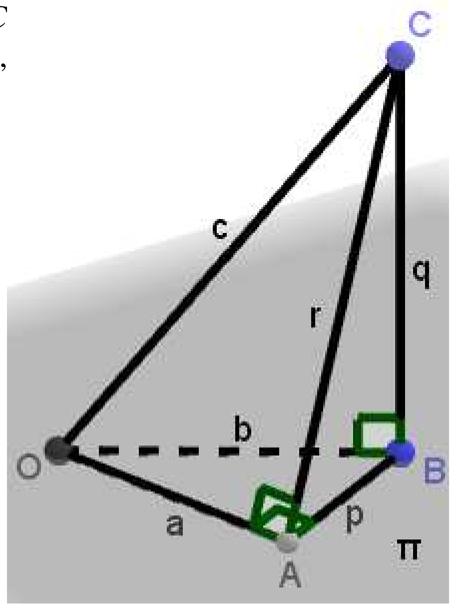
Coplanar and volume of tetrahedron

Created by Mr. Francis Hung on 2022-10-31

1. (a) In the figure, O, A, B are three points of the horizontal plane π. C is a point not lying on π. $OB \perp BC$, $OA \perp AB$, $OA \perp AC$, $OA = a$, $OB = b$, $OC = c$, $AB = p$, $BC = q$, $AC = r$.
Show that $AB \perp BC$.



(b) Let $\mathbf{e_1}$ and $\mathbf{e_2}$ be two mutually perpendicular unit vectors and $\mathbf{V}$ be any vector.
Define $c_1 = \mathbf{V}\cdot\mathbf{e_1}$ and $c_2 = \mathbf{V}\cdot\mathbf{e_2}$.
For any real numbers b_1 and b_2, prove that $|\mathbf{V} - c_1\mathbf{e_1} - c_2\mathbf{e_2}| \le |\mathbf{V} - b_1\mathbf{e_1} - b_2\boldsymbol{e_2}|$.

(c) Let the coordinates of $A(1, 2, 0)$, $B(-1, 3, 0)$, $C(0, 1, -1)$, $E(0, 0,1)$.

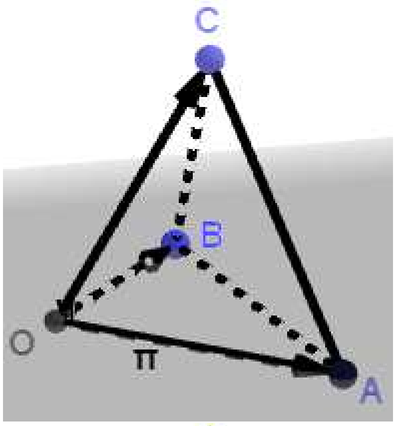
(i) Find the unit vector $\mathbf{e_1}$ in the direction of EA.

(ii) Find a unit vector $\mathbf{e_2}$ in the plane EAB such that $\mathbf{e_1} \perp \mathbf{e_2}$.

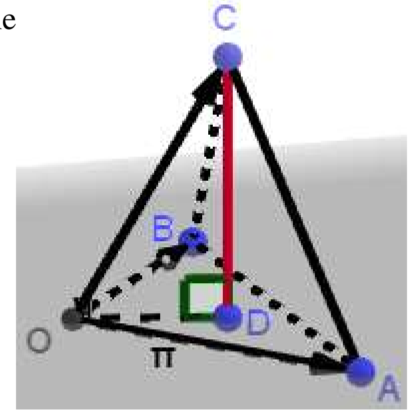
(iii) Using the result of (b), find the projection vector $\overrightarrow{ED}$ of $\overrightarrow{EC}$ on the plane EAB.

(iv) Find a point D on the plane such that D is nearest to C.

2. (a) In the figure, O, A, B are three points of the plane π.
C is any point in the 3-dimensional space.
$\overrightarrow{OA} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$,
$\overrightarrow{OB} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$,
$\overrightarrow{OC} = c_1\mathbf{i} + c_2\mathbf{j} + c_3\mathbf{k}$.
If O, A, B, C are coplanar, show that $\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = 0$.
It is known that the converse is also true.



