Gram-Schmidt process

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1. Let $\mathbf{u} = 2\mathbf{i} + 3\mathbf{j}$, $\mathbf{v} = \mathbf{i} + 2\mathbf{j}$. Find the projection vector $\mathbf{a}$ of $\mathbf{u}$ on $\mathbf{v}$.
   Let $\mathbf{b} = \mathbf{u} - \mathbf{a}$. Show that $\mathbf{b} \perp \mathbf{v}$.

1. $\mathbf{a} = \left(\dfrac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\right)\mathbf{v} = \left(\dfrac{2\times1+3\times2}{1^2+2^2}\right)(\mathbf{i}+2\mathbf{j}) = \dfrac{8}{5}\mathbf{i} + \dfrac{16}{5}\mathbf{j}$

   $\mathbf{b} = 2\mathbf{i} + 3\mathbf{j} - \left(\dfrac{8}{5}\mathbf{i} + \dfrac{16}{5}\mathbf{j}\right) = \dfrac{2}{5}\mathbf{i} - \dfrac{1}{5}\mathbf{j}$

   $\mathbf{b}\cdot\mathbf{v} = \left(\dfrac{2}{5}\mathbf{i} - \dfrac{1}{5}\mathbf{j}\right)\cdot(\mathbf{i} + 2\mathbf{j}) = \dfrac{2}{5} - \dfrac{2}{5} = 0$

   $\therefore \mathbf{b} \perp \mathbf{v}$

2. Let $\mathbf{u} = 2\mathbf{i} + 3\mathbf{j} - \mathbf{k}$, $\mathbf{v} = \mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$. Find the projection vector $\mathbf{a}$ of $\mathbf{u}$ on $\mathbf{v}$.
   Let $\mathbf{b} = \mathbf{u} - \mathbf{a}$. Show that $\mathbf{b} \perp \mathbf{v}$.

2. $\mathbf{a} = \left(\dfrac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\right)\mathbf{v} = \left(\dfrac{2\times1+3\times2-1\times2}{1^2+2^2+2^2}\right)(\mathbf{i}+2\mathbf{j}+2\mathbf{k}) = \dfrac{2}{3}\mathbf{i} + \dfrac{4}{3}\mathbf{j} + \dfrac{4}{3}\mathbf{k}$

   $\mathbf{b} = 2\mathbf{i} + 3\mathbf{j} - \mathbf{k} - \left(\dfrac{2}{3}\mathbf{i} + \dfrac{4}{3}\mathbf{j} + \dfrac{4}{3}\mathbf{k}\right) = \dfrac{4}{3}\mathbf{i} + \dfrac{5}{3}\mathbf{j} - \dfrac{7}{3}\mathbf{k}$

   $\mathbf{b}\cdot\mathbf{v} = \left(\dfrac{4}{3}\mathbf{i} + \dfrac{5}{3}\mathbf{j} - \dfrac{7}{3}\mathbf{k}\right)\cdot(\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}) = \dfrac{4}{3} + \dfrac{10}{3} - \dfrac{14}{3} = 0$

   $\therefore \mathbf{b} \perp \mathbf{v}$

3. Describe the line spanned by the vector $2\mathbf{i} - 3\mathbf{j}$.

3. Let $\mathbf{v} = x\mathbf{i} + y\mathbf{j} = s(2\mathbf{i} - 3\mathbf{j})$, where s is any real number.
   Compare coefficients,
   $x = 2s \cdots\cdots (1)$, $y = -3s \cdots\cdots (2)$
   $(2)\div(1)$: $\dfrac{y}{x} = \dfrac{-3}{2}$
   The line spanned by the vector $2\mathbf{i} - 3\mathbf{j}$ is $2y = -3x$.

4. Describe the line passes through the point (1, –2) and parallel to the vector $2\mathbf{i} - 3\mathbf{j}$.

4. Let $\mathbf{v} = x\mathbf{i} + y\mathbf{j}$ be any point on the line.
   $\mathbf{v} - (\mathbf{i} - 2\mathbf{j}) = s(2\mathbf{i} - 3\mathbf{j})$, where s is any real number.
   Compare coefficients,
   $x - 1 = 2s \cdots\cdots (1)$, $y + 2 = -3s \cdots\cdots (2)$
   $(2)\div(1)$: $\dfrac{y+2}{x-1} = \dfrac{-3}{2}$
   $2y + 4 = -3x + 3$
   The line passes through (1, –2) and parallel to the vector $2\mathbf{i} - 3\mathbf{j}$ is $3x + 2y + 1 = 0$.

