ=16\sqrt{2}$

I7 求$\left(\frac{1\times4\times16\times64+2\times8\times32\times128+3\times12\times48\times192+\cdots+2023\times8092\times32368\times129472}{1\times5\times25\times125+2\times10\times50\times250+3\times15\times75\times375+\cdots+2023\times10115\times50575\times252875}\right)^{\frac{1}{6}}$的值。

Evaluate $\left(\frac{1\times4\times16\times64+2\times8\times32\times128+3\times12\times48\times192+\cdots+2023\times8092\times32368\times129472}{1\times5\times25\times125+2\times10\times50\times250+3\times15\times75\times375+\cdots+2023\times10115\times50575\times252875}\right)^{\frac{1}{6}}$.

Reference: 2000 FI5.1, 2015 FG1.1

$$\left(\frac{1\times4\times16\times64+2\times8\times32\times128+3\times12\times48\times192+\cdots+2023\times8092\times32368\times129472}{1\times5\times25\times125+2\times10\times50\times250+3\times15\times75\times375+\cdots+2023\times10115\times50575\times252875}\right)^{\frac{1}{6}}$$

$$=\left[\frac{1\times4\times16\times64\left(1^4+2^4+3^4+\cdots+2023^4\right)}{1\times5\times25\times125\left(1^4+2^4+3^4+\cdots+2023^4\right)}\right]^{\frac{1}{6}}$$

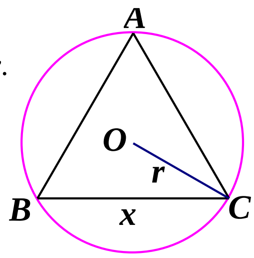
$$=\left(\frac{4\times4^2\times4^3}{5\times5^2\times5^3}\right)^{\frac{1}{6}}=\left(\frac{4^6}{5^6}\right)^{\frac{1}{6}}=\frac{4}{5}$$

I8 若一個等邊三角形的面積與其在周界在數值上相等，求該正三角形的外接圓的半徑。

If the area of an equilateral triangle is numerically equal to its perimeter, find the radius of the circumcircle of this equilateral triangle.

Let the equilateral triangle be *ABC* with *BC* = *x*.

Let *O* be the centre of the circumscribed circle with *OC* = *r*.



Perimeter = $3x$, area $=\frac{1}{2}\cdot x^2\sin 60^\circ=\frac{\sqrt{3}}{4}x^2$

$3x=\frac{\sqrt{3}}{4}x^2\Rightarrow x=4\sqrt{3}$

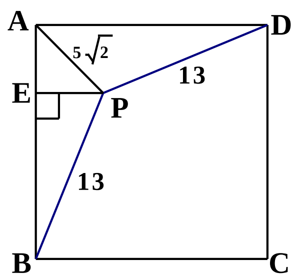
Apply cosine rule on Δ*BOC*: $2r^2-2r^2\cos 120^\circ=x^2$

$r^2\left[2-2\cdot\left(-\frac{1}{2}\right)\right]=\left(4\sqrt{3}\right)^2\Rightarrow r=4$

I9 在圖二中，*P* 為正方形 *ABCD* 內的一點使得 $\Delta ABP\cong\Delta ADP$，$AP=5\sqrt{2}$ 及 $BP=13$。求正方形 *ABCD* 的面積。

In Figure 2, *P* is a point inside the square *ABCD* such that $\Delta ABP\cong\Delta ADP$. $AP=5\sqrt{2}$ and $BP=13$.

Find the area of square *ABCD* .



圖二 Figure 2

$\angle BAD=90^\circ$ (property of square)

$\angle BAP=\angle DAP$ (corr. ∠s, ≅Δs)

$=45^\circ$

Let *E* be the foot of perpendicular from *P* to *AB*.

$AE=AP\cos 45^\circ=5\sqrt{2}\times\frac{1}{\sqrt{2}}=5$, $EP=AP\sin 45^\circ=5$

$EP^2+BE^2=BP^2$ (Pythagoras' theorem)

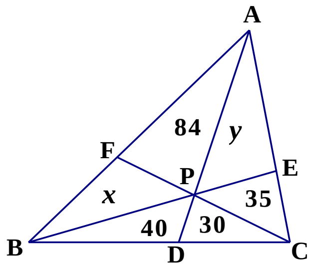
$BE=\sqrt{13^2-5^2}=12$

$AB=AE+EB=5+12=17$

Area of *ABCD* $=17^2=289$

I10 在圖三中，D、E 及 F 分別為 BC、AC 及 AB 上的點。AD、BE 及 CF 相交於 P 使得 ΔAPF 的面積= 84、ΔBPD 的面積= 40、ΔCPD 的面積= 30 及 ΔCPE 的面積= 35。求ΔABC 的面積。

In Figure 3, D, E and F are points lying on BC, AB and AB respectively. AD, BE and CF intersect at P such that such that area of ΔAPF = 84, area of ΔBPD = 40, area of ΔCPD = 30 and area of ΔCPE = 35. Find the area of ΔABC .



