Ceva's Theorem (Vectors)

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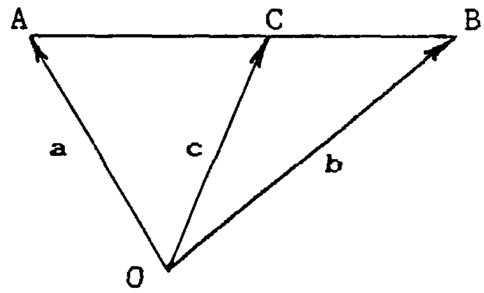
1. Let $\vec{a}$, $\vec{b}$, $\vec{c}$ be the corresponding position vectors of A, B and C respectively.
If C lies on the line AB, prove that $\vec{c} = \lambda\vec{a} + \mu\vec{b}$ with $\lambda + \mu = 1$.

Proof Let $AC : CB = m : n$, where m and n are any scalar.



$$\vec{c} = \frac{n\vec{a} + m\vec{b}}{m+n} = \frac{n}{m+n}\vec{a} + \frac{m}{m+n}\vec{b}$$

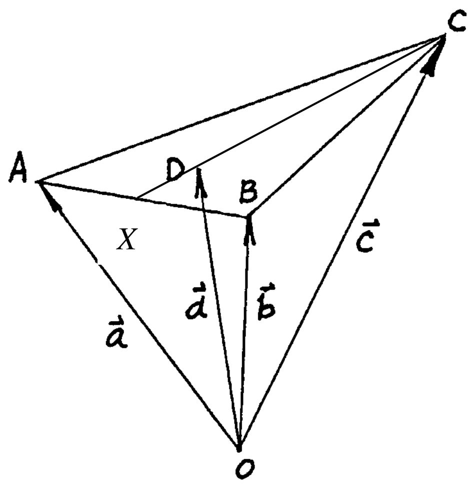
$$= \lambda\vec{a} + \mu\vec{b}, \text{ where } \lambda = \frac{n}{m+n} \text{ and } \mu = \frac{m}{m+n}$$

$$\lambda + \mu = \frac{n}{m+n} + \frac{m}{m+n} = \frac{m+n}{m+n} = 1$$

2. Let $\vec{a}$, $\vec{b}$, $\vec{c}$ be the corresponding position vectors of A, B and C respectively such that A, B and C are not collinear. If D lies in the plane of A, B and C, show that

$$\overrightarrow{OD} = \vec{d} = \lambda\vec{a} + \mu\vec{b} + \nu\vec{c} \text{ with } \lambda + \mu + \nu = 1.$$

ProofAs shown in the figure, produce CD to X on AB.



By example 1, $\overrightarrow{OX} = p\vec{a} + q\vec{b}$ with $p + q = 1$ and

$\overrightarrow{OD} = r\overrightarrow{OX} + s\vec{c}$ with $r + s = 1$

$= r(p\vec{a} + q\vec{b}) + s\vec{c}$

$= rp\vec{a} + rq\vec{b} + s\vec{c}$

$= \lambda\vec{a} + \mu\vec{b} + \nu\vec{c}$

where $\lambda = rp$, $\mu = rq$, $\nu = s$

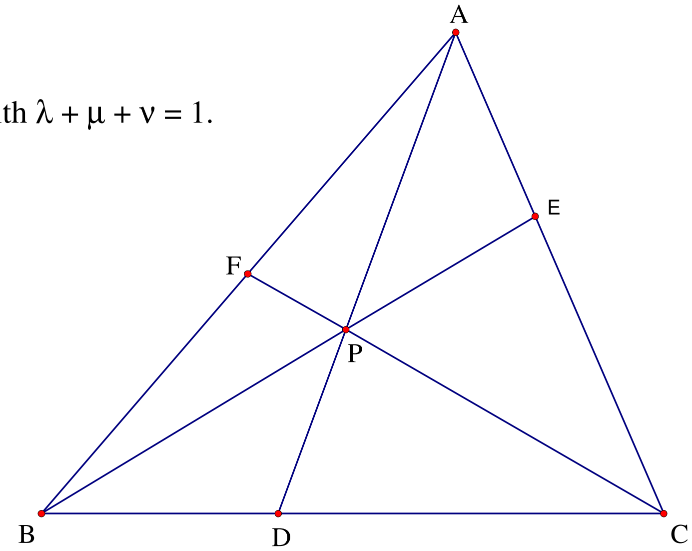
and $\lambda + \mu + \nu = rp + rq + s$

$= r(p + q) + s$

$= r + s = 1$

The proof is completed.

3. In ΔABC, D, E, F are points on BC, CA and AB respectively. AD, BE and CF are concurrent at P. Then $\frac{BD}{DC}\cdot\frac{CE}{EA}\cdot\frac{AF}{FB}=1$



Proof: By the result of 2, $\vec{p}=\lambda\vec{a}+\mu\vec{b}+\nu\vec{c}$ with $\lambda+\mu+\nu=1$.

$$\vec{p}-\lambda\vec{a}=\mu\vec{b}+\nu\vec{c}$$

$$\frac{\vec{p}-\lambda\vec{a}}{1-\lambda}=\frac{\mu\vec{b}+\nu\vec{c}}{1-\lambda}$$

$$\frac{\vec{p}-\lambda\vec{a}}{1-\lambda}=\frac{\mu\vec{b}+\nu\vec{c}}{\mu+\nu} \quad \text{....................(1)}$$

LHS of (1) is a point G on AP produced such that $AP : PG = 1 - \lambda : \lambda$

RHS of (1) is a point H on BC such that $BH : HC = \nu : \mu$

AP produced intersects BC at D which means that $AP : PD = 1 - \lambda : \lambda$ and $BD : DC = \nu : \mu$

In a similar manner, $\vec{p}=\lambda\vec{a}+\mu\vec{b}+\nu\vec{c}\Rightarrow \vec{p}-\mu\vec{b}=\lambda\vec{a}+\nu\vec{c}$

$$\frac{\vec{p}-\mu\vec{b}}{1-\mu}=\frac{\lambda\vec{a}+\nu\vec{c}}{\lambda+\nu} \quad \text{....................(2)}$$

BP produced intersects AC at E which means that $BP : PE = 1 - \mu : \mu$ and $CE : EA = \lambda : \nu$

Similarly, $\vec{p}=\lambda\vec{a}+\mu\vec{b}+\nu\vec{c}\Rightarrow \vec{p}-\nu\vec{c}=\lambda\vec{a}+\mu\vec{b}$

$$\frac{\vec{p}-\nu\vec{c}}{1-\nu}=\frac{\lambda\vec{a}+\mu\vec{b}}{\lambda+\mu} \quad \text{....................(3)}$$

CP produced intersects AB at F which means that $CP : PF = 1 - \nu : \nu$ and $AF : FB = \mu : \lambda$

$$\frac{BD}{DC}\cdot\frac{CE}{EA}\cdot\frac{AF}{FB}=\frac{\nu}{\mu}\cdot\frac{\lambda}{\nu}\cdot\frac{\mu}{\lambda}=1$$

The proof is completed.

Please refer to the other proof of Ceva's Theorem in Geometry:

http://www.hkedcity.net/ihouse/fh7878/Geometry/others/Ceva-Menelaus.pdf