23-24 Individual	**1**	4320	**2**	$\frac{1}{81}$	**3**	1001	**4**	4096	**5**	$2\sqrt{26}$
	6	1980	**7**	70	**8**	23°	**9**	$-1+\sqrt{2}$	**10**	540°
	11	41616	**12**	$\frac{85}{8}$	**13**	$\frac{2023}{4100625}$	**14**	$5\sqrt{2}$ cm	**15**	5
23-24 Group	**1**	$\frac{15}{2}=7.5$	**2**	132°	**3**	1351	**4**	3	**5**	$\frac{1025}{64}$
	6	$\frac{5\sqrt{17}}{16}$	**7**	225	**8**	74	**9**	41	**10**	$\frac{\sqrt{5}-1}{2}$

Individual Events

I1 求 2024 的所有因數之和。

Find the sum of the factors of 2024.

Reference: 2002 FG4.1

$2024 = 2^3\times 11\times 23$

The positive factors may be $2^a 11^b 23^c$, where $0 \le a \le 3$, $0 \le b, c \le 1$ are integers.

The sum of all positive factors are $(1 + 2 + 2^2 + 2^3)(1 + 11)(1 + 23) = 15\times 12\times 24 = 4320$

I2 若 $a^{3y} = 729$，求 a^{-2y} 的值。

If $a^{3y} = 729$, find the value of a^{-2y}.

Reference: 1993 FI1.1

$(a^y)^3 = 9^3$

$a^y = 9$

$a^{-2y} = \frac{1}{81}$

I3 一個 6 位數由兩個相同的 3 位數組合而成，如 256256 及 678678。求這些 6 位數的最大公因數。

A 6-digit number is formed by joining two identical 3-digit numbers, such as 256256 and 678678. Find the greatest common factor of these 6-digit numbers.

Reference: 2000 FG4.1, 2010 HG1

Let $p = \overline{abcabc} = \overline{abc}000 + \overline{abc} = \overline{abc}\times 1000 + \overline{abc} = 1001\,\overline{abc}$

The H.C.F. = 1001

I4 若 $4^{x+3} - 47 = 193 + 4^{x+1}$，求 $\left(4^{x+3}\right)\left(4^{x+1}\right)$ 的值。

If $4^{x+3} - 47 = 193 + 4^{x+1}$, find the value of $\left(4^{x+3}\right)\left(4^{x+1}\right)$.

$64\times 4^x = 240 + 4\times 4^x$

$60\times 4^x = 240$

$4^x = 4$

$x = 1$

$\left(4^{x+3}\right)\left(4^{x+1}\right) = 4^4\times 4^2 = 4^6 = 4096$

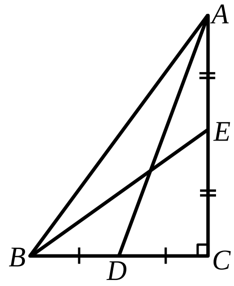
I5 在一直角三角形中，從銳角頂點所作的中線長度為 7 及 9。求三角形斜邊的長。

In a right-angled triangle, the lengths of the medians from the vertices of the acute angles are 7 and 9. Find the length of the hypotenuse of the triangle.

Reference: 1990 HI17

Let the triangle be ΔABC, $\angle C = 90°$ and D, E are the mid-points of BC and CA respectively. $AD = 7$ and $BE = 9$.

Let $BD = x = DC$, $AE = y = EC$



$x^2 + (2y)^2 = 7^2 \cdots\cdots (1)$

$(2x)^2 + y^2 = 9^2 \cdots\cdots (2)$

$4(1) - (2)$: $15y^2 = 115 \Rightarrow y^2 = \frac{23}{3}$

$4(2) - (1)$: $15x^2 = 275 \Rightarrow x^2 = \frac{55}{3}$

$AB^2 = (2x)^2 + (2y)^2 = 4\times\frac{23+55}{3} = 104$

$\Rightarrow AB = 2\sqrt{26}$

I6 志偉生於 20 世紀 (1901–2000)，於 y^2 年時的歲數為 y。求志偉的出生年份。

Eric was born in 20th century (1901–2000), and he was y years old in the year of y^2.

Find his year of birth.

Reference: 1990 HG5

Suppose Eric was born in x years **after** 1900 A.D.

$1900 + x + y = y^2$

$43^2 = 1849 < 1900$, $44^2 = 1936$, $45^2 = 2025 > 2000$

If $y = 44$, $x = 1936 - 1900 - 44 = -8$ (rejected)

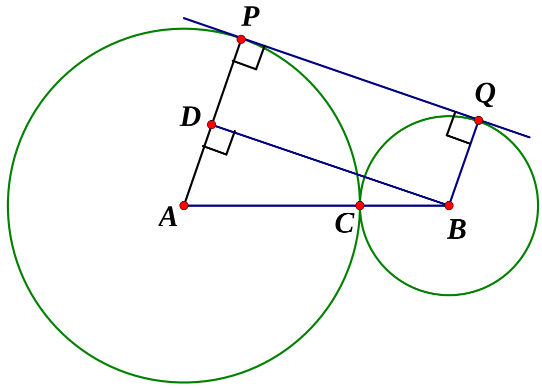
If $y = 45$, $x = 2025 - 1900 - 45 = 80$

$1900 + x = 1980$

$\Rightarrow$ He was born in A.D. 1980.

I7 如圖一所示，一條公切線與一大圓及一小圓分別相交於點 P 及 Q。已知該兩圓相交於點 C 且它們的半徑分別為 49 及 25，求 PQ 的長。

As shown in Figure 1, a common tangent touches a large circle and a small circle at P and Q respectively. Given that the two circles touch each other at C and their radii are 49 and 25 respectively, find the length of PQ.



圖一 Figure 1

As shown, let the centre of the two circles be A and B respectively. $AP\perp PQ$, $BQ\perp PQ$ (tangent ⊥ radius)

Draw $BD\perp AP$, cutting AP at D.

Then $DPQB$ is a rectangle.

$AP = 49$, $BQ = 25 = DP$ (opp. sides of rectangle)

$AD = AP - DP = 49 - 25 = 24$; $AB = AC + CB = 49 + 25 = 74$

$$PQ = DB = \sqrt{AB^2 - AD^2} = \sqrt{74^2 - 24^2}$$

$$= \sqrt{(74+24)(74-24)}$$

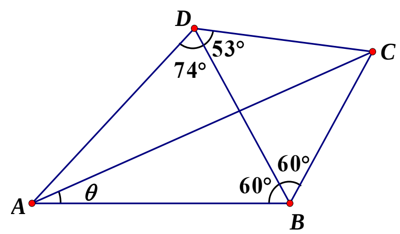
$$= \sqrt{98\times 50}$$

$$= \sqrt{49\times 100} = 70$$

I8 如圖二所示，$ABCD$ 是一個四邊形。若$\angle ABD = \angle CBD = 60°$，$\angle ADB = 74°$及$\angle CDB = 53°$，求$\angle BAC$ 的值。

As shown in Figure 2, $ABCD$ is a quadrilateral.

If $\angle ABD = \angle CBD = 60°$, $\angle ADB = 74°$ and $\angle CDB = 53°$, find the value of $\angle BAC$.