圖三 Figure 3

Let S represents the area(s), $S_{\Delta BPF} = x$, $S_{\Delta APE} = y$

ΔBPD and ΔCPD have the same height but different bases

$$\frac{S_{\Delta BPD}}{S_{\Delta CPD}} = \frac{BD}{DC} = \frac{4}{3}$$

ΔABD and ΔACD have the same height but different bases

$$\frac{S_{\Delta ABD}}{S_{\Delta ACD}} = \frac{BD}{DC} = \frac{84+x+40}{y+35+30} = \frac{4}{3}$$

$3(124 + x) = 4(y + 65)$

$372 + 3x = 4y + 260$

$112 + 3x = 4y \cdots\cdots (1)$

ΔBPC and ΔCPE have the same height but different bases

$$\frac{S_{\Delta BPC}}{S_{\Delta CPE}} = \frac{BP}{PE} = \frac{40+30}{35} = \frac{2}{1}$$

ΔABP and ΔAPE have the same height but different bases

$$\frac{S_{\Delta ABP}}{S_{\Delta APE}} = \frac{BP}{PE} = \frac{84+x}{y} = \frac{2}{1}$$

$84 + x = 2y \cdots\cdots (*)$

$168 + 2x = 4y \cdots\cdots (2)$

(1) = (2) $112 + 3x = 168 + 2x$

$x = 56$

Sub. $x = 56$ into (*): $84 + 56 = 2y$

$y = 70$

Area of $\Delta ABC = 40 + 30 + 35 + 70 + 84 + 56 = 315$

I11 已知 n 是一個少於 2023 正整數。若 n 只有三個不同的因數，求 n 的可能性的總數。

Given that n is a positive integer less than 2023.

If n has only 3 distinct factors, find the number of possible values of n.

Reference: 2004 FI1.1

n must be the square of a prime number p, i.e. $n = p^2$, the factors are 1, p and p^2.

$44^2 = 1936 < 45^2 = 2025$, $44 < \sqrt{2023} < 45$

Possible n are 2^2, 3^2, 5^2, 7^2, 11^2, 13^2, 17^2, 19^2, 23^2, 29^2, 31^2, 37^2, 41^2 and 43^2 .

Number of possible n is 14.

I12 已知 p 及 q 為正實數。若 $\log_9 p = \log_{15} q = \log_{25}(3p+2q)$，求 $\frac{p}{q}$ 的值。

Given that p and q are positive numbers. If $\log_9 p = \log_{15} q = \log_{25}(3p+2q)$, find the value of $\frac{p}{q}$.

Let $\log_9 p = \log_{15} q = \log_{25}(3p+2q) = k$

$$\frac{\log p}{\log 9} = \frac{\log q}{\log 15} = \frac{\log(3p+2q)}{\log 25} = k$$

$\log 9 = \frac{\log p}{k}$, $\log 15 = \frac{\log q}{k}$, $\log 25 = \frac{\log(3p+2q)}{k}$

$\log 3 = \frac{\log p}{2k}$, $\log 3 + \log 5 = \frac{\log q}{k}$, $\log 5 = \frac{\log(3p+2q)}{2k}$

$\frac{\log p}{2k} + \frac{\log(3p+2q)}{2k} = \frac{\log q}{k}$

$\log p + \log(3p + 2q) = 2 \log q$

$p(3p + 2q) = q^2$

$3p^2 + 2pq - q^2 = 0$

$(3p - q)(p + q) = 0$

$\frac{p}{q} = \frac{1}{3}$ or -1 (rejected)

I13 數列 $\{a_n\}$ 定義為 $a_1 = 1$、$a_2 = \frac{3}{7}$ 及對所有 $n \ge 3$，$a_n = \frac{a_{n-2}a_{n-1}}{2a_{n-2} - a_{n-1}}$。求 $\frac{1}{a_{2023}}$ 的值。

A sequence of numbers $\{a_n\}$ is defined by $a_1 = 1$, $a_2 = \frac{3}{7}$ and $a_n = \frac{a_{n-2}a_{n-1}}{2a_{n-2} - a_{n-1}}$ for all $n \ge 3$.

Find the value of $\frac{1}{a_{2023}}$.

