# Diagonalisation of matrices

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**Example 1** Let $A = \begin{pmatrix} 4 & -2 \\ 1 & 1 \end{pmatrix}$.

(*a*) Solve for $\lambda$ in the equation: $\det(A - \lambda I) = 0$.

(*b*) Let $\lambda_1$ and $\lambda_2$ be the roots of in (*a*), where $\lambda_1 < \lambda_2$.

Find $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$ and $\begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$ such that $\begin{pmatrix} 4 & -2 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \lambda_1\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$ and $\begin{pmatrix} 4 & -2 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \lambda_2\begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$,

where $x_1^2 + y_1^2 \neq 0$ and $x_2^2 + y_2^2 \neq 0$.

(*c*) Let $P = \begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix}$. Show that $P$ is non-singular. Hence find $P^{-1}$.

(*d*) Find $P^{-1}AP$. Hence find $A^{100}$.

(*a*) $\det(A - \lambda I) = 0$.

$$\det\left[\begin{pmatrix} 4 & -2 \\ 1 & 1 \end{pmatrix} - \lambda\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right] = 0$$

$$\begin{vmatrix} 4-\lambda & -2 \\ 1 & 1-\lambda \end{vmatrix} = 0$$

$(4 - \lambda)(1 - \lambda) + 2 = 0$

$\lambda^2 - 5\lambda + 6 = 0$

$(\lambda - 2)(\lambda - 3) = 0$

$\lambda = 2$ or $\lambda = 3$

(*b*) $\begin{pmatrix} 4 & -2 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = 2\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \Rightarrow x_1 = y_1$

Let $x_1 = y_1 = 1$

$\begin{pmatrix} 4 & -2 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = 3\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \Rightarrow x_2 = 2y_2$

Let $x_2 = 2$, $y_2 = 1$

(*c*) $P = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}$. $\det P = -1 \neq 0 \Rightarrow P$ is non-singular.

$$P^{-1} = -\begin{pmatrix} 1 & -2 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 2 \\ 1 & -1 \end{pmatrix}$$

(*d*) $P^{-1}AP = \begin{pmatrix} -1 & 2 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 4 & -2 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} -2 & 4 \\ 3 & -3 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$.

$$(P^{-1}AP)^{100} = \underbrace{(P^{-1}AP)(P^{-1}AP)\cdots(P^{-1}AP)}_{100 \text{ times}} = P^{-1}(APP^{-1}AP\cdots P^{-1}A)P = P^{-1}\left(\underbrace{AA\cdots A}_{100 \text{ times}}\right)P$$

$$P^{-1}A^{100}P = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}^{100} = \begin{pmatrix} 2^{100} & 0 \\ 0 & 3^{100} \end{pmatrix}$$

$$A^{100} = P\begin{pmatrix} 2^{100} & 0 \\ 0 & 3^{100} \end{pmatrix}P^{-1} = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 2^{100} & 0 \\ 0 & 3^{100} \end{pmatrix}\begin{pmatrix} -1 & 2 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 2^{100} & 2\cdot 3^{100} \\ 2^{100} & 3^{100} \end{pmatrix}\begin{pmatrix} -1 & 2 \\ 1 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} -2^{100} + 2\cdot 3^{100} & 2^{101} - 2\cdot 3^{100} \\ 3^{100} - 2^{100} & 2^{101} - 3^{100} \end{pmatrix}$$

**Example 2** Let $A = \begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix}$.

(*a*) Solve for $\lambda$ in the equation: $\det(A - \lambda I) = 0$.

(*b*) Let $\lambda_1$, $\lambda_2$ and $\lambda_3$ be the roots of in (*a*), where $\lambda_1 \le \lambda_2 < \lambda_3$.

Find $\begin{pmatrix} x_i \\ y_i \\ z_i \end{pmatrix}$ such that $\begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix}\begin{pmatrix} x_i \\ y_i \\ z_i \end{pmatrix} = \lambda_i \begin{pmatrix} x_i \\ y_i \\ z_i \end{pmatrix}$ where $x_i^2 + y_i^2 + z_i^2 \ne 0$ for $i = 1, 2, 3$.

(*c*) Let $P = \begin{pmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{pmatrix}$. Show that $P$ is non-singular. Hence find $P^{-1}$.

(*d*) Find $P^{-1}AP$. Hence find $A^{10}$.

(*a*) $\begin{vmatrix} -1-\lambda & 1 & 1 \\ 1 & -1-\lambda & 1 \\ 1 & 1 & -1-\lambda \end{vmatrix} = 0$

$-(\lambda + 1)^3 + 2 + 3(1 + \lambda) = 0$

$\lambda^3 + 3\lambda^2 + 3\lambda + 1 - 3\lambda - 5 = 0$

$\lambda^3 + 3\lambda^2 - 4 = 0$

$(\lambda + 2)^2(\lambda - 1) = 0$

$\lambda_1 = -2 = \lambda_2$ and $\lambda_3 = 1$

(*b*) $\begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = -2\begin{pmatrix} x \\ y \\ z \end{pmatrix}$

$x + y + z = 0$

$x = t, y = s, z = -t - s, t, s \in \mathbf{R}$.