Method 1 provided by Mr. Elstan Fong, Mr. Tony Cho and Mr. Ben Yip

In ΔABD, $\angle BAD = 180° - 60° - 74° = 46°$ (∠ sum of Δ)

Let $BC = 1$, $\angle BAC = \theta$, $\angle CAD = 46° - \theta$.

In ΔBCD, $\frac{DC}{\sin 60°} = \frac{BC}{\sin 53°} \Rightarrow DC = \frac{\sin 60°}{\sin 53°}$ …… (1)

In ΔACD, $\frac{AC}{\sin(74°+53°)} = \frac{DC}{\sin(46°-\theta)}$

By (1), $AC = \frac{\sin 60°}{\sin 53°}\times\frac{\sin 127°}{\sin(46°-\theta)} = \frac{\sin 60°}{\sin(46°-\theta)}$ …… (2)

In ΔABC, $\frac{BC}{\sin\theta} = \frac{AC}{\sin 120°}$

By (2), $\frac{1}{\sin\theta} = \frac{\sin 60°}{\sin(46°-\theta)}\times\frac{1}{\sin 120°}$

$$\frac{1}{\sin\theta} = \frac{1}{\sin(46°-\theta)}$$

$\therefore\ \theta = 46° - \theta$

$\angle BAC = \theta = 23°$

Method 2

Reflect ΔBCD along DC to ΔFCD.

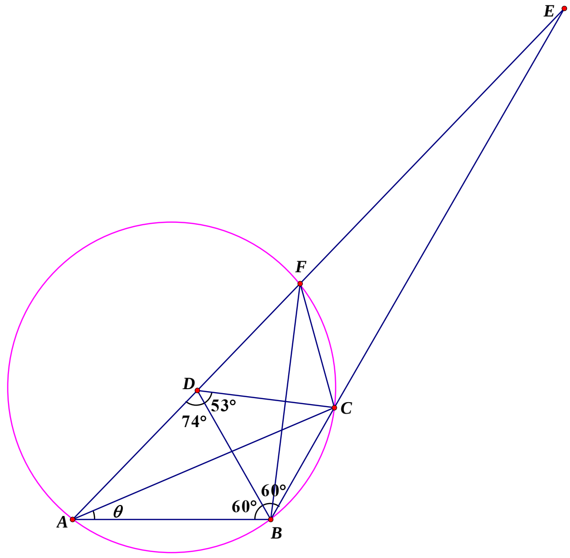
By the property of reflection,

$BD = FD$, $\angle CDF = 53°$, $\angle CFD = 60°$

$\angle ADB + \angle CDB + \angle CDF = 180°$

$\therefore$ A, D, F are collinear

Extend ADF and BC to meet at E.



$$\angle DBF = \angle DFB = \frac{180° - 53° - 53°}{2} = 37°$$

(base $\angle$s, isos. Δ, $\angle$ sum of Δ)

$\angle CFE = 180° - \angle CFD = 120°$ (adj. $\angle$s on st. line)

$= \angle ABC$

$\therefore$ A, B, C, F are concyclic (ext. $\angle$ = int. opp. $\angle$)

$\angle BAC = \theta = \angle BFC$ ($\angle$s in the same segment)

$= \angle CFD - \angle DFB = 60° - 37° = 23°$

I9 設 a 為正實數。若方程組 $\begin{cases}(a+3)x+(a+2)y=1\\(a-1)x-\quad ay=1\end{cases}$ 無解，求 a 的值。

Let a be a positive real number.

If the system of equations $\begin{cases}(a+3)x+(a+2)y=1\\(a-1)x-\quad ay=1\end{cases}$ has no solution, find the value of a.

$$\begin{vmatrix}a+3 & a+2\\ a-1 & -a\end{vmatrix} = 0$$

$-a(a + 3) - (a - 1)(a + 2) = 0$

$-a^2 - 3a - a^2 - a + 2 = 0$

$2a^2 + 4a - 2 = 0 \Rightarrow a^2 + 2a - 1 = 0$

$a = -1 \pm \sqrt{2}$

$\because a > 0$ $\therefore a = -1 + \sqrt{2}$

I10 圖三所示為圓 $ABCDEFGH$，求 $a+b+c+d$ 的值。

Figure 3 shows the circle $ABCDEFGH$.

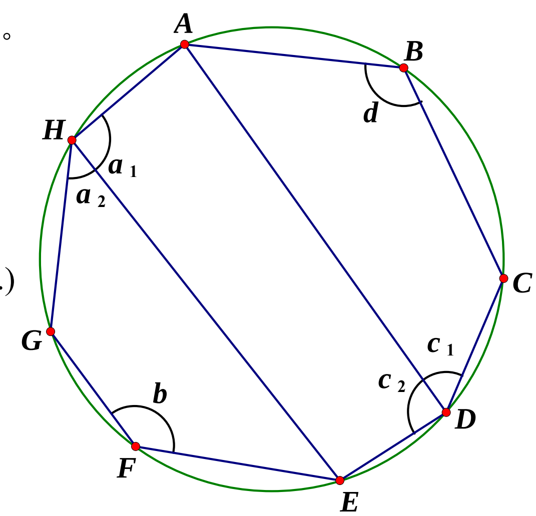
Find the value of $a + b + c + d$.

As shown in the figure, let $a = a_1 + a_2$, $c = c_1 + c_2$

$a_2 + b = 180°$. $a_1 + c_2 = 180°$, $c_1 + d = 180°$

(opp. $\angle$s cyclic quad.)

$a + b + c + d = 180°\times 3 = 540°$



圖三 **Figure 3**

I11 若 $1+2+3+\cdots+k$ 的和為一完全平方 N，其中 $N < 250\,000$，求 N 的最大可能值。

If $1+2+3+\cdots+k$ is a perfect square N, where $N < 250\,000$,

find the largest possible value of N.

Reference: Discovering the square-triangular numbers by Phil Lafer, Washington State University

The first few square-triangular numbers are 1, 36, 1225 and 41616.

$T_1 = 1 \qquad = 1^2 \qquad = 1^2\times(1+0)^2$

$T_2 = 1+2+3+4+5+6 = 36 \qquad = 2^2\times 3^2 \qquad = 2^2\times(2+1)^2$

$T_3 = 1+2+\cdots+49 = \dfrac{49\times 50}{2} = 35^2 = 1225 \qquad = 5^2\times 7^2 \qquad = 5^2\times(5+2)^2$

$T_4 = 1+2+\cdots+288 = \dfrac{288\times 289}{2} = 204^2 = 41616 \qquad = 12^2\times 17^2 \qquad = 12^2\times(12+5)^2$

Define the recurrence sequence $\{a_n\}$ as follows: $a_0 = 0$, $a_1 = 1$, $a_{n+2} = 2a_{n+1} + a_n$ for $n \ge 0$.

Then $a_2 = 2\times 1 + 0 = 2$, $a_3 = 2\times 2 + 1 = 5$, $a_4 = 2\times 5 + 2 = 12$

It **seems that** $a_1, a_2, a_3, a_4, \cdots, a_n$ are the first square factor of $T_1, T_2, T_3, T_4, \cdots, T_n$.

It **also seems that** the second square factor are $(a_1 + a_0), (a_2 + a_1), \cdots, (a_n + a_{n-1})$.

We shall prove this by induction. We have already proved that it is true for $n = 1, 2, 3$ and 4.