Let $b_1 = \frac{1}{a_1} = 1$, $b_2 = \frac{1}{a_2} = \frac{7}{3}$, $b_n = \frac{1}{a_n} = \frac{2}{a_{n-1}} - \frac{1}{a_{n-2}} = 2b_{n-1} - b_{n-2}$

The characteristics equation for b_n is $\lambda^2 = 2\lambda - 1 \Rightarrow \lambda^2 - 2\lambda + 1 = 0 \Rightarrow \lambda = 1$

$\therefore$ The general solution for $b_n = A + Bn$, where A and B are constants

$b_1 = A + B = 1$ …… (1)

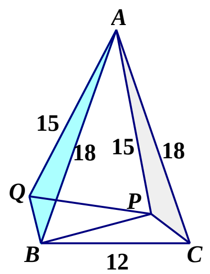
$b_2 = A + 2B = \frac{7}{3}$ … (2)

(2) – (1): $B = \frac{4}{3}$, $A = -\frac{1}{3}$

$b_n = -\frac{1}{3} + \frac{4n}{3}$

$b_{2023} = \frac{1}{a_{2023}} = -\frac{1}{3} + \frac{4 \times 2023}{3} = 2697$

I14 ABC 是一個等腰三角形，其中 $AB = AC = 18$ 及 $BC = 12$。P 為 ΔABC 內的任意一點使得 $\angle ABP + \angle ACP = 90°$ 及 $AP = 15$。求 $BP^2 + CP^2$ 的值。



ABC is an isosceles triangle with $AB = AC = 18$ and $BC = 12$. P is any interior point of ΔABC such that $\angle ABP + \angle ACP = 90°$ and $AP = 15$.

Find the value of $BP^2 + CP^2$.

Let $\angle BAC = \theta$. Rotate ΔAPC clockwise about A through θ to ΔAQB.

$\Delta APQ \sim \Delta ACB$ (S.A.S.) and $\Delta ABQ \cong \Delta ACP$

$QP : 12 = 15 : 18$ (corr. sides, $\sim\Delta$s)

$\Rightarrow QP = 10$

$\angle PBQ = \angle ABP + \angle ABQ$

$= \angle ABP + \angle ACP = 90°$ (corr. $\angle$s, $\cong\Delta$s)

$BP^2 + CP^2 = PQ^2 = 100$ (Pythagoras' theorem)

I15 求方程 $\sqrt[3]{x}+\sqrt[3]{x-4}=\sqrt[3]{x-2}$ 的根之積。

Find the product of roots of the equation $\sqrt[3]{x}+\sqrt[3]{x-4}=\sqrt[3]{x-2}$.

Let $t = x - 2$, then $x = t + 2$, $x - 4 = t - 2$, the equation becomes $\sqrt[3]{t+2}+\sqrt[3]{t-2}=\sqrt[3]{t}$

$$\left(\sqrt[3]{t+2}+\sqrt[3]{t-2}\right)^3=\left(\sqrt[3]{t}\right)^3$$

$$t+2+3\sqrt[3]{(t+2)^2}\cdot\sqrt[3]{t-2}+3\sqrt[3]{t+2}\cdot\sqrt[3]{(t-2)^2}+t-2=t$$

$$t+3\sqrt[3]{(t^2-4)}\cdot\sqrt[3]{t+2}+3\sqrt[3]{t-2}\cdot\sqrt[3]{(t^2-4)}=0$$

$$t+3\sqrt[3]{(t^2-4)}\cdot\left(\sqrt[3]{t+2}+\sqrt[3]{t-2}\right)=0$$

$$t+3\sqrt[3]{(t^2-4)}\cdot\sqrt[3]{t}=0$$

$$3\sqrt[3]{(t^3-4t)}=-t$$

$27(t^3 - 4t) = -t^3$

$28t^3 - 108t = 0$

$t(7t^2 - 27) = 0$

$$t=0,\ \sqrt{\frac{27}{7}}\ \text{ or }\ -\sqrt{\frac{27}{7}}$$

$$x=2,\ 2+\sqrt{\frac{27}{7}}\ \text{ or }\ 2-\sqrt{\frac{27}{7}}$$

$$\text{Product of roots}=2\left(2+\sqrt{\frac{27}{7}}\right)\left(2-\sqrt{\frac{27}{7}}\right)$$

$$=2\left(4-\frac{27}{7}\right)$$

$$=\frac{2}{7}$$

Group Events

G1 求 3^{2023} 的最尾兩位數字。Find the last two digits of 3^{2023}.