5. Describe the line spanned by the vector $2\mathbf{i} - 3\mathbf{j} + \mathbf{k}$.

5. Let $\mathbf{v} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} = t(2\mathbf{i} - 3\mathbf{j} + \mathbf{k})$, where t is any real number.
   Compare coefficients,
   $x = 2t \cdots\cdots (1)$, $y = -3t \cdots\cdots (2)$, $z = t \cdots\cdots (3)$
   $\dfrac{x}{2} = \dfrac{y}{-3} = z$
   The line spanned by the vector $2\mathbf{i} - 3\mathbf{j} + \mathbf{k}$ is $\dfrac{x}{2} = \dfrac{y}{-3} = z$.

6. Describe the line passes through the point (1, –2, 4) and parallel to the vector $2\mathbf{i} - 3\mathbf{j} + \mathbf{k}$.

6. Let $\mathbf{v} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ be any point on the line.
   $\mathbf{v} - (\mathbf{i} - 2\mathbf{j} + 4\mathbf{k}) = t(2\mathbf{i} - 3\mathbf{j} + \mathbf{k})$, where t is any real number.
   Compare coefficients,

$x - 1 = 2t$ ⋯⋯ (1), $y + 2 = -3t$ ⋯⋯ (2), $z - 4 = t$ ⋯⋯ (3)

$$\frac{x-1}{2} = \frac{y+2}{-3} = z - 4$$

The line passes through (1, –2, 4) and parallel to 2**i** – 3**j** + **k** is $\frac{x-1}{2} = \frac{y+2}{-3} = z - 4$.

7. Describe the plane spanned by the vectors **i** + **j**, –**i** + **k**.

7. Let **v** = x**i** + y**j** + z**k** = s(**i** + **j**) + t(–**i** + **k**), where s and t are any real number.
Compare coefficients,
$x = s - t$ ⋯⋯ (1), $y = s$ ⋯⋯ (2), $z = t$ ⋯⋯ (3)
Sub. (2), (3) into (1): $x = y - z$
$x - y + z = 0$
The plane spanned by the vector **i** + **j**, –**i** + **k** is $x - y + z = 0$.

8. Describe the plane passes through the point (2, –1, 3) and parallel to the plane spanned by the vectors **i** + **j** + **k**, –**i** + 2**j** + **k**.

8. Let **v** = x**i** + y**j** + z**k**, then **v** – (2**i** – **j** + 3**k**) = s(**i** + **j** + **k**) + t(–**i** + 2**j** + **k**), where $s, t \in \mathbf{R}$.
Compare coefficients,
$x - 2 = s - t$ ⋯⋯ (1), $y + 1 = s + 2t$ ⋯⋯ (2), $z - 3 = s + t$ ⋯⋯ (3)
(1) + (3): $x + z - 5 = 2s$ ⋯⋯ (4)
(3) – (1): $z - x - 1 = 2t$ ⋯⋯ (5)
Sub. (4) and (5) into (2): $2y + 2 = x + z - 5 + 2z - 2x - 2$
$x + 2y - 3z + 9 = 0$
The plane passes through (2, –1, 3) and parallel to the plane spanned by the vector **i** + **j** + **k**, –**i** + 2**j** + **k** is $x + 2y - 3z + 9 = 0$.

9. Consider the plane spanned by the vectors **i** + **j** + **k**, –**i** + 2**j** + **k**. Find two mutually perpendicular unit vectors $\mathbf{e}_1$ and $\mathbf{e}_2$ on the plane such that $\mathbf{e}_1$ // (**i** + **j** + **k**).