Suppose $T_k = 1 + 2 + \cdots + m = \dfrac{m\times(m+1)}{2} = \dfrac{2a_k^2\times(a_k+a_{k-1})^2}{2}$ and $\left|2a_k^2-(a_k+a_{k-1})^2\right| = 1$

Clearly $a_{k+1}^2\times(a_{k+1}+a_k)^2$ is a square number.

$$a_{k+1}^2\times(a_{k+1}+a_k)^2 = \frac{2a_{k+1}^2\times(a_{k+1}+a_k)^2}{2}$$

$$\left|2a_{k+1}^2-(a_{k+1}+a_k)^2\right| = \left|2(2a_k+a_{k-1})^2-(2a_k+a_{k-1}+a_k)^2\right|$$

$$= \left|2(4a_k^2+4a_ka_{k-1}+a_{k-1}^2)-(9a_k^2+6a_ka_{k-1}+a_{k-1}^2)\right|$$

$$= \left|-a_k^2+2a_ka_{k-1}+a_{k-1}^2\right| = \left|a_k^2-2a_ka_{k-1}-a_{k-1}^2\right|$$

$$= \left|2a_k^2-(a_k+a_{k-1})^2\right| = 1 \text{ (by induction assumption)}$$

By mathematical induction, $a_{k+1}^2\times(a_{k+1}+a_k)^2$ is a triangular number for all positive integer n.

Claim: Any square-triangular number is given by $T_n = a_n^2\times(a_n+a_{n-1})^2$.

Proof: Let $m_1^2 = \dfrac{k_1(k_1+1)}{2}$ be an arbitrary square-triangular number. (m_1, k_1 are +ve integers.)

If k_1 is odd, then $\frac{k_1+1}{2}$ is an integer and k_1 and $\frac{k_1+1}{2}$ are relatively prime ($2\times\frac{k_1+1}{2}-k_1=1$)

$\therefore$ Both k_1 and $\frac{k_1+1}{2}$ are perfect squares

Let $\frac{k_1+1}{2}=b_1^2$ and $k_1=c_1^2$ with $b_1, c_1 \ge 1$.

Now $b_1^2=\frac{k_1+1}{2} \Rightarrow k_1 = \boldsymbol{c_1^2 = 2b_1^2 - 1} \ge 1 \Rightarrow c_1^2 - b_1^2 = b_1^2 - 1 \ge 0 \Rightarrow b_1 \le c_1$

$b_1 = c_1 \Leftrightarrow k_1 = \frac{k_1+1}{2} \Leftrightarrow k_1 = 1 \Leftrightarrow m_1^2 = 1 \Leftrightarrow m_1^2 = a_1^2\times(a_1+a_0)^2 = T_1$ and we are done.

Consider $b_1 < c_1$ and define $b_2 = c_1 - b_1$, $c_2 = 2b_1 - c_1$ and $m_2^2 = b_2^2\times c_2^2$.

$\because 2b_1^2 - c_1^2 = k_1 + 1 - k_1 = 1$

We factor and get $\left(\sqrt{2}b_1 - c_1\right)\left(\sqrt{2}b_1 + c_1\right)=1$,

where $b_1, c_1 \ge 1$ implies that $\sqrt{2}b_1 + c_1 > 0$, which implies that $\sqrt{2}b_1 - c_1 > 0$.

So $\sqrt{2}b_1 > c_1$. Now $\frac{3}{2} > \sqrt{2}$ so it also follows that $\frac{3}{2}b_1 > c_1$.

This is equivalent to $3b_1 > 2c_1$ and thus $2b_1 - c_1 > c_1 - b_1$.

Also $c_1 > b_1 > 2b_1 - c_1 > c_1 - b_1 > 0$, or equivalently $\boldsymbol{c_1 > b_1 > c_2 > b_2 > 0}$.

Furthermore, $m_2^2 = b_2^2\times c_2^2 = \frac{2b_2^2\times c_2^2}{2}$

where $\left|2b_2^2 - c_2^2\right| = \left|2(c_1-b_1)^2 - (2b_1-c_1)^2\right|$

$= \left|c_1^2 - 2b_1^2\right| = |-1|^2 = 1$

So, m_2^2 is also a square-triangular number and is necessarily smaller than m_1^2.

It might be observed that in this case $m_2^2 \ne 1$ which is equivalent to the fact that $\boldsymbol{c_2 > b_2}$.

Now let $m_2^2 = \frac{2b_2^2\times c_2^2}{2} = \frac{k_2(k_2+1)}{2}$ with $2b_2^2 = k_2$ (since $2b_2^2 - c_2^2 = -1$ gives $c_2^2 = 2b_2^2 + 1$.)

Continue in the same manner by defining $b_3 = c_2 - b_2$, $c_3 = 2b_2 - c_2$, and $m_3^2 = b_3^2\times c_3^2$.

In this case, $b_3 \le c_3$ since **if** $\boldsymbol{b_3 > c_3}$, then by substitution $c_2 - b_2 > 2b_2 - c_2$ which implies

$c_2 > \frac{3}{2}b_2$

Recalling that $2b_2^2 - c_2^2 = c_1^2 - 2b_1^2 = -1$,

which is equivalent to $c_2^2 - 2b_2^2 = 1$, we get by using $c_2 > \frac{3}{2}b_2$ that $\left(\frac{3}{2}b_2\right)^2 - 2b_2^2 < 1$.

This implies $\frac{b_2^2}{4} < 1$,

which implies b_2 is a positive integer with square less than 4, or that $b_2 = 1$.

This, however, yields $c_2^2 - 2(1)^2 = 1$ or $c_2^2 = 3$ in which case $c_2 = \sqrt{3}$, a +ve integer, false.

Thus the hypothesis that $b_3 \le c_3$ as claimed.

We might note that $b_3 = c_3$ is equivalent to $c_2 = \frac{3}{2}b_2$ which implies $1 = c_2^2 - 2b_2^2 = \frac{b_2^2}{4}$.

This implies that $b_2 = 2$ and also $c_2 = \frac{3}{2} \times 2 = 3$.

This, in turn, gives $b_3 = c_3 = 1$ as well as $b_1 = 5$, $c_1 = 7$, so $m_1^2 = 5^2 \cdot 7^2 = a_3^2 \times (a_3 + a_2)^2 = T_3$.

In general, with $b_3 \le c_3$, $m_3^2 = b_3^2 \times c_3^2 = \frac{2b_3^2 \times c_3^2}{2}$,

with $\left|2b_3^2 - c_3^2\right| = \left|2(c_2 - b_2)^2 - (2b_2 - c_2)^2\right| = \left|c_2^2 - 2b_2^2\right| = 1$.

So m_3^2 is again a square triangular number.

Since $c_2 > b_2 > 2b_2 - c_2 \ge c_2 - b_2 > 0$, or equivalently $c_2 > b_2 > c_3 \ge b_3 > 0$,

m_3^2 is again smaller than m_2^2.

If we let $m_3^2 = \frac{2b_3^2 \times c_3^2}{2} = \frac{(k_3 + 1)k_3}{2}$ with $k_3 = c_3^2$ (since $2b_3^2 - c_3^2 = 1$ gives $2b_3^2 = c_3^2 + 1$.)