(b) If C does not lie on the plane OAB (π), find the height (CD) of the tetrahedron. Hence the point D on (π) which is nearest to C.



(c) Let the coordinates of $A(1, 2, 0)$, $B(-1, 3, 0)$, $C(0, 1, -1)$, $E(0, 0,1)$.
(i) Show that A, B, C and E are not coplanar.
(ii) Find the volume of the tetrahedron $ABCE$.

1. (a) In ΔOBC, $b^2 + q^2 = c^2$ (Pythagoras' theorem) $\cdots\cdots$ (1)

In ΔOAC, $a^2 + r^2 = c^2$ (Pythagoras' theorem) $\cdots\cdots$ (2)

In ΔOAB, $a^2 + p^2 = b^2$ (Pythagoras' theorem) $\cdots\cdots$ (3)

(1) – (2) + (3): $b^2 + q^2 - r^2 + p^2 = b^2$

$p^2 + q^2 = r^2$

$\therefore AB \perp BC$ (converse, Pythagoras' theorem)

(b) **Method 1**

$|\mathbf{V} - b_1\mathbf{e_1} - b_2\mathbf{e_2}|^2 - |\mathbf{V} - c_1\mathbf{e_1} - c_2\mathbf{e_2}|^2$

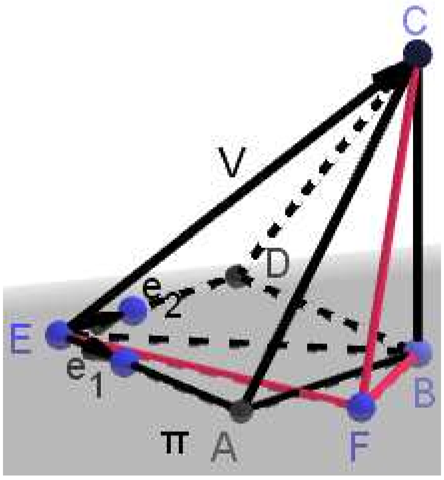
$= |\mathbf{V}|^2 + b_1^2 + b_2^2 - 2b_1\mathbf{V}\cdot\mathbf{e_1} - 2b_2\mathbf{V}\cdot\mathbf{e_2} + 2b_1b_2\mathbf{e_1}\cdot\mathbf{e_2} - [|\mathbf{V}|^2 + c_1^2 + c_2^2 - 2c_1\mathbf{V}\cdot\mathbf{e_1} - 2c_2\mathbf{V}\cdot\mathbf{e_2} + 2c_1c_2\mathbf{e_1}\cdot\mathbf{e_2}]$

$= b_1^2 + b_2^2 - 2b_1 c_1 - 2b_2 c_2 - (c_1^2 + c_2^2 - 2c_1^2 - 2c_2^2)$

$= (b_1 - c_1)^2 + (b_2 - c_2)^2 \ge 0$

$\therefore |\mathbf{V} - c_1\mathbf{e_1} - c_2\mathbf{e_2}| \le |\mathbf{V} - b_1\mathbf{e_1} - b_2\mathbf{e_2}|$

Method 2 Let the plane determined by $\mathbf{e_1}$ and $\mathbf{e_2}$ be π. Let $\mathbf{V} = \overrightarrow{EC}$.



$c_1\mathbf{e_1} = (\mathbf{V}\cdot\mathbf{e_1})\mathbf{e_1}$ = projection vector of $\mathbf{V}$ on $\mathbf{e_1} = \overrightarrow{EA}$ ($EA \perp AC$)

$c_2\mathbf{e_2} = (\mathbf{V}\cdot\mathbf{e_2})\mathbf{e_2}$ = projection vector of $\mathbf{V}$ on $\mathbf{e_2} = \overrightarrow{ED}$ ($ED \perp DC$)

$c_1\mathbf{e_1} + c_2\mathbf{e_2} = \overrightarrow{EA} + \overrightarrow{ED} = \overrightarrow{EB}$ (where $EABD$ is a parallelogram)

$\mathbf{V} - c_1\mathbf{e_1} - c_2\mathbf{e_2} = \overrightarrow{EC} - \overrightarrow{EB} = \overrightarrow{BC}$