9. $$\mathbf{e}_1 = \frac{\mathbf{i}+\mathbf{j}+\mathbf{k}}{\sqrt{1^2+1^2+1^2}} = \frac{\mathbf{i}}{\sqrt{3}} + \frac{\mathbf{j}}{\sqrt{3}} + \frac{\mathbf{k}}{\sqrt{3}}$$

$$\mathbf{a} = \left[\left(\frac{\mathbf{i}}{\sqrt{3}} + \frac{\mathbf{j}}{\sqrt{3}} + \frac{\mathbf{k}}{\sqrt{3}}\right)\cdot(-\mathbf{i}+2\mathbf{j}+\mathbf{k})\right]\left(\frac{\mathbf{i}}{\sqrt{3}} + \frac{\mathbf{j}}{\sqrt{3}} + \frac{\mathbf{k}}{\sqrt{3}}\right) = \frac{2}{3}\mathbf{i} + \frac{2}{3}\mathbf{j} + \frac{2}{3}\mathbf{k}$$

$$\mathbf{b} = (-\mathbf{i} + 2\mathbf{j} + \mathbf{k}) - \left(\frac{2}{3}\mathbf{i} + \frac{2}{3}\mathbf{j} + \frac{2}{3}\mathbf{k}\right) = -\frac{5}{3}\mathbf{i} + \frac{4}{3}\mathbf{j} + \frac{1}{3}\mathbf{k}$$

$$\mathbf{e}_2 = \frac{-5\mathbf{i}+4\mathbf{j}+\mathbf{k}}{\sqrt{(-5)^2+4^2+1^2}} = \frac{-5\mathbf{i}}{\sqrt{42}} + \frac{4\mathbf{j}}{\sqrt{42}} + \frac{\mathbf{k}}{\sqrt{42}}$$

$$\mathbf{e}_1\cdot\mathbf{e}_2 = \frac{1}{\sqrt{126}}(-5+4+1) = 0$$

Then $|\mathbf{e}_1| = 1$, $|\mathbf{e}_2| = 1$ and $\mathbf{e}_1 \perp \mathbf{e}_2$

10. Find the projection vector **a** of **u** = 2**i** – **j** + 3**k** on the plane spanned by the vectors **i** + **j** + **k**, –**i** + 2**j** + **k**. Find also the normal vector **b** such that **a** + **b** = **u**. Verify that **a** ⊥ **b**.

10. From Q9, $\mathbf{e}_1 = \frac{\mathbf{i}}{\sqrt{3}} + \frac{\mathbf{j}}{\sqrt{3}} + \frac{\mathbf{k}}{\sqrt{3}}$, $\mathbf{e}_2 = \frac{-5\mathbf{i}}{\sqrt{42}} + \frac{4\mathbf{j}}{\sqrt{42}} + \frac{\mathbf{k}}{\sqrt{42}}$ are the base vectors of the plane.
The projection vector $\mathbf{a} = (\mathbf{u}\cdot\mathbf{e}_1)\mathbf{e}_1 + (\mathbf{u}\cdot\mathbf{e}_2)\mathbf{e}_2$

$$=(2\mathbf{i}-\mathbf{j}+3\mathbf{k})\cdot\left(\frac{\mathbf{i}}{\sqrt{3}} + \frac{\mathbf{j}}{\sqrt{3}} + \frac{\mathbf{k}}{\sqrt{3}}\right)\left(\frac{\mathbf{i}}{\sqrt{3}} + \frac{\mathbf{j}}{\sqrt{3}} + \frac{\mathbf{k}}{\sqrt{3}}\right) + (2\mathbf{i}-\mathbf{j}+3\mathbf{k})\cdot\left(\frac{-5\mathbf{i}}{\sqrt{42}} + \frac{4\mathbf{j}}{\sqrt{42}} + \frac{\mathbf{k}}{\sqrt{42}}\right)\left(\frac{-5\mathbf{i}}{\sqrt{42}} + \frac{4\mathbf{j}}{\sqrt{42}} + \frac{\mathbf{k}}{\sqrt{42}}\right)$$

$$= \frac{1}{3}(2-1+3)(\mathbf{i}+\mathbf{j}+\mathbf{k}) + \frac{1}{42}(-10-4+3)(-5\mathbf{i}+4\mathbf{j}+\mathbf{k})$$

$=\frac{4}{3}(\mathbf{i}+\mathbf{j}+\mathbf{k})-\frac{11}{42}(-5\mathbf{i}+4\mathbf{j}+\mathbf{k})=\frac{56}{42}(\mathbf{i}+\mathbf{j}+\mathbf{k})-\frac{11}{42}(-5\mathbf{i}+4\mathbf{j}+\mathbf{k})=\frac{37}{14}\mathbf{i}+\frac{4}{14}\mathbf{j}+\frac{15}{14}\mathbf{k}$