We have $2b_3^2 = k_3 + 1$. Thus k_3 is now odd as in the first case and one can proceed in exactly the same manner generating new and smaller square-triangular number until we finally arrive at $m_n^2 = b_n^2 \times c_n^2 = 1$ with $b_n = c_n = 1$.

This gives us $1 = b_n = c_{n-1} - b_{n-1}$ and $1 = c_n = 2b_{n-1} - c_{n-1}$

which gives $b_{n-1} = 2$ and $c_{n-1} = 3$ when solved.

It follows that $2 = c_{n-2} - b_{n-2}$ and $3 = c_{n-1} = 2b_{n-2} - c_{n-2}$, which yields b_{n-2} and $c_{n-2} = 7$, $\cdots$.

In general, for $j \ge 2$, $b_{j+1} = c_j - b_j$, $c_{j+1} = 2b_j - c_j$, and $b_j = c_{j-1} - b_{j-1}$, $c_j = 2b_{j-1} - c_{j-1}$.

Therefore, $2b_j + b_{j+1} = 2(c_{j-1} - b_{j-1}) + (c_j - b_j) = 2c_{j-1} - 2b_{j-1} + (2b_{j-1} - c_{j-1}) - (c_{j-1} - b_{j-1}) = b_{j-1}$

and $b_j + b_{j-1} = b_j + (2b_j + b_{j+1}) = 3b_j + b_{j+1} = 3c_{j-1} - 3b_{j-1} + (2b_{j-1} - c_{j+1}) - (c_{j-1} - b_{j-1}) = c_{j-1}$.

We have done the computation for the induction proof $b_j = a_{n-j+1}$ and $c_j = a_{n-j+1} + a_{n-j}$ for $j = 1$, $2, \cdots, n$. In particular, for $j = 1$, it follows that $m_1^2 = b_1^2 c_1^2 = a_n^2 (a_n + a_{n-1})^2$ and m_1^2 is our sequence as claimed.

Since all a_n are positive integers and so $T_n = a_n^2 \times (a_n + a_{n-1})^2$ is a strictly increasing sequence of squared-triangular number.

$a_5 = 2\times12 + 5 = 29$, $a_6 = 2\times29 + 12 = 70$

$T_4 = 41616 < 250000$

$T_5 = 29^2\times(29 + 12)^2 = \frac{1682\times1681}{2} = 1413721 > 250000$

The largest N is 41616.

I12 若 ΔABC 的邊長為 9、10 及 17，求 ΔABC 外接圓的半徑。

If the lengths of the three sides of a ΔABC are 9, 10 and 17, find the radius of the circum-circle of ΔABC.

Reference: 1991 HI19

$a = 9$, $b = 10$, $c = 17$

$$\cos C = \frac{9^2 + 10^2 - 17^2}{2\times 9\times 10} = -\frac{3}{5}$$

$$\sin C = \sqrt{1-\cos^2 C} = \frac{4}{5} \quad (\because 0 < C < 180^\circ \therefore \sin C > 0)$$

By sine formula, $\frac{c}{\sin C} = 2R$, where R is the radius of the circumscribed circle.

$$R = \frac{17}{2\times\frac{4}{5}} = \frac{85}{8}$$

I13 求 $S = \frac{1}{2024} - \frac{3}{2024^2} + \frac{5}{2024^3} - \frac{7}{2024^4} + \frac{9}{2024^5} - \cdots$ 的值。

Find the value of $S = \frac{1}{2024} - \frac{3}{2024^2} + \frac{5}{2024^3} - \frac{7}{2024^4} + \frac{9}{2024^5} - \cdots$.

$$S = \frac{1}{2024} - \frac{3}{2024^2} + \frac{5}{2024^3} - \frac{7}{2024^4} + \frac{9}{2024^5} - \cdots \qquad \cdots(1)$$

$$\frac{1}{2024}S = \frac{1}{2024^2} - \frac{3}{2024^3} + \frac{5}{2024^4} - \frac{7}{2024^5} + \frac{9}{2024^6} - \cdots \qquad \cdots(2)$$

(1)+(2): $\frac{2025}{2024}S = \frac{1}{2024} - \frac{2}{2024^2} + \frac{2}{2024^3} - \frac{2}{2024^4} + \frac{2}{2024^5} + \cdots$

$$2025S = 1 - \frac{2}{2024} + \frac{2}{2024^2} - \frac{2}{2024^3} + \frac{2}{2024^4} + \cdots$$

$$S = \frac{1}{2025}\left[1 - \frac{2}{2024}\left(1 - \frac{1}{2024} + \frac{1}{2024^3} - \frac{1}{2024^4} + \cdots\right)\right]$$

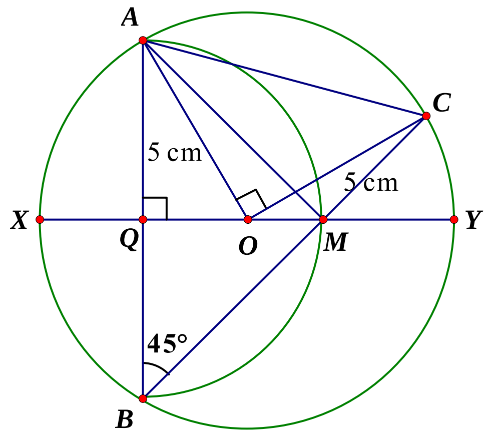
$$= \frac{1}{2025}\left[1 - \frac{2}{2024}\left(\frac{1}{1+\frac{1}{2024}}\right)\right]$$

$$= \frac{1}{2025}\left[1 - \frac{2}{2024}\left(\frac{2024}{2025}\right)\right]$$

$$= \frac{1}{2025}\cdot\frac{2023}{2025} = \frac{2023}{4100625}$$

I14 在圖四中，XY 是一個以 O 為圓心及半徑為 5 cm 的圓的直徑。 XY 與弦 AB 相交於點 Q，使得 $\angle AQO = 90°$ 及 $XQ = QO$。 以 AB 為直徑的半圓與 XY 相交於 M，延線 BM 與圓相交於點 C，求 AC 的長。

In Figure 4, XY is a diameter of the circle with centre at O and radius 5 cm. XY intersects the chord AB at Q such that $\angle AQO = 90°$ and $XQ = QO$. A semi-circle with diameter AB intersects XY at M. BM produced intersects the circle at C. Find the length of AC.



圖四 Figure 4

Join OA, OC and AC. Then $OA = OC = 5$ cm (radii)

$QA = QB = QM$ (radii of the semi-circle)

$\therefore$ ΔQBM is a right-angled isosceles triangle.

$\angle QBM = \angle QMB = 45°$ (base $\angle$s isos. Δ, $\angle$ sum of Δ)

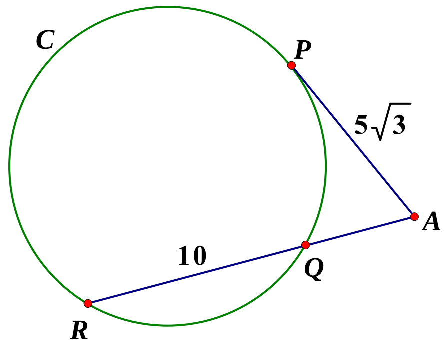
$\angle AOC = 2\angle ABC = 2\angle QBM = 90°$ ($\angle$ at centre twice $\angle$ at $\odot^{ce}$)

Apply Pythagoras' theorem on ΔOAC: $AC = \sqrt{5^2 + 5^2}$ cm $= 5\sqrt{2}$ cm

I15 在圖五中，點 P 及 R 均在圓 C 上。AP 是 C 的切線及 AR 相交於 Q。若 $QR = 10$ 及 $PA = 5\sqrt{3}$，求 AQ 的長。

In Figure 5, P and R are points on the circle C. AP is the tangent to C at P and AR intersects at Q.

