Apollonius Theorem (proved by vector method)

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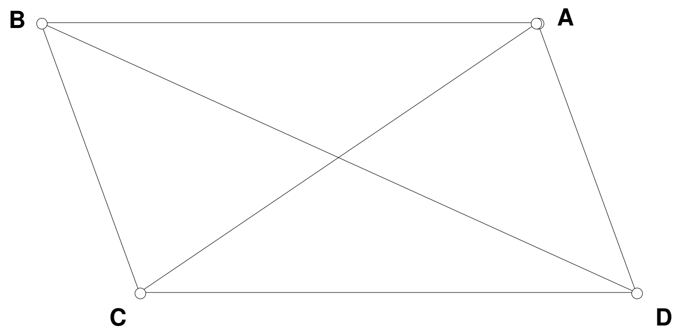
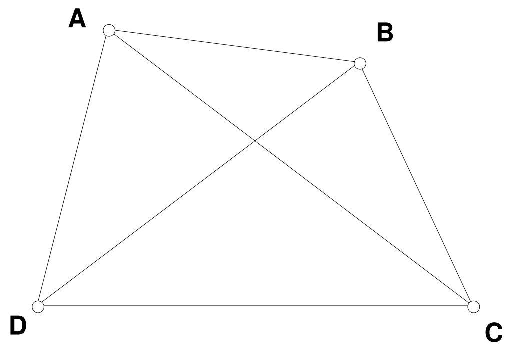


Let ABCD is a parallelogram.
Let AB = CD = p; AD = BC = q.
Let AC = x; BD = y.
Then $2(p^2 + q^2) = x^2 + y^2$
Proof:

$$x^2 + y^2 = \overrightarrow{AC} \cdot \overrightarrow{AC} + \overrightarrow{BD} \cdot \overrightarrow{BD}$$
$$= \left(\overrightarrow{AB} + \overrightarrow{AD}\right) \cdot \left(\overrightarrow{AB} + \overrightarrow{AD}\right) + \left(\overrightarrow{AD} - \overrightarrow{AB}\right) \cdot \left(\overrightarrow{AD} - \overrightarrow{AB}\right)$$
$$= \left|\overrightarrow{AB}\right|^2 + 2\overrightarrow{AB} \cdot \overrightarrow{AD} + \left|\overrightarrow{AD}\right|^2 + \left|\overrightarrow{AD}\right|^2 - 2\overrightarrow{AB} \cdot \overrightarrow{AD} + \left|\overrightarrow{AB}\right|^2$$
$$= 2(p^2 + q^2)$$

Converse of Apollonius Theorem



Let ABCD be a quadrilateral.
Let AB = p, BC = q, CD = r, DA = s;
Let AC = x, BD = y.
If $p^2 + q^2 + r^2 + s^2 = x^2 + y^2$,
then ABCD is a parallelogram.

Proof: $\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC}$

$\overrightarrow{BD} = \overrightarrow{AD} - \overrightarrow{AB}$

$$x^2 + y^2 = \overrightarrow{AC} \cdot \overrightarrow{AC} + \overrightarrow{BD} \cdot \overrightarrow{BD}$$
$$= \left(\overrightarrow{AB} + \overrightarrow{BC}\right) \cdot \left(\overrightarrow{AB} + \overrightarrow{BC}\right) + \left(\overrightarrow{AD} - \overrightarrow{AB}\right) \cdot \left(\overrightarrow{AD} - \overrightarrow{AB}\right)$$
$$= \left|\overrightarrow{AB}\right|^2 + 2\overrightarrow{AB} \cdot \overrightarrow{BC} + \left|\overrightarrow{BC}\right|^2 + \left|\overrightarrow{AD}\right|^2 - 2\overrightarrow{AB} \cdot \overrightarrow{AD} + \left|\overrightarrow{AB}\right|^2$$
$$p^2+q^2+r^2+s^2 = 2p^2 + q^2 + s^2 + 2\,\overrightarrow{AB} \cdot \left(\overrightarrow{BC} - \overrightarrow{AD}\right)$$
$$r^2 - p^2 = 2\,\overrightarrow{AB} \cdot \left(\overrightarrow{BC} - \overrightarrow{AB} - \overrightarrow{BC} - \overrightarrow{CD}\right)$$
$$= -2\,\overrightarrow{AB} \cdot \left(\overrightarrow{AB} + \overrightarrow{CD}\right)$$
$$\left|\overrightarrow{CD}\right|^2 - \left|\overrightarrow{AB}\right|^2 = -2\left|\overrightarrow{AB}\right|^2 - 2\overrightarrow{AB} \cdot \overrightarrow{CD}$$
$$\left|\overrightarrow{CD}\right|^2 + 2\overrightarrow{AB} \cdot \overrightarrow{CD} + \left|\overrightarrow{AB}\right|^2 = 0$$
$$\left(\overrightarrow{CD} + \overrightarrow{AB}\right) \cdot \left(\overrightarrow{CD} + \overrightarrow{AB}\right) = 0$$
$$\left|\overrightarrow{CD} + \overrightarrow{AB}\right|^2 = 0$$
$$\overrightarrow{CD} = -\overrightarrow{AB}$$

So $CD = AB$ and AB // CD
$ABCD$ is a parallelogram (opp. sides are eq. and parallel)