Reference 2021 P1Q2

$3^1 = 03$, $3^2 = 09$, $3^3 = 27$, $3^4 = 81$, $3^5 \equiv 43$, $3^6 \equiv 29$, $3^7 \equiv 87$, $3^8 \equiv 61$, $3^9 \equiv 83$, $3^{10} \equiv 49$

$3^{11} \equiv 47$, $3^{12} \equiv 41$, $3^{13} \equiv 23$, $3^{14} \equiv 69$, $3^{15} \equiv 07$, $3^{16} \equiv 21$, $3^{17} \equiv 63$, $3^{18} \equiv 89$, $3^{19} \equiv 67$, $3^{20} \equiv 01$

$3^{2023} = (3^{20})^{101} \cdot 3^3 \equiv 27 \pmod{100}$

Method 2 (provided by Mr. Tam Chi Leung)

$$3^{2023} = 3\times(3^2)^{1011} = 3\times(10-1)^{1011}$$

$$= 3\sum_{k=0}^{1011} C_k^{1011}(-1)^k 10^{1011-k}$$

$$= 3\left(10^{1011} - C_1^{1011}\cdot 10^{1010} + C_2^{1011}\cdot 10^{1009} + \cdots - C_{1009}^{1011}\cdot 10^2 + C_{1010}^{1011}\cdot 10 - 1\right)$$

$= 3(100m + 10110 - 1)$, where m is an integer

$= 300m + 30327$

The last two digits = 27.

G2 對於 $0 < x < 2$，求 $\left(\frac{\sqrt{2+x}}{\sqrt{2+x}-\sqrt{2-x}} + \frac{2-x}{\sqrt{4-x^2}+x-2}\right)\left(\sqrt{\frac{4}{x^2}-1}-\frac{2}{x}\right)$ 的值。

For $0 < x < 2$, find the value of $\left(\frac{\sqrt{2+x}}{\sqrt{2+x}-\sqrt{2-x}} + \frac{2-x}{\sqrt{4-x^2}+x-2}\right)\left(\sqrt{\frac{4}{x^2}-1}-\frac{2}{x}\right)$.

Reference: 2017 FG3.2

Expression

$$= \left[\frac{\sqrt{2+x}}{\sqrt{2+x}-\sqrt{2-x}}\cdot\frac{\sqrt{2+x}+\sqrt{2-x}}{\sqrt{2+x}+\sqrt{2-x}} + \frac{2-x}{\sqrt{4-x^2}-(2-x)}\cdot\frac{\sqrt{4-x^2}+(2+x)}{\sqrt{4-x^2}+(2+x)}\right]\left(\frac{\sqrt{4-x^2}}{x}-\frac{2}{x}\right)$$

$$= \left[\frac{2+x+\sqrt{4-x^2}}{2+x-(2-x)} + \frac{(2-x)\left(\sqrt{4-x^2}+2-x\right)}{4-x^2-(2-x)^2}\right]\left(\frac{\sqrt{4-x^2}-2}{x}\right)$$

$$= \left[\frac{2+x+\sqrt{4-x^2}}{2x} + \frac{\sqrt{4-x^2}+2-x}{2x}\right]\cdot\frac{\sqrt{4-x^2}-2}{x}$$

$$= \frac{2\sqrt{4-x^2}+4}{2x}\cdot\frac{\sqrt{4-x^2}-2}{x} = \frac{\sqrt{4-x^2}+2}{x}\cdot\frac{\sqrt{4-x^2}-2}{x} = \frac{4-x^2-4}{x^2} = -1$$

G3 已知 $\tan\alpha$ 和 $\tan\beta$ 是二次方程 $x^2 - 4x - 2 = 0$ 的根。

求 $\sin^2(\alpha+\beta) + 2\sin(\alpha+\beta)\cos(\alpha+\beta) + 3\cos^2(\alpha+\beta)$ 的值。

Given that $\tan\alpha$ and $\tan\beta$ are the roots of the quadratic equation $x^2 - 4x - 2 = 0$.

Find the value of $\sin^2(\alpha+\beta) + 2\sin(\alpha+\beta)\cos(\alpha+\beta) + 3\cos^2(\alpha+\beta)$.

$\tan\alpha + \tan\beta = 4$, $\tan\alpha\tan\beta = -2$

$$\tan(\alpha+\beta) = \frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta} = \frac{4}{1-(-2)} = \frac{4}{3} > 0$$

$\alpha + \beta$ lies in the first or third quadrant.

When $\alpha + \beta$ lies in the first quadrant, $\sin(\alpha+\beta) > 0$, $\cos(\alpha+\beta) > 0$

$\sin(\alpha+\beta) = \frac{4}{5}$, $\cos(\alpha+\beta) = \frac{3}{5}$

$$\text{Expression} = \left(\frac{4}{5}\right)^2 + 2\left(\frac{4}{5}\right)\left(\frac{3}{5}\right) + 3\left(\frac{3}{5}\right)^2 = \frac{67}{25}$$

When $\alpha + \beta$ lies in the second quadrant, the answer is the same.