If $QR = 10$ and $PA = 5\sqrt{3}$, find the length of AQ.



圖五 Figure 5

Reference: 2000 FG1.4

By intersecting chords theorem, $AQ \times AR = AP^2$

Let $AQ = x$, then $x(x + 10) = \left(5\sqrt{3}\right)^2$

$x^2 + 10x - 75 = 0$

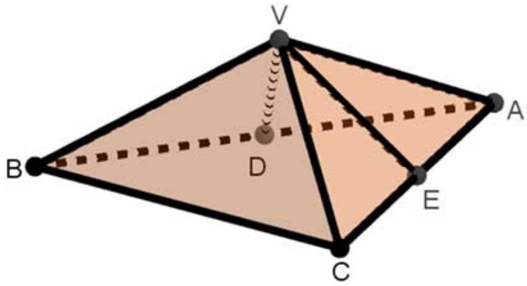
$(x - 5)(x + 15) = 0$

$AQ = x = 5$

Group Events

G1 在圖一中，$VABC$ 是一個四面體，使得 $VA \perp VB$、$VB \perp VC$ 及 $VA \perp VC$。$VA = 5$，$VB = 4$ 及 $VC = 3$。若 D 和 E 分別為 AB 和 AC 的中點，求角錐 $VBCED$ 的體積。

In Figure 1, $VABC$ is a tetrahedron such that $VA \perp VB$, $VB \perp VC$ and $VA \perp VC$. $VA = 5$, $VB = 4$ and $VC = 3$. If D and E are the mid-points of AB and AC respectively, find the volume of pyramid $VBCED$.



圖一 Figure 1

Similar questions: HKCEE Mathematics 2006 Q17, Additional Mathematics 2001 Q15

Volume of the tetrahedron $VABC = \frac{1}{3} \times$ area of $\Delta VAB \times VC = \frac{1}{3} \times \frac{1}{2} \times 5 \times 4 \times 3 = 10$

Let the area of ΔABC be S. Let the height of the pyramid $VABC$ with base ABC be h.
By calculating the volume of the pyramid $VABC$ in two different ways:

$$\frac{1}{3} \times S \times h = 10$$

$$h = \frac{30}{S}$$

$\because$ D and E are the mid-points of AB and AC

$\therefore$ $DE \parallel BC$ and $DE = \frac{1}{2} BC$ (mid-points theorem)

$\Delta ADE \sim \Delta ABC$ (equiangular)

$$\frac{\text{Area of } \Delta ADE}{\text{Area of } \Delta ABC} = \left(\frac{DE}{BC}\right)^2 = \left(\frac{1}{2}\right)^2 = \frac{1}{4}$$

$$\frac{\text{Area of trapezium } BCED}{\text{Area of } \Delta ABC} = \frac{\text{Area of } \Delta ABC - \text{Area of } \Delta ADE}{\text{Area of } \Delta ABC} = 1 - \frac{1}{4} = \frac{3}{4}$$

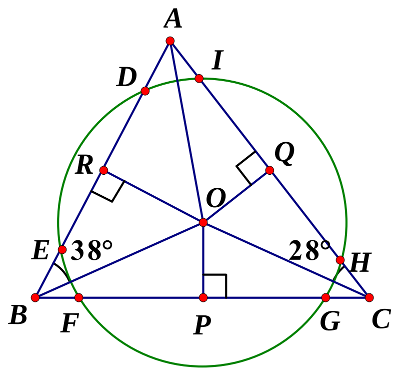
Area of trapezium $BCED = \frac{3}{4} S$

Volume of pyramid $VBCED = \frac{1}{3} \times \frac{3}{4} S \times h = \frac{1}{3} \times \frac{3}{4} S \times \frac{30}{S} = \frac{15}{2} = 7.5$

G2 在圖二中，O 是圓 $DEFGHI$ 的圓心，ΔABC 與該圓相交於 D、E、F、G、H 及 I，使得 $AD = EB$ 及 $BF = CG$，已知 $\angle ABO = 38°$及$\angle ACO = 28°$，求 $\angle BOC$。

In Figure 2, O is the centre of the circle $DEFGHI$. ΔABC intersects the circle at D, E, F, G, H and I such that $AD = EB$ and $BF = CG$. If $\angle ABO = 38°$ and $\angle ACO = 28°$, find $\angle BOC$.

Reference: HKDSE 2014 Paper 2 Q21



圖二 Figure 2

Join OA. Let the feet of perpenculars from O to AB, BC and CA be P, Q and R respectively. ($OP \perp BC$, $OQ \perp CA$, $OR \perp AB$)

$AR = AD + DR = BE + ER = BR$ (line from centre $\perp$ chord bisect chord)

$\therefore$ $\Delta AOR \cong \Delta BOR$ (S.A.S.)

$BP = BF + FP = CG + GP = CP$ (line from centre $\perp$ chord bisect chord)

$\therefore$ $\Delta BOP \cong \Delta COP$ (S.A.S.)

$OA = OB = OC$ (corr. sides, $\cong \Delta$s)

$\therefore$ $\Delta COQ \cong \Delta AOQ$ (R.H.S.)

$\angle BOR = \angle AOR = 90° - 38° = 52°$, $\angle COQ = \angle AOQ = 90° - 28° = 62°$ (corr. $\angle$s, $\cong \Delta$s)

$\angle BOC = 360° - 2 \times 52° - 2 \times 62° = 132°$ ($\angle$s at a pt.)

G3 設 a、b 及 c 為正整數。若 $ab + c = 2023$ 及 $a + bc = 2024$，求 $a + b + c$ 的值。

Let a, b and c be positive integers.

If $ab + c = 2023$ and $a + bc = 2024$, find the value of $a + b + c$.

$ab^2 + bc = 2023b$

$ab^2 + 2024 - a = 2023b$

$a(b + 1)(b - 1) + 1 = 2023(b - 1)$

$1 = (2023 - ab - a)(b - 1)$

$(c - a)(b - 1) = 1$

$(c - a = 1 \text{ and } b - 1 = 1)$ or $(c - a = -1 \text{ and } b - 1 = -1)$

$(c = a + 1 \text{ and } b = 2)$ or $(a = c + 1 \text{ and } b = 0)$ (rejected)

Sub. $c = a + 1$ and $b = 2$ into $ab + c = 2023$

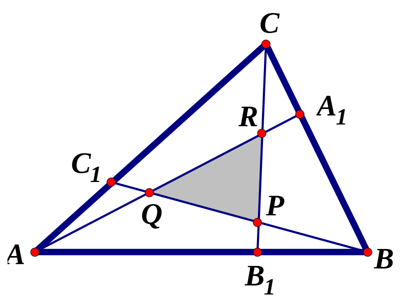
$2a + a + 1 = 2023$

$a = 674$, $c = 675$, $b = 2$

$a + b + c = 674 + 2 + 675 = 1351$

G4 在圖三中，A_1、B_1 及 C_1 分別為 BC、AC 及 AB 上的點，使得 $AC_1 = 2C_1B$，$BA_1 = 2A_1C$ 及 $CB_1 = 2B_1A$。若 ΔABC 的面積是 21 平方單位，求陰影部分的面積。

In Figure 3, A_1, B_1 and C_1 are points on BC, AC and AB respectively such that $AC_1 = 2C_1B$, $BA_1 = 2A_1C$ and $CB_1 = 2B_1A$. If the area of ΔABC is 21 square units, find the area of the shaded region.



