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ROW AND COLUMN SPACE OF A MATRIX; RANK; APPLICATIONS TO FINDING BASES

Definition. Consider the $m \times n$ matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

The vectors

$$\begin{aligned} \mathbf{r}_1 &= (a_{11}, a_{12}, \ldots, a_{1n}) \\ \mathbf{r}_2 &= (a_{21}, a_{22}, \ldots, a_{2n}) \\ &\vdots \\ \mathbf{r}_m &= (a_{m1}, a_{m2}, \ldots, a_{mn}) \end{aligned}$$

formed from the rows of A, are called the ***row vectors*** of A and the vectors

$$\mathbf{c}_1 = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}, \mathbf{c}_2 = \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix}, \ldots, \mathbf{c}_n = \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}$$

formed from the columns of A are called the ***column vectors*** of A. The subspace of R^n spanned by the row vectors is called the ***row space*** of A and the subspace of R^m spanned by the column vectors is called the ***column space*** of A.

Example 37

Let

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 3 & -1 & 4 \end{bmatrix}$$

The row vectors of A are

$$\mathbf{r}_1 = (2, 1, 0) \quad \text{and} \quad \mathbf{r}_2 = (3, -1, 4)$$

and the column vectors of A are

$$\mathbf{c}_1 = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \quad \mathbf{c}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \quad \text{and} \quad \mathbf{c}_3 = \begin{bmatrix} 0 \\ 4 \end{bmatrix}$$

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Theorem 10. *Elementary row operations do not change the row space of a matrix.*

Proof of Theorem 10. Suppose that the row vectors of a matrix A are $\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_m$ and let B be obtained from A by performing an elementary row operation. We shall show that every vector in the row space of B is also in the row space of A, and conversely that every vector in the row space of A is in the row space of B. We can then conclude that A and B have the same row space.

Consider the possibilities. If the row operation is a row interchange, then B and A have the same row vectors, and consequently the same row space. If the row operation is multiplication of a row by a scalar or addition of a multiple of one row to another, then the row vectors $\mathbf{r}'_1, \mathbf{r}'_2, \ldots, \mathbf{r}'_m$ of B are linear combinations of $\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_m$; thus they lie in the row space of A. Since a vector space is closed under addition and scalar multiplication, all linear combinations of $\mathbf{r}'_1, \mathbf{r}'_2, \ldots, \mathbf{r}'_m$ will also lie in the row space of A. Therefore each vector in the row space of B is in the row space of A.

Since B is obtained from A by performing a row operation, A can be obtained from B by performing the inverse operation (Section 1.7). Thus the argument above shows that row space of A is contained in the row space of B.

Theorem 11. *The nonzero row vectors in a row-echelon form of a matrix A form a basis for the row space of A.*

Example 38

Find a basis for the space spanned by the vectors

$$\mathbf{v}_1 = (1, -2, 0, 0, 3) \quad \mathbf{v}_2 = (2, -5, -3, -2, 6) \quad \mathbf{v}_3 = (0, 5, 15, 10, 0)$$
$$\mathbf{v}_4 = (2, 6, 18, 8, 6)$$

Solution. The space spanned by these vectors is the row space of the matrix

$$\begin{bmatrix} 1 & -2 & 0 & 0 & 3 \\ 2 & -5 & -3 & -2 & 6 \\ 0 & 5 & 15 & 10 & 0 \\ 2 & 6 & 18 & 8 & 6 \end{bmatrix}$$

Putting this matrix in row-echelon form we obtain (verify):

$$\begin{bmatrix} 1 & -2 & 0 & 0 & 3 \\ 0 & 1 & 3 & 2 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The nonzero row vectors in this matrix are

$$\mathbf{w}_1 = (1, -2, 0, 0, 3) \quad \mathbf{w}_2 = (0, 1, 3, 2, 0) \quad \mathbf{w}_3 = (0, 0, 1, 1, 0)$$

These form a basis for the row space and consequently a basis for the space spanned by $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$. and $\mathbf{v}_4$.

Example 39

Find a basis for the column space of

$$A = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 3 & 2 & 5 & 1 \\ 0 & 4 & 4 & -4 \end{bmatrix}$$

Solution. Transposing we obtain

$$A^t = \begin{bmatrix} 1 & 3 & 0 \\ 0 & 2 & 4 \\ 1 & 5 & 4 \\ 1 & 1 & -4 \end{bmatrix}$$

and reducing to row-echelon form yields (verify)

$$\begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Thus the vectors (1, 3, 0) and (0, 1, 2) form a basis for the row space of A^t or equivalently

$$\mathbf{w}_1 = \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix} \quad \text{and} \quad \mathbf{w}_2 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$$

form a basis for the column space of A.