G4 排列 5 個不同的單數及 5 個不同的雙數在同一行使得任意兩個相鄰數的積必為雙數。求所有排列的可能性數目。

Five distinct odd numbers and five distinct even numbers are arranged in a row such that the product of any two consecutive numbers is always even.

Find the number of all possible arrangements.

Method 1 Arrange the five odd numbers in a row, number of permutations = 5! = 120

The following pattern shows one possible arrangement:

	O_2		O_5		O_1		O_4		O_3	

G_1 G_2 G_3 G_4 G_5 G_6

Mark the gaps amongst these odd numbers as G_1, G_2, G_3, G_4, G_5 and G_6 .

If E_1 is put into G_1, then only the following pattern is possible:

E	O_2	E	O_5	E	O_1	E	O_4	E	O_3

Number of permutations for putting "É"s = 5! = 120

If E_5 is put into G_6, then only the following pattern is possible:

	O_2	E	O_5	E	O_1	E	O_4	E	O_3	E

Number of permutations for putting "É"s = 5! = 120

If E does not put in G_1 and G_6, then, the 5 "E"s must be inserted into G_2, G_3, G_4 and G_5.

One of G_2, G_3, G_4, G_5 must contain 2 "E"s, number of arrangements = $P_2^5 \times 4!$ = 480

Total number of permutations = 120×(120 + 120 + 480) = 86400

Method 2 If these 10 numbers are arranged in any order, number of permutation = 10! = 3628800

Let **O** be an odd number, **E** be an even number. Then **E**× **E** = **O**× **E** = **E**×**O** = even, **O**×**O** = odd

We find the number of ways so that **the product of at least one pair of consecutive number is odd**.

Case 1 Divide these five odd numbers into groups of (O_1O_2), O_3, O_4, O_5 .

Number of ways of arranging these odd numbers = $P_2^5 \times 4! = 480$

Next, arrange the five even numbers in a row, number of permutations = 5! = 120

Then insert the 4 groups of odd numbers amongst the 6 gaps of even numbers so that every group of odd numbers is separated by even numbers. Number of combinations = C_4^6 = 15

Number of different arrangement in case 1 = 480×120×15 = 864000

Case 2 Divide these five odd numbers into groups of (O_1O_2), (O_3O_4), O_5 .

Similarly, the number of different arrangements in case 2 = $5\times P_2^4 \times P_2^2 \div 2 \times 3! \times 5! \times C_3^6 = 864000$

Case 3 Divide these five odd numbers into groups of (O_1O_2), ($O_3O_4O_5$).

The number of different arrangements in case 3 = $P_3^5 \times P_2^2 \times 2! \times 5! \times C_2^6 = 432000$

Case 4 Divide these five odd numbers into groups of O_1 , O_2 , ($O_3O_4O_5$).

The number of different arrangements in case 4 = $P_3^5 \times 3! \times 5! \times C_3^6 = 864000$

Case 5 Divide these five odd numbers into groups of ($O_1O_2O_3O_4$), O_5.

The number of different arrangements in case 5 = $P_4^5 \times 2! \times 5! \times C_2^6 = 432000$

Case 6 Divide these five odd numbers into one group = ($O_1O_2O_3O_4O_5$).

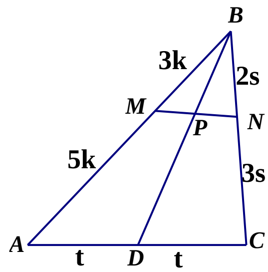
The number of different arrangements in case 6 = $5! \times 5! \times C_1^6 = 86400$

Total number of arrangements = 3628800 – (864000 + 864000 + 432000 + 864000 + 432000 + 86400)
= 3628800 – 3542400 = 86400

G5 在圖一中，M 和 N 分別是ΔABC 的邊 AB 和 BC 上的點。MN 與ΔABC 的中線相交於 P。若 $\frac{AM}{BM}=\frac{5}{3}$ 及 $\frac{CN}{BN}=\frac{3}{2}$。求 $\frac{DP}{BP}$ 的值。

In Figure 1, M and N are points on AB and BC of ΔABC respectively. MN and the median of ΔABC intersect at P.

If $\frac{AM}{BM}=\frac{5}{3}$ and $\frac{CN}{BN}=\frac{3}{2}$, find the value of $\frac{DP}{BP}$.