圖三 Figure 3

Reference: 1986 FG10.2

Let P, Q and R be as shown in the figure.

By considering the areas of ΔACA_1 and ΔABA_1.

$$\frac{\frac{1}{2}AC \cdot AA_1 \sin\angle CAA_1}{\frac{1}{2}AB \cdot AA_1 \sin\angle BAA_1} = \frac{CA_1}{BA_1} = \frac{1}{2}$$

$$\Rightarrow \frac{AC\sin\angle CAA_1}{AB\sin\angle BAA_1} = \frac{1}{2} \quad \cdots\cdots (1)$$

By considering the areas of ΔAC_1Q and ΔABQ

$$\frac{\frac{1}{2}AC_1 \cdot AQ\sin\angle CAA_1}{\frac{1}{2}AB \cdot AQ\sin\angle BAA_1} = \frac{C_1Q}{BQ}$$

$$\frac{AC_1\sin\angle CAA_1}{AB\sin\angle BAA_1} = \frac{C_1Q}{BQ}$$

$$\frac{\frac{1}{3}AC\sin\angle CAA_1}{AB\sin\angle BAA_1} = \frac{C_1Q}{BQ}$$

By (1), $\frac{1}{3}\times\frac{1}{2} = \frac{C_1Q}{BQ} \Rightarrow \frac{C_1Q}{BQ} = \frac{1}{6} \cdots\cdots (2)$

By considering the areas of ΔCAB_1 and ΔCBB_1.

$$\frac{\frac{1}{2}AC \cdot CB_1\sin\angle ACB_1}{\frac{1}{2}BC \cdot CB_1\sin\angle BCB_1} = \frac{AB_1}{BB_1} = \frac{2}{1} \Rightarrow \frac{AC\sin\angle ACB_1}{BC\sin\angle BCB_1} = 2 \quad \cdots\cdots (3)$$

By considering the areas of ΔCC_1P and ΔCBP.

$$\frac{\frac{1}{2}CC_1 \cdot CP\sin\angle ACB_1}{\frac{1}{2}CB \cdot CP\sin\angle BCB_1} = \frac{C_1P}{BP}$$

$$\frac{CC_1\sin\angle ACB_1}{CB\sin\angle BCB_1} = \frac{C_1P}{BP}$$

$$\frac{\frac{2}{3}AC\sin\angle ACB_1}{BC\sin\angle BCB_1} = \frac{C_1P}{BP}$$

By (3), $\frac{2}{3}\times 2=\frac{C_1P}{BP}$

$\Rightarrow \frac{C_1P}{BP}=\frac{4}{3}$ .... (4)

By (2) and (4), $C_1Q : QP : PB = 1 : 3 : 3$

By symmetry $B_1P : PR : RC = 1 : 3 : 3$ and $A_1R : RQ : QA = 1 : 3 : 3$

Let S stands for the area, x = area of ΔPQR.

$S_{\Delta ABC_1}=S_{\Delta BCB_1}=S_{\Delta ACA_1}=\frac{21}{3}=7$

and $S_{\Delta AQC_1}=S_{\Delta BPB_1}=S_{\Delta CRA_1}=\frac{1}{7}\times\frac{1}{3}\times S_{\Delta ABC}=1$ ($\because C_1Q : BQ = 1 : 6 \Rightarrow C_1Q=\frac{1}{7}BC_1$)

The total area of ΔABC: $21=S_{\Delta ABC_1}+S_{\Delta BCB_1}+S_{\Delta ACA_1}+S_{\Delta PQR}-3\,S_{\Delta AQC_1}$

$21 = 7 + 7 + 7 + x - 3$

x = area of $\Delta PQR = 3$

G5 求方程 $\left(\log_4 x^2\right)^2+9\log_x 64=\pi^{3\log_\pi 3}$ 的所有根之和。

Find the sum of the roots of the equation $\left(\log_4 x^2\right)^2+9\log_x 64=\pi^{3\log_\pi 3}$.

$\left(\log_4 x^2\right)^2+9\log_x 64=\pi^{3\log_\pi 3}$

$\left(2\log_4 x\right)^2+9\log_x 4^3=\left(\pi^{\log_\pi 3}\right)^3$

$4\left(\log_4 x\right)^2+27\log_x 4=3^3$

$4\left(\log_4 x\right)^2+\frac{27}{\log_4 x}=27$

$4\left(\log_4 x\right)^3-27\log_4 x+27=0$

$\left(\log_4 x+3\right)\left(2\log_4 x-3\right)^2=0$

$\log_4 x=-3,\ \frac{3}{2}$ or $\frac{3}{2}$

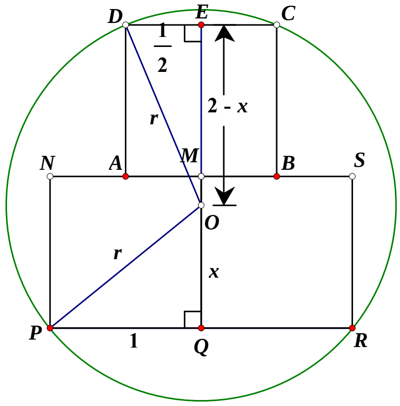
$x=4^{-3},\ 4^{\frac{3}{2}}$ or $4^{\frac{3}{2}}$

$x=\frac{1}{64}$, 8 or 8

sum of roots $=\frac{1}{64}+8+8=16\frac{1}{64}=\frac{1025}{64}$.

G6 在圖四中，三個邊長為 1 cm 的正方形 *ABCD*、*MNPQ*、*MQRS* 併在一起使得 *A* 及 *B* 分別為 *MN* 及 *MS* 的中點。已知一圓包這三個正方形，且通過 *C*、*D*、*P* 及 *R*，求該圓的半徑。

In Figure 4, three squares *ABCD*, *MNPQ* and *MQRS* of sides1 cm toucheach other so that points *A* and *B* are the mid-points of *MN* and *MS* respectively. Given that a circle contains all three squares and passes through points *C*, *D*, *P* and *R*, find the radius of the circle.



圖四 Figure 4

By symmetry, the centre of the circle(*O*) lies on *MQ*.

Extend *QM* to meet *CD* at *E*. Then $DE=\frac{1}{2}$, $OE = 2-x$

Let the radius be *r*. Apply Pythagoras' theorem on ΔODE and ΔOPQ.

$$x^2+1=(2-x)^2+\left(\frac{1}{2}\right)^2=r^2$$

$$1=4-4x+\frac{1}{4}$$

$$x=\frac{13}{16}$$

$$r^2=\left(\frac{13}{16}\right)^2+1=\frac{425}{256}$$

$$r=\frac{5\sqrt{17}}{16}$$

G7 若 *x*、*y* 及 *z* 為正整數，求能滿足 $xyz = 10000$ 的 (x, y, z)組的數目。

If *x*, *y* and *z* positive integers, find the number of sets of (x, y, z) satisfying $xyz = 10000$.