$\overrightarrow{BC}\cdot\mathbf{e_1} = (\mathbf{V} - c_1\mathbf{e_1} - c_2\mathbf{e_2})\cdot\mathbf{e_1}$

$= \mathbf{V}\cdot\mathbf{e_1} - (\mathbf{V}\cdot\mathbf{e_1})\mathbf{e_1}\cdot\mathbf{e_1} - (\mathbf{V}\cdot\mathbf{e_2})\mathbf{e_2}\cdot\mathbf{e_1}$

$= \mathbf{V}\cdot\mathbf{e_1} - \mathbf{V}\cdot\mathbf{e_1} = 0$

$\therefore \overrightarrow{BC} \perp \mathbf{e_1}$

$\overrightarrow{BC}\cdot\mathbf{e_2} = (\mathbf{V} - c_1\mathbf{e_1} - c_2\mathbf{e_2})\cdot\mathbf{e_2}$

$= \mathbf{V}\cdot\mathbf{e_2} - (\mathbf{V}\cdot\mathbf{e_1})\mathbf{e_1}\cdot\mathbf{e_2} - (\mathbf{V}\cdot\mathbf{e_2})\mathbf{e_2}\cdot\mathbf{e_2}$

$= \mathbf{V}\cdot\mathbf{e_2} - \mathbf{V}\cdot\mathbf{e_2} = 0$

$\therefore \overrightarrow{BC} \perp \mathbf{e_2}$

$\overrightarrow{BC}\cdot\overrightarrow{EB} = \overrightarrow{BC}\cdot(c_1\mathbf{e_1} + c_2\mathbf{e_2})$

$= c_1\overrightarrow{BC}\cdot\mathbf{e_1} + c_2\overrightarrow{BC}\cdot\mathbf{e_2}$

$= 0$

$\therefore EB \perp BC$

$\because$ $\mathbf{e_1}$ and $\mathbf{e_2}$ are mutually perpendicular unit vectors

$\therefore EA \perp ED$

$\because$ $EABD$ is a parallelogram

$\therefore EA \perp AB$(int. $\angle$s, AB // ED), also $ED \perp DB$

By the result of (a), $AB \perp BC$ and $DB \perp BC$

$\therefore BC \perp \pi$

For any real numbers b_1 and b_2,

$\overrightarrow{EF} = b_1\mathbf{e_1} + b_2\mathbf{e_2}$ is any vector lying on π

$\mathbf{V} - b_1\mathbf{e_1} - b_2\mathbf{e_2} = \overrightarrow{FC}$

Consider ΔBCF, $BF \perp BC$

$BC^2 + BF^2 = CF^2$ (Pythagoras' theorem)

$BC \le CE$

$|\mathbf{V} - c_1\mathbf{e_1} - c_2\mathbf{e_2}| \le |\mathbf{V} - b_1\mathbf{e_1} - b_2\mathbf{e_2}|$

(c) (i) $\overrightarrow{EA} = (1,2,-1)$, $\overrightarrow{EB} = (-1,3,-1)$, $\overrightarrow{EC} = (0,1,-2)$

$|\overrightarrow{EA}| = \sqrt{1^2+2^2+(-1)^2} = \sqrt{6}$

$\mathbf{e_1} = \frac{1}{\sqrt{6}}(\mathbf{i}+2\mathbf{j}-\mathbf{k})$

(ii) The projection vector of $\overrightarrow{EB}$ on $\mathbf{e_1}$ is $(\overrightarrow{EB}\cdot\mathbf{e_1})\mathbf{e_1}$