圖一 Figure 1

Let $AM = 5k$, $BM = 3k$, $CN = 3s$, $BN = 2s$, $AD = CD = t$

Join AP, CP, MD. Let the areas be S. $S_{\Delta BMP} = p$, $S_{\Delta BNP} = q$

ΔAMP and ΔBMP have the same height but different bases

$\frac{S_{\Delta AMP}}{S_{\Delta BMP}}=\frac{AM}{BM}=\frac{5}{3}\Rightarrow S_{\Delta AMP}=\frac{5p}{3}$

ΔCNP and ΔBNP have the same height but different bases

$\frac{S_{\Delta CNP}}{S_{\Delta BNP}}=\frac{CN}{BN}=\frac{3}{2}\Rightarrow S_{\Delta CNP}=\frac{3q}{2}$

ΔADP and ΔCDP have the same height and same base

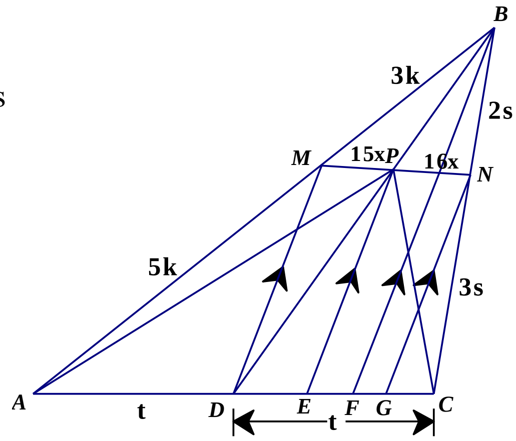
$\Rightarrow S_{\Delta ADP} = S_{\Delta CDP}$

ΔABD and ΔCBD have the same height and same base

$\Rightarrow S_{\Delta ABD} = S_{\Delta CBD}$

$S_{\Delta ABD} - S_{\Delta ADP} = S_{\Delta CBD} - S_{\Delta CDP} \Rightarrow S_{\Delta ABP} = S_{\Delta CBP}$

$p+\frac{5p}{3}=q+\frac{3q}{2}\Rightarrow\frac{8p}{3}=\frac{5q}{2}\Rightarrow\frac{p}{q}=\frac{15}{16}$



ΔBMP and ΔBNP have the same height but different bases

$\frac{S_{\Delta BMP}}{S_{\Delta BNP}}=\frac{MP}{NP}\Rightarrow\frac{p}{q}=\frac{MP}{NP}=\frac{15}{16}$. Let $MP = 15x$, $NP = 16x$

Draw PE, BF and NG // MD, cutting AC at E, F and G respectively.

By the theorem of equal ratios,

$AD : DF = AM : BM = 5 : 3$, $FG : GC = BN : CN = 2 : 3$, $DE : GE = MP : NP = 15 : 16$

$DF=\frac{3t}{5}=0.6t$, $CF = CD - DF = t - 0.6t = 0.4t$

$GC=\frac{3}{5}CF=\frac{3}{5}\times 0.4t=0.24t$, $FG=\frac{2}{5}CF=\frac{2}{5}\times 0.4t=0.16t$

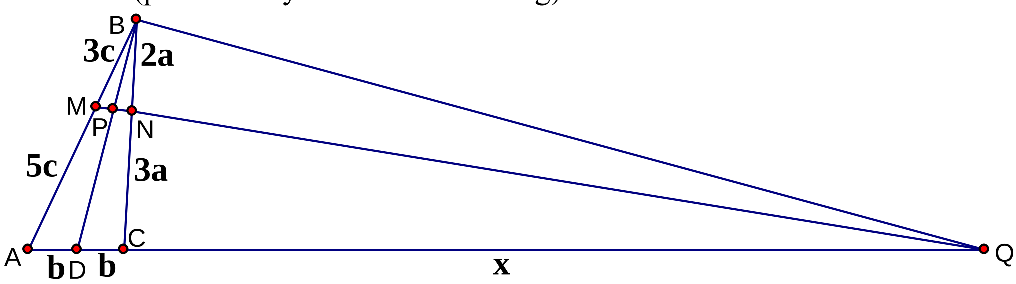
$DG = DF + FG = 0.6t + 0.16t = 0.76t$

$DE=\frac{15}{15+16}DG=\frac{15}{31}\times 0.76t=\frac{11.4}{31}t$, $GE=\frac{16}{31}\times 0.76t=\frac{12.16}{31}t$

$FE=GE-FG=\frac{12.16}{31}t-0.16t=\frac{7.2}{31}t$

$\frac{DP}{BP}=\frac{DE}{FE}=\frac{11.4}{31}t\div\frac{7.2}{31}t=\frac{19}{12}$

Method 2 (provided by Mr.Tam Chi Leung)



Produce MN and AC to intersect at Q. Let CQ be x.