$10000 = 2^4\times5^4$

$x = 2^a\times5^c$, $y = 2^b\times5^d$, $z = 2^{4-a-b}\times5^{4-c-d}$

where *a*, *b*, *c* and *d* are integers such that $0 \le a, b, c, d \le 4$, $0 \le a + b \le 4$ and $0 \le c + d \le 4$

We count different combinations for *a* and *b* and different combinations for *c* and *d* separately.

Arrange the four '2's in a row:

2	2	2	2

There are 5 gaps between the four numbers, including the front and the end.

	2		2		2		2	

Insert two vertical sticks in these five gaps, so as to divide the four numbers into 3 groups.

The following is an example:

|	2	|	2		2		2	

The first group has no number (left of the leftmost stick), the second group has one '2' (between two sticks), the third group has three '2' (right of the right most stick).

Then $2^a = 2^0 = 1$, $2^b = 2^1 = 2$, $2^{4-a-b} = 8$

The following is another example:

	2	||	2		2		2	

The first group has one '2' (left of the leftmost stick), the second group has no number (between two sticks), the third group has three '2' (right of the right most stick).

Then $2^a = 2^1 = 2$, $2^0 = 1$, $2^{4-a-b} = 8$

It is equivalent to arrange 6 objects in a row, choose two object as sticks and the rest and '2'.

X	X	X	X	X	X

$\therefore$ Number of combinations for '2's is $C_2^6 = 15$.

Similarly, the number of combinations for '5's is $C_2^6 = 15$.

The total number of combinations $= 15\times15 = 225$.

G8 設 a 為實數。若方程 $x^2 + ax + 6a = 0$ 有兩個整數解，求 a 的最大和最小值之差。

Let a be a real number. If the equation $x^2 + ax + 6a = 0$ has two integral roots, find the difference between the largest and the smallest values of a .

Reference: 2001 FI2.2

$\because$ The equation have two integral roots $\therefore$ It can be factorised as $x^2 + ax + 6a \equiv (x - \alpha)(x - \beta)$

where α and β are integers. $a = -(\alpha + \beta)$ and $\alpha\beta = 6a \Rightarrow a$ is an integer

$\Delta = a^2 - 4(6a) = m^2$, where m is an integer.

$a^2 - 24a + 12^2 = m^2 + 12^2$

$(a - 12)^2 - m^2 = 144$

$(a + m - 12)(a - m - 12) = 1\times144 = 2\times72 = 3\times48 = 4\times36 = 6\times24 = 8\times18 = 9\times16 = 12\times12$

Let $a + m - 12 = p \cdots(1)$ and $a - m - 12 = q \cdots (2)$, where p and q are integers so that $pq = 144$

$\frac{(1)+(2)}{2}: a - 12 = \frac{p+q}{2}$

$a = 12 + \frac{p+q}{2}$

a is the largest when p, q are positive integers and the difference between p and q is the greatest i.e. ($p = 1$, $q = 144$) or ($p = 144$, $q = 1$)

But this will gives $a = 12 + \frac{145}{2}$ which is not an integer $\therefore$ rejected

The second choice is ($p = 2$, $q = 72$) or vice versa, then $a = 12 + \frac{2+72}{2} = 49$

$\therefore$ The largest possible $a = 49$

a is the smallest when p, q are negative integers and the difference between them is the greatest i.e. ($p = -1$, $q = -144$) or ($p = -144$, $q = -1$)

But this will gives $a = 12 - \frac{145}{2}$ which is not an integer $\therefore$ rejected

Another choice is ($p = -2$, $q = -72$) or vice versa, then $a = 12 - \frac{2+72}{2} = -25$

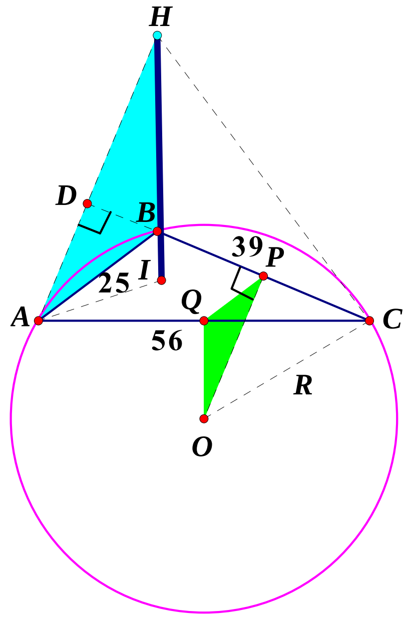
$\therefore$ The smallest possible $a = -25$

The difference between the largest and the smallest a is $49 - (-25) = 74$

G9 若三角形的邊長為 25、39 及 56，求該三角形的內心及垂心之間的距離。

If the lengths of the three sides of a triangle are 25, 39 and 56, find the distance between the incentre and the orthocentre of the triangle.

Reference: 2022 P1Q15



Let the sides of ΔABC be $AB = 25$, $BC = 39$, $AC = 56$.

$\cos A=\frac{25^2+56^2-39^2}{2\times 25\times 56}=\frac{4}{5}\Rightarrow \sin A=\frac{3}{5}$

$\cos B=\frac{25^2+39^2-56^2}{2\times 25\times 39}=-\frac{33}{65}<0\Rightarrow B>90°$

$\cos C=\frac{56^2+39^2-25^2}{2\times 56\times 39}=\frac{12}{13}$

The orthocentre H lies outside ΔABC.

Construct a circum-circle, centre at O with circumradius R.

By sine rule, $\frac{a}{\sin A}=2R\Rightarrow R=\frac{39}{2\times\frac{3}{5}}=32.5$

Let the incentre be I.

Join AI and HI (note that H, B, I are not collinear)

$\angle BOC = 2\angle A$ ($\angle$ at centre twice $\angle$ at $\odot^{ce}$)

Let P and Q be the mid-points of BC and AC respectively.

Then $\Delta BOP \cong \Delta COP$ (R.H.S.), PQ // AB and $PQ=\frac{1}{2}AB$ (mid-point theorem) …… (1)

$\angle COP = \angle BOP$ (corr. $\angle$s, $\cong\Delta$s), $OP \perp BC$, $OQ \perp AC$ (line centre centre to mid-pt of chords⊥ chords)

$=\frac{1}{2}\angle BOC=\angle A$

$OP = R\cos\angle COP = R\cos\angle A = 32.5\times\frac{4}{5}=26$ …… (2)

CB produced cut AH at D.

$\because$ H is the orthcentre. $ADH \perp CBD$, $BH \perp AC$

OP // AH and BH // OQ (int. $\angle$s supp.)