Theorem 12. *If A is any matrix, then the row space and column space of A have the same dimension.*

Proof of Theorem 12. Denote the row vectors of

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

by

$$\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_m$$

Suppose the row space of A has dimension k and that $S = \{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_k\}$ is a basis for the row space, where $\mathbf{b}_i = (b_{i1}, b_{i2}, \ldots, b_{in})$. Since S is a basis for the row space, each row vector is expressible as a linear combination of $\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_k$; thus

$$\begin{aligned} \mathbf{r}_1 &= c_{11}\mathbf{b}_1 + c_{12}\mathbf{b}_2 + \cdots + c_{1k}\mathbf{b}_k \\ \mathbf{r}_2 &= c_{21}\mathbf{b}_1 + c_{22}\mathbf{b}_2 + \cdots + c_{2k}\mathbf{b}_k \\ &\vdots \\ \mathbf{r}_m &= c_{m1}\mathbf{b}_1 + c_{m2}\mathbf{b}_2 + \cdots + c_{mk}\mathbf{b}_k \end{aligned} \tag{4.11}$$

Since two vectors in R^n are equal if and only if corresponding components are equal, we can equate the jth component on each side of (4.11) to obtain

$$\begin{aligned} a_{1j} &= c_{11}b_{1j} + c_{12}b_{2j} + \cdots + c_{1k}b_{kj} \\ a_{2j} &= c_{21}b_{1j} + c_{22}b_{2j} + \cdots + c_{2k}b_{kj} \\ &\vdots \\ a_{mj} &= c_{m1}b_{1j} + c_{m2}b_{2j} + \cdots + c_{mk}b_{kj} \end{aligned}$$

or equivalently

$$\begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{bmatrix} = b_{1j}\begin{bmatrix} c_{11} \\ c_{21} \\ \vdots \\ c_{m1} \end{bmatrix} + b_{2j}\begin{bmatrix} c_{12} \\ c_{22} \\ \vdots \\ c_{m2} \end{bmatrix} + \cdots + b_{kj}\begin{bmatrix} c_{1k} \\ c_{2k} \\ \vdots \\ c_{mk} \end{bmatrix} \tag{4.12}$$

The left side of this equation is the jth column vector of A, and $j = 1, 2, \ldots, n$ is arbitrary; therefore each column vector of A lies in the space spanned by the k vectors on the right side of (4.12). Thus the column space of A has dimension $\leq k$.

Since

$$k = \dim(\text{row space of } A)$$

we have

$$\dim(\text{column space of } A) \leq \dim(\text{row space of } A). \tag{4.13}$$

Since the matrix A is completely arbitrary, this same conclusion applies to A^t, that is

$$\dim(\text{column space of } A^t) \leq \dim(\text{row space of } A^t) \tag{4.14}$$

But transposing a matrix converts columns to rows and rows to columns so that

$$\text{column space of } A^t = \text{row space of } A$$

and

$$\text{row space of } A^t = \text{column space of } A$$

Thus (4.14) can be rewritten as

$$\dim(\text{row space of } A) \leq \dim(\text{column space of } A).$$

From this result and (4.13) we conclude that

$$\dim(\text{row space of } A) = \dim(\text{column space of } A).$$

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Theorem 13. *If A is an $n \times n$ matrix, then the following statements are equivalent.*

(a) *A is invertible.*
(b) *$A\mathbf{x} = \mathbf{0}$ has only the trivial solution.*
(c) *A is row equivalent to I_n.*
(d) *$A\mathbf{x} = \mathbf{b}$ is consistent for every $n \times 1$ matrix $\mathbf{b}$.*
(e) *$\det(A) \neq 0$*
(f) *A has rank n*
(g) *The row vectors of A are linearly independent.*
(h) *The column vectors of A are linearly independent.*

Definition. The dimension of the row and column space of a matrix A is called the ***rank*** of A.

Proof. We shall prove the equivalence by establishing the following chain of implications: $(a) \Rightarrow (b) \Rightarrow (c) \Rightarrow (a)$.

$(a) \Rightarrow (b)$: Assume A is invertible and let X_0 be any solution to $AX = 0$. Thus $AX_0 = 0$. Multiplying both sides of this equation by A^{-1} gives $A^{-1}(AX_0) = A^{-1}0$, or $(A^{-1}A)X_0 = 0$, or $IX_0 = 0$, or $X_0 = 0$. Thus $AX = 0$ has only the trivial solution.

