Second problem of a triangle (by vector method)

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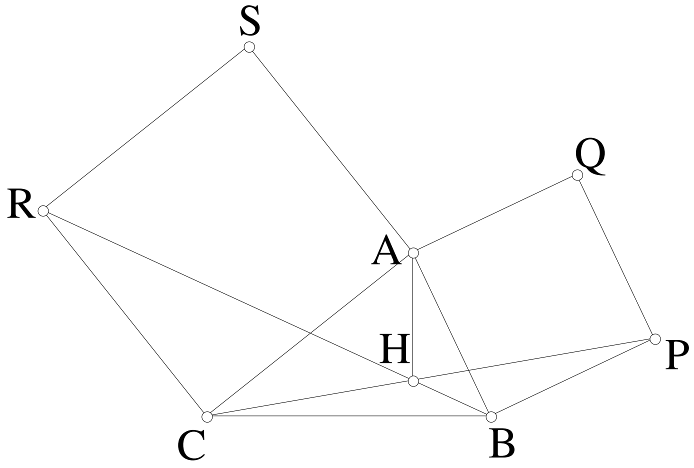
In ΔABC, squares $ABPQ$, $ACRS$ are drawn outwards as shown. The lines BR, CP intersect at H. Prove that $AH\perp BC$.

Let A be the reference point.



$\vec{b}=\overrightarrow{AB}$; $\vec{c}=\overrightarrow{AC}$; $\vec{h}=\overrightarrow{AH}$; $\vec{p}=\overrightarrow{AP}$; $\vec{q}=\overrightarrow{AQ}$

$\vec{r}=\overrightarrow{AR}$; $\vec{s}=\overrightarrow{AS}$

Suppose $BH : HR = x : 1-x$; $CH : HP = y : 1-y$

$\vec{h}=y\,\vec{p}+(1-y)\ \vec{c}=x\,\vec{r}+(1-x)\ \vec{b}$

$\vec{h}=y(\vec{b}+\vec{q})+(1-y)\vec{c}=x(\vec{c}+\vec{s})+(1-x)\vec{b}$ ...(1)

$\vec{c}+y(\vec{b}-\vec{c})+y\vec{q}\ =\ \vec{b}+x(\vec{c}-\vec{b})+x\vec{s}$

$(\vec{c}-\vec{b})(1-x-y)=x\vec{s}-y\vec{q}$

clearly $1-x-y\neq 0$,

otherwise $x\vec{s}=y\vec{q}$ and $\vec{s},\vec{q}$ are not parallel

$\Rightarrow x=y=0$ which is impossible for $1-x-y=0$

$\therefore \vec{c}-\vec{b}=\dfrac{x\vec{s}-y\vec{q}}{1-x-y}$ ...... (2)

To prove $AH\perp BC$

$$\overrightarrow{AH}\cdot\overrightarrow{BC}=\vec{h}\cdot\left(\vec{c}-\vec{b}\right)$$

$$=\vec{h}\cdot\vec{c}-\vec{h}\cdot\vec{b}$$

$$=[x(\vec{c}+\vec{s})+(1-x)\vec{b}]\cdot\vec{c}\ -[y(\vec{b}+\vec{q})+(1-y)\vec{c}]\cdot\vec{b}\qquad\text{by (1)}$$

$$=xc^2+(1-x)\vec{b}\cdot\vec{c}\ -yb^2-(1-y)\vec{b}\cdot\vec{c}\qquad(\because \vec{c}\cdot\vec{s}=0,\ \vec{b}\cdot\vec{q}=0)$$

$$=xc^2-x\vec{b}\cdot\vec{c}\ -yb^2+y\vec{b}\cdot\vec{c}$$

$$=x\vec{c}\cdot(\vec{c}-\vec{b})+y\vec{b}\cdot(\vec{c}-\vec{b})$$

$$=(\vec{c}-\vec{b})\,(x\vec{c}+y\vec{b})$$

$$=\frac{x\vec{s}-y\vec{q}}{1-x-y}\cdot(x\vec{c}+y\vec{b})\qquad\text{by (2)}$$

$$=\frac{x^2\vec{s}\cdot\vec{c}-xy\vec{q}\cdot\vec{c}+xy\vec{s}\cdot\vec{b}-y^2\vec{q}\cdot\vec{b}}{1-x-y}$$

$$=\frac{xy\left(\vec{s}\cdot\vec{b}-\vec{q}\cdot\vec{c}\right)}{1-x-y}\qquad(\because \vec{c}\cdot\vec{s}=0,\ \vec{b}\cdot\vec{q}=0)$$

$$=\frac{xy}{1-x-y}\,[sb\cos(A+90^\circ)-qc\cos(A+90^\circ)]$$

$$=0\qquad(\because s=c,\ q=b)$$

$\therefore AH\perp BC$