$\therefore$ $\Delta OPQ \sim \Delta HAB$ (A.A.A., 3 pairs of //-lines)

$\frac{OP}{AH}=\frac{PQ}{AB}=\frac{1}{2}$ (corr. sides, ~Δs and by (1))

$AH = 2OP = 2\times 26 = 52$ (by (2)) …… (3)

In ΔAIB, $\angle BAI=\frac{A}{2}$, $\angle ABI=\frac{B}{2}$ ($\because$ I is the incentre)

$\angle AIB = 180°-\frac{A}{2}-\frac{B}{2}$ ($\angle$ sum of Δ)

$= 180°-\frac{180°-C}{2}=90°+\frac{C}{2}$

Apply sine rule on ΔAIB, $\frac{25}{\sin\angle AIB}=\frac{AI}{\sin\angle ABI}$

$$AI=\frac{25\sin\frac{B}{2}}{\sin\left(90^\circ+\frac{C}{2}\right)}=\frac{25\sin\frac{B}{2}}{\cos\frac{C}{2}}=\frac{25\sqrt{\frac{1-\cos B}{2}}}{\sqrt{\frac{1+\cos C}{2}}}=\frac{25\sqrt{\frac{1-\left(-\frac{33}{65}\right)}{2}}}{\sqrt{\frac{1+\frac{12}{13}}{2}}}=25\sqrt{\frac{98}{65}\times\frac{13}{25}}=7\sqrt{10}\cdots(4)$$

In ΔABD, $\angle BAD = 180° - 90° - \angle ABD$ ($\angle$ sum of Δ)
$= 90° - (A + C)$ (ext. $\angle$ of Δ)

$$\angle HAI=\angle BAD+\angle BAI=90^\circ-(A+C)+\frac{A}{2}=\frac{A+B+C}{2}-\frac{A}{2}-C=\frac{B-C}{2}$$

$$\cos\angle HAI=\cos\frac{B-C}{2}=\cos\frac{B}{2}\cos\frac{C}{2}+\sin\frac{B}{2}\sin\frac{C}{2}=\sqrt{\frac{1+\left(-\frac{33}{65}\right)}{2}}\cdot\sqrt{\frac{1+\frac{12}{13}}{2}}+\sqrt{\frac{1-\left(-\frac{33}{65}\right)}{2}}\cdot\sqrt{\frac{1-\frac{12}{13}}{2}}$$

$$=\frac{1}{2}\left(\sqrt{\frac{32}{65}}\cdot\sqrt{\frac{25}{13}}+\sqrt{\frac{98}{65}}\cdot\sqrt{\frac{1}{13}}\right)=\frac{1}{2}\left(\frac{4\sqrt{10}}{13}+\frac{7\sqrt{10}}{65}\right)=\frac{27\sqrt{10}}{130}\cdots\cdots(5)$$

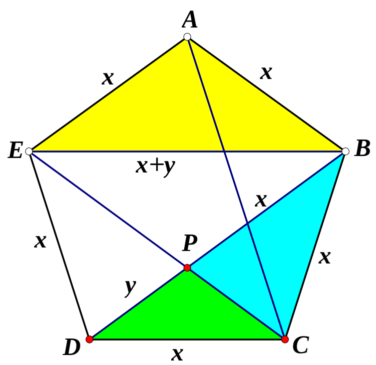
Apply cosine rule on ΔAHI: $HI^2=52^2+\left(7\sqrt{10}\right)^2-2(52)\left(7\sqrt{10}\right)\times\frac{27\sqrt{10}}{130}$ by (3), (4) and (5)

$HI^2 = 1682$

$HI = 41$

G10 在圖五中，$ABCDE$ 為一正五邊形，BD 及 CE 相交於 P。若ΔABE 的面積為 1，求ΔBPC 的面積。

In Figure 5, $ABCDE$ is a regular pentagon, BD and CE intersect at P.
If the area of ΔABE is 1, find the area of ΔBPC.



圖五 Figure 5

Let $AB = BC = CD = DE = EA = x$.

$$\angle BAE=\angle BCD=\frac{180^\circ\times(5-2)}{5}=108^\circ \text{ } (\angle \text{ sum of polygon})$$

$$\angle CBD=\angle CDB=\frac{180^\circ-108^\circ}{2}=36^\circ \text{ (base } \angle\text{s isos. } \Delta)$$

Denote the areas by S. $S_{\Delta ABE}=\frac{1}{2}x^2\sin 108^\circ=1$

$$\frac{1}{2}x^2=\frac{1}{\sin 108^\circ}$$

$$S_{\Delta BPC}=\frac{1}{2}x^2\sin 36^\circ$$

$$=\frac{\sin 36^\circ}{\sin 108^\circ}$$

$$=\frac{\sin 36^\circ}{\sin 72^\circ}$$

$$=\frac{\sin 36^\circ}{2\sin 36^\circ\cos 36^\circ}$$

$$=\frac{1}{2\cos 36^\circ}$$

Let $\theta = 36°$, $5\theta = 180°$

$3\theta = 180° - 2\theta$

$\sin 3\theta = \sin(180° - 2\theta)$

$3 \sin \theta - 4 \sin^3 \theta = \sin 2\theta = 2 \sin \theta \cos \theta$

$\because \sin \theta = \sin 36° \neq 0$, divide both sides by $\sin \theta$, we have

$3 - 4 \sin^2 \theta = 2 \cos \theta$

$3 - 4(1 - \cos^2 \theta) = 2 \cos \theta$

$4 \cos^2 \theta - 2\cos \theta - 1 = 0$

$\cos \theta = \frac{1+\sqrt{5}}{4}$ or $\frac{1-\sqrt{5}}{4}$ (< 0, rejected)

$$S_{\Delta BPC} = \frac{1}{2\times\frac{1+\sqrt{5}}{4}}$$

$$= \frac{2}{1+\sqrt{5}}\cdot\frac{\sqrt{5}-1}{\sqrt{5}-1}$$

$$= \frac{\sqrt{5}-1}{2}$$

Method 2

It is easy to show that $\Delta ABE \cong \Delta CDB$ and their angles are 108°, 36°, 36°.

It is easy to show that ΔBCP is a 36°, 72°, 72° triangle.

Also, it is easy to show that ΔPCD is a 108°, 36°, 36° triangle.

Let $AB = BC = CD = DE = EA = x$. Then $BP = x$ (base $\angle$s isos. Δ), let $DP = y$.

Then $BE = BD = BP + PD = x + y$ (corr. sides, $\cong\Delta$s)

$\Delta PCD \sim \Delta ABE$ (A.A.A.)

$\frac{BE}{CD} = \frac{AE}{PD} \Rightarrow \frac{x+y}{x} = \frac{x}{y}$ (corr. sides, $\sim\Delta$s)

$1 + \frac{y}{x} = \frac{x}{y}$, let $t = \frac{y}{x}$, then $1 + t = \frac{1}{t}$

$t^2 + t - 1 = 0$

$t = \frac{y}{x} = \frac{-1+\sqrt{5}}{2}$ or $\frac{-1-\sqrt{5}}{2}$ (< 0, rejected)

Denote the areas by S.

$$\frac{S_{\Delta PCD}}{S_{\Delta ABE}} = \left(\frac{PD}{AE}\right)^2 = \left(\frac{y}{x}\right)^2 = t^2 = 1 - t = 1 - \frac{-1+\sqrt{5}}{2} = \frac{3-\sqrt{5}}{2}$$

$S_{\Delta PCD} = \frac{3-\sqrt{5}}{2}$ ($\because S_{\Delta ABE} = 1$)

$S_{\Delta PCD} + S_{\Delta BPC} = S_{\Delta CBD} = S_{\Delta ABE} = 1$

$\frac{3-\sqrt{5}}{2} + S_{\Delta BPC} = 1$

$S_{\Delta BPC} = \frac{-1+\sqrt{5}}{2}$