$\mathbf{b}=\mathbf{u}-\mathbf{a}=(2\mathbf{i}-\mathbf{j}+3\mathbf{k})-\left(\frac{37}{14}\mathbf{i}+\frac{4}{14}\mathbf{j}+\frac{15}{14}\mathbf{k}\right)=\frac{-9}{14}\mathbf{i}-\frac{18}{14}\mathbf{j}+\frac{27}{14}\mathbf{k}$

$\mathbf{a}\cdot\mathbf{b}=\left(\frac{37}{14}\mathbf{i}+\frac{4}{14}\mathbf{j}+\frac{15}{14}\mathbf{k}\right)\cdot\left(\frac{-9}{14}\mathbf{i}-\frac{18}{14}\mathbf{j}+\frac{27}{14}\mathbf{k}\right)=\frac{1}{14^2}(-37\times 9-4\times 18+15\times 27)=0$

$\therefore$ $\mathbf{a}\perp\mathbf{b}$

11. Let $\mathbf{u}=\mathbf{j}$, $\mathbf{v}=\frac{1}{\sqrt{2}}\mathbf{i}+\frac{1}{\sqrt{2}}\mathbf{k}$, $\mathbf{w}=\frac{1}{\sqrt{2}}\mathbf{i}-\frac{1}{\sqrt{2}}\mathbf{k}$. Show that {**u**, **v**, **w**} form an orthonormal set of vectors in $\mathbf{R}^3$. i.e. $|\mathbf{u}|=1$, $|\mathbf{v}|=1$, $|\mathbf{w}|=1$ and $\mathbf{u}\cdot\mathbf{v}=\mathbf{v}\cdot\mathbf{w}=\mathbf{w}\cdot\mathbf{u}=1$.

$|\mathbf{u}|=1$, $|\mathbf{v}|=\sqrt{\left(\frac{1}{\sqrt{2}}\right)^2+\left(\frac{1}{\sqrt{2}}\right)^2}=1$, $|\mathbf{w}|=\sqrt{\left(\frac{1}{\sqrt{2}}\right)^2+\left(-\frac{1}{\sqrt{2}}\right)^2}=1$

$\mathbf{u}\cdot\mathbf{v}=\mathbf{j}\cdot\left(\frac{1}{\sqrt{2}}\mathbf{i}+\frac{1}{\sqrt{2}}\mathbf{k}\right)=0$; $\mathbf{v}\cdot\mathbf{w}=\left(\frac{1}{\sqrt{2}}\mathbf{i}+\frac{1}{\sqrt{2}}\mathbf{k}\right)\cdot\left(\frac{1}{\sqrt{2}}\mathbf{i}-\frac{1}{\sqrt{2}}\mathbf{k}\right)=\frac{1}{2}-\frac{1}{2}=0$; $\mathbf{w}\cdot\mathbf{u}=\left(\frac{1}{\sqrt{2}}\mathbf{i}-\frac{1}{\sqrt{2}}\mathbf{k}\right)\cdot\mathbf{j}=0$

$\therefore$ {**u**, **v**, **w**} form an orthonormal set of vectors in $\mathbf{R}^3$.

12. Let {**u**, **v**, **w**} form an orthonormal set of vectors in $\mathbf{R}^3$. For any vector **p** in $\mathbf{R}^3$, show that $\mathbf{p}=(\mathbf{u}\cdot\mathbf{p})\mathbf{u}+(\mathbf{v}\cdot\mathbf{p})\mathbf{v}+(\mathbf{w}\cdot\mathbf{p})\mathbf{w}$.

12. Let $\mathbf{p}=a\mathbf{u}+b\mathbf{v}+c\mathbf{w}$, where a, b and c are any real numbers.

$\mathbf{u}\cdot\mathbf{p}=\mathbf{u}\cdot(a\mathbf{u}+b\mathbf{v}+c\mathbf{w})=a\mathbf{u}\cdot\mathbf{u}+b\mathbf{u}\cdot\mathbf{v}+c\mathbf{u}\cdot\mathbf{w}=a(1)+0+0=a$

$\mathbf{v}\cdot\mathbf{p}=\mathbf{v}\cdot(a\mathbf{u}+b\mathbf{v}+c\mathbf{w})=a\mathbf{v}\cdot\mathbf{u}+b\mathbf{v}\cdot\mathbf{v}+c\mathbf{v}\cdot\mathbf{w}=0+b(1)+0=b$