The normal vector perpendicular to this projection vector is

$$\overrightarrow{EB} - (\overrightarrow{EB}\cdot\mathbf{e_1})\mathbf{e_1} = (-1,3,-1) - [(-1,3,-1)\cdot\frac{1}{\sqrt{6}}(\mathbf{i}+2\mathbf{j}-\mathbf{k})]\frac{1}{\sqrt{6}}(\mathbf{i}+2\mathbf{j}-\mathbf{k})$$

$$= (-1,3,-1) - \frac{-1+6+1}{6}(\mathbf{i}+2\mathbf{j}-\mathbf{k})$$

$$= (-1,3,-1) - (1,2,-1)$$

$$= (-2,1,0)$$

$$\mathbf{e_2} = \frac{1}{\sqrt{(-2)^2+1^2+0^2}}(-2\mathbf{i}+\mathbf{j}+0\mathbf{k}) = \frac{1}{\sqrt{5}}(-2\mathbf{i}+\mathbf{j})$$

(iii) The projection vector $\overrightarrow{ED} = c_1\mathbf{e_1} + c_2\mathbf{e_2}$

$$c_1 = \mathbf{V}\cdot\mathbf{e_1} = \overrightarrow{EC}\cdot\mathbf{e_1} = (0,1,-2)\cdot\frac{1}{\sqrt{6}}(\mathbf{i}+2\mathbf{j}-\mathbf{k}) = \frac{1}{\sqrt{6}}(0+2+2) = \frac{2\sqrt{6}}{3}$$

$$c_2 = \mathbf{V}\cdot\mathbf{e_2} = \overrightarrow{EC}\cdot\mathbf{e_2} = (0,1,-2)\cdot\frac{1}{\sqrt{5}}(-2\mathbf{i}+\mathbf{j}) = \frac{1}{\sqrt{5}}(0+1+0) = \frac{\sqrt{5}}{5}$$

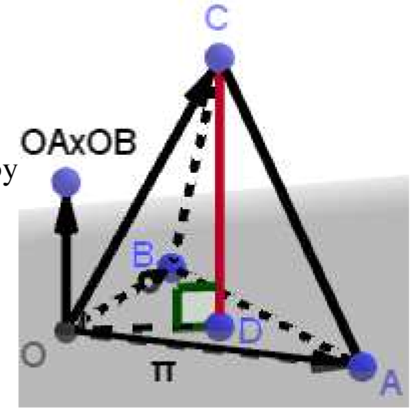
$$\overrightarrow{ED} = \frac{2\sqrt{6}}{3}\cdot\frac{1}{\sqrt{6}}(\mathbf{i}+2\mathbf{j}-\mathbf{k}) + \frac{\sqrt{5}}{5}\cdot\frac{1}{\sqrt{5}}(-2\mathbf{i}+\mathbf{j})$$

$$= \frac{2}{3}(\mathbf{i}+2\mathbf{j}-\mathbf{k}) + \frac{1}{5}(-2\mathbf{i}+\mathbf{j})$$

$$= \frac{4}{15}\mathbf{i} + \frac{23}{15}\mathbf{j} - \frac{2}{3}\mathbf{k}$$

(iv) $\overrightarrow{OD} = \overrightarrow{OE} + \overrightarrow{ED} = (0, 0, 1) + \frac{4}{15}\mathbf{i} + \frac{23}{15}\mathbf{j} - \frac{2}{3}\mathbf{k} = \frac{4}{15}\mathbf{i} + \frac{23}{15}\mathbf{j} + \frac{1}{3}\mathbf{k}$

2. (a) $\overrightarrow{OA}\times\overrightarrow{OB}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{vmatrix}=\begin{vmatrix}a_2 & a_3\\ b_2 & b_3\end{vmatrix}\mathbf{i}-\begin{vmatrix}a_1 & a_3\\ b_1 & b_3\end{vmatrix}\mathbf{j}+\begin{vmatrix}a_1 & a_2\\ b_1 & b_2\end{vmatrix}\mathbf{k}$



This is the normal vector perpendicular to the plane (π) determined by *O*, *A*, *B*.