Apply Menelus theorem, ΔABC is cut by MNQ.

$$\frac{AM}{MB}\cdot\frac{BN}{NC}\cdot\frac{CQ}{QA}=1$$

$$\frac{5}{3}\cdot\frac{2}{3}\cdot\frac{x}{x+2b}=1$$

$10x = 9(x + 2b)$

$x = 18b$

Apply Menelus theorem, ΔABD is cut by MPQ.

$$\frac{AM}{MB}\cdot\frac{BP}{PD}\cdot\frac{DQ}{QA}=1$$

$$\frac{5}{3}\cdot\frac{BP}{PD}\cdot\frac{x+b}{x+2b}=1$$

$$\frac{5}{3}\cdot\frac{BP}{PD}\cdot\frac{18b+b}{18b+2b}=1$$

$$\frac{DP}{BP}=\frac{19}{12}$$

G6 設 x、y 及 z 為實數，且滿足方程 $\begin{cases} x+yz=6 & \cdots\cdots(1) \\ y+zx=6 & \cdots\cdots(2) \\ z+xy=6 & \cdots\cdots(3) \end{cases}$，求 xyz 的最大值。

If x, y and z are real numbers that satisfy the system of equations $\begin{cases} x+yz=6 & \cdots\cdots(1) \\ y+zx=6 & \cdots\cdots(2) \\ z+xy=6 & \cdots\cdots(3) \end{cases}$, find the largest possible value of xyz.

From (1): $x = 6 - yz$ …… (*)

Sub. (*) into (2): $y + z(6 - yz) = 6 \Rightarrow y + 6z - yz^2 = 6 \Rightarrow y(1 - z^2) = 6(1 - z)$ …… (4)

If $z = 1$, then (1) becomes $x + y = 6$ and (3) becomes $xy = 5 \Rightarrow (x, y) = (1, 5)$ or $(5, 1) \Rightarrow xyz = 5$

If $z \neq 1$, then (4) becomes $y(1 + z) = 6 \Rightarrow y=\dfrac{6}{1+z}$ …… (5)

Sub. (5) into (*) $x=6-\dfrac{6z}{1+z}=\dfrac{6}{1+z}$ …… (6)

Sub. (5), (6) into (3): $z+\dfrac{6}{1+z}\cdot\dfrac{6}{1+z}=6$

$z(z^2 + 2z + 1) + 36 = 6(z^2 + 2z + 1)$

$z^3 - 4z^2 - 11z + 30 = 0$

$(z - 5)(z^2 + z - 6) = 0$

$(z - 5)(z + 3)(z - 2) = 0$

$z = 5, -3$ or 2

When $z = 5$, $x = y = 1$, $xyz = 5$

When $z = -3$, $x = y = -3$, $xyz = -27$

When $z = 2$, $x = y = 2$, $xyz = 8$

The largest possible value of xyz is 8.

G7 整數數列 $\{a_n\}$ 定義為 $a_n = 100 + n^2$，其中 n 為正整數。設 d_n 為 a_n 和 a_{n+1} 的最大公因數。求 d_n 的最大值。

A sequence of integers $\{a_n\}$ is defined by $a_n = 100 + n^2$, where n is a positive integer. Let d_n be the greatest common divisor of a_n and a_{n+1}. Find the greatest possible value of d_n.

The solution is provided by Mr. Tam Chi Leung.

$$\begin{cases} n^2 + 100 \equiv 0 \pmod{d_n} & \cdots\cdots(1) \\ n^2 + 2n + 101 \equiv 0 \pmod{d_n} & \cdots\cdots(2) \end{cases}$$

(2) – (1): $2n + 1 \equiv 0 \pmod{d_n}$

$2n \equiv -1 \pmod{d_n}$

$4n^2 \equiv 1 \pmod{d_n}$ …… (3)

From (1) $n^2 \equiv -100 \pmod{d_n}$

$4n^2 \equiv -400 \pmod{d_n}$ …… (4)

(3) – (4): $0 \equiv 401 \pmod{d_n}$

As 401 is a prime number, $d_n = 401$ or 1

$\therefore$ The largest possible value of d_n is 401.

G8 已知 x 及 y 為正實數且滿足 $x^2 - y^2 = 4\cdots(1)$及 $xy = 2\cdots(2)$。

若 $x + y$ 可寫成 $a\sqrt{b+\sqrt{c}}$，其中 a、b 及 c 均為正整數，求 $100a + 10b + c$ 的最小值。

Given that x and y are positive real numbers satisfying $x^2 - y^2 = 4\cdots(1)$ and $xy = 2\cdots(2)$.