$\mathbf{w}\cdot\mathbf{p}=\mathbf{w}\cdot(a\mathbf{u}+b\mathbf{v}+c\mathbf{w})=a\mathbf{w}\cdot\mathbf{u}+b\mathbf{w}\cdot\mathbf{v}+c\mathbf{w}\cdot\mathbf{w}=0+0+c(1)=c$

$\therefore$ $\mathbf{p}=(\mathbf{u}\cdot\mathbf{p})\mathbf{u}+(\mathbf{v}\cdot\mathbf{p})\mathbf{v}+(\mathbf{w}\cdot\mathbf{p})\mathbf{w}$

13. Let $\mathbf{u}=\mathbf{j}$, $\mathbf{v}=-\frac{4}{5}\mathbf{i}+\frac{3}{5}\mathbf{k}$, $\mathbf{w}=\frac{3}{5}\mathbf{i}+\frac{4}{5}\mathbf{k}$.

(a) Show that {**u**, **v**, **w**} form an orthonormal set of vectors in $\mathbf{R}^3$.

(b) Express $\mathbf{p}=\mathbf{i}+\mathbf{j}+\mathbf{k}$ in terms of **u**, **v** and **w**.

13. (a) $|\mathbf{u}|=1$, $|\mathbf{v}|=\sqrt{\left(-\frac{4}{5}\right)^2+\left(\frac{3}{5}\right)^2}=1$, $|\mathbf{w}|=\sqrt{\left(\frac{3}{5}\right)^2+\left(\frac{4}{5}\right)^2}=1$

$\mathbf{u}\cdot\mathbf{v}=\mathbf{j}\cdot\left(-\frac{4}{5}\mathbf{i}+\frac{3}{5}\mathbf{k}\right)=0$; $\mathbf{v}\cdot\mathbf{w}=\left(-\frac{4}{5}\mathbf{i}+\frac{3}{5}\mathbf{k}\right)\cdot\left(\frac{3}{5}\mathbf{i}+\frac{4}{5}\mathbf{k}\right)=0$; $\mathbf{w}\cdot\mathbf{u}=\left(\frac{3}{5}\mathbf{i}+\frac{4}{5}\mathbf{k}\right)\cdot\mathbf{j}=0$

$\therefore$ {**u**, **v**, **w**} form an orthonormal set of vectors in $\mathbf{R}^3$.

(b) $\mathbf{p}=\mathbf{i}+\mathbf{j}+\mathbf{k}=(\mathbf{u}\cdot\mathbf{p})\mathbf{u}+(\mathbf{v}\cdot\mathbf{p})\mathbf{v}+(\mathbf{w}\cdot\mathbf{p})\mathbf{w}$

$=\mathbf{j}\cdot(\mathbf{i}+\mathbf{j}+\mathbf{k})\mathbf{j}+\left(-\frac{4}{5}\mathbf{i}+\frac{3}{5}\mathbf{k}\right)\cdot(\mathbf{i}+\mathbf{j}+\mathbf{k})\left(-\frac{4}{5}\mathbf{i}+\frac{3}{5}\mathbf{k}\right)+\left(\frac{3}{5}\mathbf{i}+\frac{4}{5}\mathbf{k}\right)\cdot(\mathbf{i}+\mathbf{j}+\mathbf{k})\left(\frac{3}{5}\mathbf{i}+\frac{4}{5}\mathbf{k}\right)$

$=\mathbf{j}+\left(-\frac{4}{5}+\frac{3}{5}\right)\left(-\frac{4}{5}\mathbf{i}+\frac{3}{5}\mathbf{k}\right)+\left(\frac{3}{5}+\frac{4}{5}\right)\left(\frac{3}{5}\mathbf{i}+\frac{4}{5}\mathbf{k}\right)$

$=\mathbf{u}-\frac{1}{5}\mathbf{v}+\frac{7}{5}\mathbf{w}$

14. Find the projection vector **a** of $\mathbf{p}=\mathbf{i}+\mathbf{j}+\mathbf{k}$ on the plane spanned by the vectors $\mathbf{v}=\mathbf{j}$, $\mathbf{w}=-\frac{4}{5}\mathbf{i}+\frac{3}{5}\mathbf{k}$. Find also the normal vector **b** such that $\mathbf{a}+\mathbf{b}=\mathbf{p}$. Verify that $\mathbf{a}\perp\mathbf{b}$.