Let the angle between $\overrightarrow{OA}\times\overrightarrow{OB}$ and $\overrightarrow{OC}$ be θ.

$$\cos\theta=\left|\frac{(\overrightarrow{OA}\times\overrightarrow{OB})\cdot\overrightarrow{OC}}{|\overrightarrow{OA}\times\overrightarrow{OB}||\overrightarrow{OC}|}\right|$$

$$=\left|\frac{\left(\begin{vmatrix}a_2 & a_3\\ b_2 & b_3\end{vmatrix}\mathbf{i}-\begin{vmatrix}a_1 & a_3\\ b_1 & b_3\end{vmatrix}\mathbf{j}+\begin{vmatrix}a_1 & a_2\\ b_1 & b_2\end{vmatrix}\mathbf{k}\right)\cdot(c_1\mathbf{i}+c_2\mathbf{j}+c_3\mathbf{k})}{|\overrightarrow{OA}\times\overrightarrow{OB}||\overrightarrow{OC}|}\right|$$

If *O*, *A*, *B*, *C* are coplanar, then θ = 90°, so cos θ = 0

$$\begin{vmatrix}a_2 & a_3\\ b_2 & b_3\end{vmatrix}c_1-\begin{vmatrix}a_1 & a_3\\ b_1 & b_3\end{vmatrix}c_2+\begin{vmatrix}a_1 & a_2\\ b_1 & b_2\end{vmatrix}c_3=0$$

$\begin{vmatrix}a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\\ c_1 & c_2 & c_3\end{vmatrix}=0$ (cofactor expansion along the third row)

(b) The projection vector of $\overrightarrow{OC}$ on the normal vector $\overrightarrow{OA}\times\overrightarrow{OB}$ is $\overrightarrow{DC}$

$$|\overrightarrow{DC}|=OC\cos\theta=\left|\frac{(\overrightarrow{OA}\times\overrightarrow{OB})\cdot\overrightarrow{OC}}{|\overrightarrow{OA}\times\overrightarrow{OB}|}\right|=\frac{1}{\sqrt{(a_2b_3-a_3b_2)^2+(a_1b_3-a_3b_1)^2+(a_1b_2-a_2b_1)^2}}\left\|\begin{matrix}a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\\ c_1 & c_2 & c_3\end{matrix}\right\|$$

$$\overrightarrow{DC}=\left|\frac{(\overrightarrow{OA}\times\overrightarrow{OB})\cdot\overrightarrow{OC}}{|\overrightarrow{OA}\times\overrightarrow{OB}|}\right|\frac{\overrightarrow{OA}\times\overrightarrow{OB}}{|\overrightarrow{OA}\times\overrightarrow{OB}|}=\left|\frac{(\overrightarrow{OA}\times\overrightarrow{OB})\cdot\overrightarrow{OC}}{|\overrightarrow{OA}\times\overrightarrow{OB}|^2}\right|\overrightarrow{OA}\times\overrightarrow{OB}$$

$$\overrightarrow{OD}=\overrightarrow{OC}-\overrightarrow{DC}$$

$$=c_1\mathbf{i}+c_2\mathbf{j}+c_3\mathbf{k}-\frac{1}{\sqrt{(a_2b_3-a_3b_2)^2+(a_1b_3-a_3b_1)^2+(a_1b_2-a_2b_1)^2}}\left\|\begin{matrix}a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\\ c_1 & c_2 & c_3\end{matrix}\right\|\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{vmatrix}$$

(c) (i) $\overrightarrow{EA}=(1,2,-1)$, $\overrightarrow{EB}=(-1,3,-1)$, $\overrightarrow{EC}=(0,1,-2)$

$\begin{vmatrix}1 & 2 & -1\\ -1 & 3 & -1\\ 0 & 1 & -2\end{vmatrix}=-5-2(2)+1=-8\neq 0$ ∴ *A*, *B*, *C*, *E* are not coplanar

(ii) $\overrightarrow{EA}\times\overrightarrow{EB}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 1 & 2 & -1\\ -1 & 3 & -1\end{vmatrix}=\mathbf{i}+2\mathbf{j}+5\mathbf{k}$, $|\overrightarrow{EA}\times\overrightarrow{EB}|=\sqrt{30}$

Height $=\left|\frac{-8}{\sqrt{30}}\right|=\frac{4\sqrt{30}}{15}$

Area of the triangle *EAB* (the base) $=\frac{1}{2}\sqrt{30}$

Volume of the tetrahedron $=\frac{1}{3}\times\frac{1}{2}\sqrt{30}\times\frac{4\sqrt{30}}{15}=\frac{4}{3}$