If the value of $x + y$ can be expressed in the form of $a\sqrt{b+\sqrt{c}}$, where a, b and c are positive integers, find the least value of $100a + 10b + c$.

From (2), $y = \frac{2}{x}\cdots(3)$

Sub. (3) into (1): $x^2 - \frac{4}{x^2} = 4$

$x^4 - 4x^2 - 4 = 0$

$x^2 = 2 + 2\sqrt{2}$ or $2 - 2\sqrt{2}$ (< 0, rejected)

$x = \sqrt{2+2\sqrt{2}}$ or $-\sqrt{2+2\sqrt{2}}$ (< 0, rejected)

$= \sqrt{2\left(1+\sqrt{2}\right)} = \sqrt{2\left(\sqrt{2}+1\right)}$

$y = \frac{2}{\sqrt{2\left(\sqrt{2}+1\right)}}$

$= \frac{\sqrt{2}}{\sqrt{\sqrt{2}+1}} \cdot \frac{\sqrt{\sqrt{2}-1}}{\sqrt{\sqrt{2}-1}} = \sqrt{2\left(\sqrt{2}-1\right)}$

$x + y = \sqrt{2\left(\sqrt{2}+1\right)} + \sqrt{2\left(\sqrt{2}-1\right)}$

$(x+y)^2 = 2\left(\sqrt{2}+1\right) + 2\sqrt{4\sqrt{2-1}} + 2\left(\sqrt{2}-1\right)$

$= 4 + 4\sqrt{2}$

$x + y = \sqrt{4+4\sqrt{2}} = 2\sqrt{1+\sqrt{2}}$

$(a, b, c) = (1, 4, 32)$, or $(2, 1, 2)$.

$100a + 10b + c = 140 + 32 = 172$ or $200 + 10 + 2 = 212$

The least value of $100a + 10b + c$ is 172.

G9 定義 $f(z)=z^2+4z$，其中 z 是一個複數，設 $z=x+2i$，當中 x 為非零實數。

若 $\frac{f\left(f\left(z\right)\right)-f\left(z\right)}{z-f\left(z\right)}$ 是一個純虛數，求 x 的值。

Define $f(z)=z^2+4z$, where z is a complex number. let $z=x+2i$, where x is a non-zero real number. If $\frac{f\left(f\left(z\right)\right)-f\left(z\right)}{z-f\left(z\right)}$ is a purely imaginary number, find the value of x.

$f(f(z))=(z^2+4z)^2+4(z^2+4z)$

$=z^4+8z^3+16z^2+4z^2+16z$

$=z^4+8z^3+20z^2+16z$

$f(f(z))-f(z)=z^4+8z^3+20z^2+16z-(z^2+4z)$

$=z^4+8z^3+19z^2+12z$

$z-f(z)=z-(z^2+4z)=-(z^2+3z)$

$$\frac{f\left(f\left(z\right)\right)-f\left(z\right)}{z-f\left(z\right)}=-\frac{z^4+8z^3+20z^2+16z}{z^2+3z}$$

$$=-\frac{z^3+8z^2+20z+16}{z+3}$$

$=-(z^2+5z+4)$

$=-[(x+2i)^2+5(x+2i)+4]$

$=-(x^2+4xi-4+5x+10i+4)$

$=-[(x^2+5x)+(4x+10)i]$

It is a purely imaginary number $\Rightarrow$ Real part = 0

$x^2+5x=0$

$x=0$ or -5

$\because$ x is non-zero $\therefore$ $x=-5$ only

G10 下列方程有一個實數解：$\begin{cases}3\log_a\left(\sqrt{x}\log_a x\right)=26\cdots\cdots(1)\\ \log_{\log_a x} x=24\cdots\cdots(2)\end{cases}$，求 a 的值。

The following system of equations has one real number solution $\begin{cases}3\log_a\left(\sqrt{x}\log_a x\right)=26\cdots(1)\\ \log_{\log_a x} x=24\cdots(2)\end{cases}$, find the value of a.

From (1), $\sqrt{x}\log_a x=a^{\frac{26}{3}}\cdots(3)$

From (3), $\log_a x=x^{\frac{1}{24}}\cdots\cdots(4)$

Sub. (4) into (3): $x^{\frac{1}{2}}\cdot x^{\frac{1}{24}}=a^{\frac{26}{3}}$

$x^{\frac{13}{24}}=a^{\frac{26}{3}}$

$x=a^{16}\cdots\cdots(5)$

Sub. (5) into (4): $\log_a a^{16}=\left(a^{16}\right)^{\frac{1}{24}}$

$16=a^{\frac{2}{3}}$

$a=64$