Note that $|\mathbf{v}|=1$, $|\mathbf{w}|=\sqrt{\left(-\frac{4}{5}\right)^2+\left(\frac{3}{5}\right)^2}=1$ and $\mathbf{u}\cdot\mathbf{v}=0+0+0=0$

$\therefore$ {**v**, **w**} form an orthonormal set of vectors in $\mathbf{R}^3$.

The projection vector $\mathbf{a} = (\mathbf{v}\cdot\mathbf{p})\mathbf{v} + (\mathbf{w}\cdot\mathbf{p})\mathbf{w} = \mathbf{j}\cdot(\mathbf{i}+\mathbf{j}+\mathbf{k})\mathbf{j} + \left(-\frac{4}{5}\mathbf{i}+\frac{3}{5}\mathbf{k}\right)\cdot(\mathbf{i}+\mathbf{j}+\mathbf{k})\ \left(-\frac{4}{5}\mathbf{i}+\frac{3}{5}\mathbf{k}\right)$

$$= \mathbf{j} - \frac{1}{5}\left(-\frac{4}{5}\mathbf{i}+\frac{3}{5}\mathbf{k}\right) = \frac{4}{25}\mathbf{i}+\mathbf{j}-\frac{3}{25}\mathbf{k}$$

$\mathbf{b} = \mathbf{p} - \mathbf{a} = \mathbf{i}+\mathbf{j}+\mathbf{k} - \left(\frac{4}{25}\mathbf{i}+\mathbf{j}-\frac{3}{25}\mathbf{k}\right) = \frac{21}{25}\mathbf{i}+\frac{28}{25}\mathbf{k}$

$\mathbf{a}\cdot\mathbf{b} = \left(\frac{4}{25}\mathbf{i}+\mathbf{j}-\frac{3}{25}\mathbf{k}\right)\cdot\left(\frac{21}{25}\mathbf{i}+\frac{28}{25}\mathbf{k}\right) = \frac{1}{25^2}(4\times 21+0-3\times 28) = 0$

$\therefore\ \mathbf{a}\perp\mathbf{b}$.

15. Let $\mathbf{u} = \mathbf{i}+\mathbf{j}+\mathbf{k}$, $\mathbf{v} = \mathbf{j}+\mathbf{k}$, $\mathbf{w} = \mathbf{k}$. Apply Gram-Schmidt Process to transform $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ into an orthonormal basis.

$\mathbf{e}_1 = \frac{\mathbf{u}}{|\mathbf{u}|} = \frac{\mathbf{i}+\mathbf{j}+\mathbf{k}}{\sqrt{1^2+1^2+1^2}} = \frac{1}{\sqrt{3}}\mathbf{i}+\frac{1}{\sqrt{3}}\mathbf{j}+\frac{1}{\sqrt{3}}\mathbf{k}$

$\mathbf{a} = \mathbf{v} - (\mathbf{e}_1\cdot\mathbf{v})\mathbf{e}_1 = \mathbf{j}+\mathbf{k} - \left(\frac{1}{\sqrt{3}}\mathbf{i}+\frac{1}{\sqrt{3}}\mathbf{j}+\frac{1}{\sqrt{3}}\mathbf{k}\right)\cdot(\mathbf{j}+\mathbf{k})\left(\frac{1}{\sqrt{3}}\mathbf{i}+\frac{1}{\sqrt{3}}\mathbf{j}+\frac{1}{\sqrt{3}}\mathbf{k}\right)$

$= \mathbf{j}+\mathbf{k} - \frac{1}{3}(1+1)(\mathbf{i}+\mathbf{j}+\mathbf{k}) = -\frac{2}{3}\mathbf{i}+\frac{1}{3}\mathbf{j}+\frac{1}{3}\mathbf{k}$

$\mathbf{e}_2 = \frac{\mathbf{a}}{|\mathbf{a}|} = \frac{-2\mathbf{i}+\mathbf{j}+\mathbf{k}}{\sqrt{2^2+1^2+1^2}} = -\frac{2}{\sqrt{6}}\mathbf{i}+\frac{1}{\sqrt{6}}\mathbf{j}+\frac{1}{\sqrt{6}}\mathbf{k}$

$\mathbf{b} = \mathbf{w} - (\mathbf{e}_1\cdot\mathbf{w})\mathbf{e}_1 - (\mathbf{e}_2\cdot\mathbf{w})\mathbf{e}_2$