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} t + \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} s$$

$$\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

$$\begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix}\begin{pmatrix} x_3 \\ y_3 \\ z_3 \end{pmatrix} = \begin{pmatrix} x_3 \\ y_3 \\ z_3 \end{pmatrix} \Rightarrow \left(\begin{array}{ccc|c} -2 & 1 & 1 & 0 \\ 1 & -2 & 1 & 0 \\ 1 & 1 & -2 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 2 & -1 & -1 & 0 \\ 0 & -3 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \begin{array}{l} -R_1 \\ R_1 + 2R_2 \\ R_1 + R_2 + R_3 \end{array}$$

$$\sim \left(\begin{array}{ccc|c} 6 & 0 & -6 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \begin{array}{l} 3R_1 - R_2 \\ -\frac{1}{3}R_2 \\ \end{array}$$

From (2), $y_3 = k$, $z_3 = k$, where $k \in \mathbf{R}$

Sub. $z_3 = k$ into (1), we have $x_3 = k$

$$\begin{pmatrix} x_3 \\ y_3 \\ z_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

(c) $P=\begin{pmatrix}1 & 0 & 1\\ 0 & 1 & 1\\ -1 & -1 & 1\end{pmatrix}$

$\det P=\begin{vmatrix}1 & 0 & 1\\ 0 & 1 & 1\\ -1 & -1 & 1\end{vmatrix}=3\neq 0\Rightarrow P$ is non-singular.

$A_{11}=2, A_{12}=-1, A_{13}=1$
$A_{21}=-1, A_{22}=2, A_{23}=1$
$A_{31}=-1, A_{32}=-1, A_{33}=1$

$$P^{-1}=\frac{1}{\det P}\cdot adj(P)=\frac{1}{3}\begin{pmatrix}2 & -1 & 1\\ -1 & 2 & 1\\ -1 & -1 & 1\end{pmatrix}^t=\frac{1}{3}\begin{pmatrix}2 & -1 & -1\\ -1 & 2 & -1\\ 1 & 1 & 1\end{pmatrix}$$

(d) $$P^{-1}AP=\frac{1}{3}\begin{pmatrix}2 & -1 & -1\\ -1 & 2 & -1\\ 1 & 1 & 1\end{pmatrix}\begin{pmatrix}-1 & 1 & 1\\ 1 & -1 & 1\\ 1 & 1 & -1\end{pmatrix}\begin{pmatrix}1 & 0 & 1\\ 0 & 1 & 1\\ -1 & -1 & 1\end{pmatrix}=\frac{1}{3}\begin{pmatrix}-4 & 2 & 2\\ 2 & -4 & 2\\ 1 & 1 & 1\end{pmatrix}\begin{pmatrix}1 & 0 & 1\\ 0 & 1 & 1\\ -1 & -1 & 1\end{pmatrix}$$

$$=\frac{1}{3}\begin{pmatrix}-6 & 0 & 0\\ 0 & -6 & 0\\ 0 & 0 & 3\end{pmatrix}=\begin{pmatrix}-2 & 0 & 0\\ 0 & -2 & 0\\ 0 & 0 & 1\end{pmatrix}$$

$$(P^{-1}AP)^{10}=\underbrace{(P^{-1}AP)(P^{-1}AP)\cdots(P^{-1}AP)}_{10\ \text{times}}=P^{-1}(APP^{-1}AP\cdots P^{-1}A)P=P^{-1}\Big(\underbrace{AA\cdots A}_{10\ \text{times}}\Big)P$$

$$P^{-1}A^{10}P=\begin{pmatrix}-2 & 0 & 0\\ 0 & -2 & 0\\ 0 & 0 & 1\end{pmatrix}^{10}=\begin{pmatrix}1024 & 0 & 0\\ 0 & 1024 & 0\\ 0 & 0 & 1\end{pmatrix}$$

$$A^{10}=P\begin{pmatrix}1024 & 0 & 0\\ 0 & 1024 & 0\\ 0 & 0 & 1\end{pmatrix}P^{-1}=\begin{pmatrix}1 & 0 & 1\\ 0 & 1 & 1\\ -1 & -1 & 1\end{pmatrix}\begin{pmatrix}1024 & 0 & 0\\ 0 & 1024 & 0\\ 0 & 0 & 1\end{pmatrix}\frac{1}{3}\begin{pmatrix}2 & -1 & -1\\ -1 & 2 & -1\\ 1 & 1 & 1\end{pmatrix}$$

$$=\frac{1}{3}\begin{pmatrix}1024 & 0 & 1\\ 0 & 1024 & 1\\ -1024 & -1024 & 1\end{pmatrix}\begin{pmatrix}2 & -1 & -1\\ -1 & 2 & -1\\ 1 & 1 & 1\end{pmatrix}=\frac{1}{3}\begin{pmatrix}2049 & -1023 & -1023\\ -1023 & 2049 & -1023\\ -1023 & -1023 & 2049\end{pmatrix}$$

$$=\begin{pmatrix}683 & -341 & -341\\ -341 & 683 & -341\\ -341 & -341 & 683\end{pmatrix}$$