$(b) \Rightarrow (c)$: Let $AX = 0$ be the matrix form of the system

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= 0 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= 0 \\ \vdots \qquad \vdots \qquad\quad \vdots \quad & \;\; \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= 0 \end{aligned} \tag{1.8}$$

and assume the system has only the trivial solution. If we solve by Gauss-Jordan elimination, then the system of equations corresponding to the reduced row-echelon form of the augmented matrix will be

$$\begin{aligned} x_1 \qquad\qquad &= 0 \\ x_2 \qquad &= 0 \\ \ddots \quad & \\ x_n &= 0 \end{aligned} \tag{1.9}$$

Thus the augmented matrix

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & 0 \\ a_{21} & a_{22} & \cdots & a_{2n} & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} & 0 \end{bmatrix}$$

for (1.8) can be reduced to the augmented matrix

$$\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix}$$

for (1.9) by a sequence of elementary row operations. If we disregard the last column (of zeros) in each of these matrices, we can conclude that A can be reduced to I_n by a sequence of elementary row operations; that is, A is row equivalent to I_n.

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$(c) \Rightarrow (a)$: Assume A is row equivalent to I_n, so that A can be reduced to I_n by a finite sequence of elementary row operations. By Theorem 8 each of these operations can be accomplished by multiplying on the left by an appropriate elementary matrix. Thus, we can find elementary matrices $E_1, E_2, \ldots, E_k$ such that

$$E_k \cdots E_2 E_1 A = I_n \tag{1.10}$$

By Theorem 9, $E_1, E_2, \ldots, E_k$ are invertible. Multiplying both sides of equation (1.10) on the left successively by $E_k^{-1}, \ldots, E_2^{-1}, E_1^{-1}$ we obtain

$$A = E_1^{-1}E_2^{-1} \cdots E_k^{-1} I_n = E_1^{-1}E_2^{-1} \cdots E_k^{-1} \tag{1.11}$$

Since (1.11) expresses A as a product of invertible matrices, we can conclude that A is invertible.

REMARK. Because I_n is in reduced row-echelon form and because the reduced row-echelon form of a matrix A is unique, part (c) of Theorem 10 is equivalent to stating that I_n is the reduced row-echelon form of A.

Proof. Since we proved in Theorem 10 that (a), (b), and (c) are equivalent, it will be sufficient to prove $(a) \Rightarrow (d)$ and $(d) \Rightarrow (a)$.

$(a) \Rightarrow (d)$: If A is invertible and B is any $n \times 1$ matrix then $X = A^{-1}B$ is a solution of $AX = B$ by Theorem 11. Thus $AX = B$ is consistent.

$(d) \Rightarrow (a)$: If the system $AX = B$ is consistent for every $n \times 1$ matrix B then, in particular, the systems

$$AX = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad AX = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \ldots, \quad AX = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

will be consistent. Let X_1 be a solution of the first system, X_2 a solution of the second system, ..., and X_n a solution of the last system, and let us form an $n \times n$ matrix C having these solutions as columns. Thus C has the form

$$C = [X_1 \mid X_2 \mid \cdots \mid X_n]$$

As discussed in Example 17, the successive columns of the product AC will be

$$AX_1, AX_2, \ldots, AX_n$$

Thus

$$AC = [AX_1 \mid AX_2 \mid \cdots \mid AX_n] = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} = I$$

By part (b) of Theorem 12 it follows that $C = A^{-1}$. Thus A is invertible.

Proof. We shall show that (*c*), (*f*), (*g*) and (*h*) are equivalent by proving the sequence of implications (*c*) ⇒ (*f*) ⇒ (*g*) ⇒ (*h*) ⇒ (*c*). This will complete the proof since we already know that (*c*) is equivalent to (*a*), (*b*), (*d*), and (*e*).

(*c*) ⇒ (*f*) Since A is row equivalent to I_n, and I_n has n nonzero rows, the row space of A is n-dimensional by Theorem 11. Thus A has rank n.

(*f*) ⇒ (*g*) Since A has rank n, the row space of A is n dimensional. Since the n row vectors of A span the row space of A, it follows from Theorem 9 in Section 4.5 that the row vectors of A are linearly independent.

(*g*) ⇒ (*h*) Assume the row vectors of A are linearly independent. Thus the row space of A is n-dimensional. By Theorem 12 the column space of A is also n-dimensional. Since the column vectors of A span the column space, the column vectors of A are linearly independent by Theorem 9 in Section 4.5.

(*h*) ⇒ (*c*) Assume the column vectors of A are linearly independent. Thus the column space of A is n-dimensional and consequently the row space of A is n-dimensional by Theorem 12. This means that the reduced row-echelon form of A has n nonzero rows, that is all rows are nonzero. As noted in Example 24 of Section 2.3 this implies that the reduced row-echelon form of A is I_n. Thus A is row equivalent to I_n.