$= \mathbf{k} - \left(\frac{1}{\sqrt{3}}\mathbf{i}+\frac{1}{\sqrt{3}}\mathbf{j}+\frac{1}{\sqrt{3}}\mathbf{k}\right)\cdot\mathbf{k}\left(\frac{1}{\sqrt{3}}\mathbf{i}+\frac{1}{\sqrt{3}}\mathbf{j}+\frac{1}{\sqrt{3}}\mathbf{k}\right) - \left(-\frac{2}{\sqrt{6}}\mathbf{i}+\frac{1}{\sqrt{6}}\mathbf{j}+\frac{1}{\sqrt{6}}\mathbf{k}\right)\cdot\mathbf{k}\left(-\frac{2}{\sqrt{6}}\mathbf{i}+\frac{1}{\sqrt{6}}\mathbf{j}+\frac{1}{\sqrt{6}}\mathbf{k}\right)$

$= \mathbf{k} - \frac{1}{3}(\mathbf{i}+\mathbf{j}+\mathbf{k}) - \frac{1}{6}(-2\mathbf{i}+\mathbf{j}+\mathbf{k}) = -\frac{1}{2}\mathbf{j}+\frac{1}{2}\mathbf{k}$

$\mathbf{e}_3 = \frac{\mathbf{b}}{|\mathbf{b}|} = \frac{-\mathbf{j}+\mathbf{k}}{\sqrt{(-1)^2+1^2}} = -\frac{1}{\sqrt{2}}\mathbf{j}+\frac{1}{\sqrt{2}}\mathbf{k}$

Then $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ will form an orthonormal basis of vectors in $\mathbf{R}^3$.

16. Determine whether the following points are collinear. If they are not collinear, determine the shortest distance from C to the line determined by AB. Find the point D on the line AB which is nearest to C.

(a) $A(1, -2, 4)$, $B(5, -8, 6)$, $C(-1, 1, 3)$.

(b) $A(1, -2, 4)$, $B(5, -8, 6)$, $C(0, 1, 3)$.

(a) $\overrightarrow{AB} = 5\mathbf{i} - 8\mathbf{j} + 6\mathbf{k} - (\mathbf{i} - 2\mathbf{j} + 4\mathbf{k}) = 4\mathbf{i} - 6\mathbf{j} + 2\mathbf{k}$

$\overrightarrow{AC} = -\mathbf{i} + \mathbf{j} + 3\mathbf{k} - (\mathbf{i} - 2\mathbf{j} + 4\mathbf{k}) = -2\mathbf{i} + 3\mathbf{j} - \mathbf{k}$

$\overrightarrow{AB} = -2\overrightarrow{AC}$

$\therefore A, B, C$ are collinear.

(b) $\overrightarrow{AB} = 5\mathbf{i} - 8\mathbf{j} + 6\mathbf{k} - (\mathbf{i} - 2\mathbf{j} + 4\mathbf{k}) = 4\mathbf{i} - 6\mathbf{j} + 2\mathbf{k}$

$\overrightarrow{AC} = \mathbf{j} + 3\mathbf{k} - (\mathbf{i} - 2\mathbf{j} + 4\mathbf{k}) = -\mathbf{i} + 3\mathbf{j} - \mathbf{k}$

If $\overrightarrow{AB} = m\overrightarrow{AC}$, then $4 = -m$ …… (1), $-6 = 3m$ …… (2), $2 = -m$ …… (3)

(1) contradicts with (3)

$\therefore A, B, C$ are not collinear.

The projection vector of $\overrightarrow{AC}$ on $\overrightarrow{AB}$ is $\left(\dfrac{\overrightarrow{AC}\cdot\overrightarrow{AB}}{\overrightarrow{AB}\cdot\overrightarrow{AB}}\right)\overrightarrow{AB}$.

The normal vector is $\overrightarrow{AC} - \left(\dfrac{\overrightarrow{AC}\cdot\overrightarrow{AB}}{\overrightarrow{AB}\cdot\overrightarrow{AB}}\right)\overrightarrow{AB}$.

$= (-\mathbf{i} + 3\mathbf{j} - \mathbf{k}) - \dfrac{(-\mathbf{i}+3\mathbf{j}-\mathbf{k})\cdot(4\mathbf{i}-6\mathbf{j}+2\mathbf{k})}{(4\mathbf{i}-6\mathbf{j}+2\mathbf{k})\cdot(4\mathbf{i}-6\mathbf{j}+2\mathbf{k})}(4\mathbf{i} - 6\mathbf{j} + 2\mathbf{k})$

$= (-\mathbf{i} + 3\mathbf{j} - \mathbf{k}) - \dfrac{-4-18-2}{4^2+(-6)^2+2^2}(4\mathbf{i} - 6\mathbf{j} + 2\mathbf{k})$

$= (-\mathbf{i} + 3\mathbf{j} - \mathbf{k}) + \dfrac{3}{7}(4\mathbf{i} - 6\mathbf{j} + 2\mathbf{k})$

$= \dfrac{5}{7}\mathbf{i} + \dfrac{3}{7}\mathbf{j} - \dfrac{1}{7}\mathbf{k}$

The shortest distance $= \sqrt{\left(\dfrac{5}{7}\right)^2 + \left(\dfrac{3}{7}\right)^2 + \left(-\dfrac{1}{7}\right)^2} = \dfrac{1}{7}\sqrt{35}$

$\overrightarrow{AD} = \overrightarrow{OD} - \overrightarrow{OA} = \left(\dfrac{\overrightarrow{AC}\cdot\overrightarrow{AB}}{\overrightarrow{AB}\cdot\overrightarrow{AB}}\right)\overrightarrow{AB} = -\dfrac{3}{7}(4\mathbf{i} - 6\mathbf{j} + 2\mathbf{k})$

$\overrightarrow{OD} = \mathbf{i} - 2\mathbf{j} + 4\mathbf{k} - \dfrac{3}{7}(4\mathbf{i} - 6\mathbf{j} + 2\mathbf{k}) = -\dfrac{5}{7}\mathbf{i} + \dfrac{4}{7}\mathbf{j} + \dfrac{22}{7}\mathbf{k}$

17. Determine whether the following points are coplanar. If they are not coplanar, determine the shortest distance from D to the plane determined by ABC. Find the coordinates of the projection E of D on the plane ABC.

(a) $A(2, -1, 3)$, $B(-9, 0, 0)$, $C(0, 0, 3)$, $D(0, -4.5, 0)$.

(b) $A(2, -1, 3)$, $B(-9, 0, 0)$, $C(0, 0, 3)$, $D(0, -5, 0)$.

(a) $\overrightarrow{AB} = (-9, 0, 0) - (2, -1, 3) = (-11, 1, -3)$

$\overrightarrow{AC} = (0, 0, 3) - (2, -1, 3) = (-2, 1, 0)$

$\overrightarrow{AD} = (0, -4.5, 0) - (2, -1, 3) = (-2, -3.5, -3)$

The volume of parallelepiped formed by $\overrightarrow{AB}$, $\overrightarrow{AC}$ and $\overrightarrow{AD}$:

$= \overrightarrow{AD} \cdot (\overrightarrow{AB} \times \overrightarrow{AC})$

$= \begin{vmatrix} -2 & -3.5 & -3 \\ -11 & 1 & -3 \\ -2 & 1 & 0 \end{vmatrix} = \begin{vmatrix} -9 & -3.5 & -3 \\ -9 & 1 & -3 \\ 0 & 1 & 0 \end{vmatrix}$ $(C_1 + 2C_2 \to C_1)$

$= 0$

$\therefore A, B, C, D$ are coplanar.

(b) $\overrightarrow{AB} = (-11, 1, -3)$, $\overrightarrow{AC} = (-2, 1, 0)$, $\overrightarrow{AD} = (0, -5, 0) - (2, -1, 3) = (-2, -4, -3)$

The volume of parallelepiped formed by $\overrightarrow{AB}$, $\overrightarrow{AC}$ and $\overrightarrow{AD}$:

$= \overrightarrow{AD} \cdot (\overrightarrow{AB} \times \overrightarrow{AC})$

$= \left\|\begin{matrix} -2 & -4 & -3 \\ -11 & 1 & -3 \\ -2 & 1 & 0 \end{matrix}\right\| = \left\|\begin{matrix} -10 & -4 & -3 \\ -9 & 1 & -3 \\ 0 & 1 & 0 \end{matrix}\right\|$ $(C_1 + 2C_2 \to C_1)$

$= |-(30 - 27)| = 3 \neq 0$

$\therefore A, B, C, D$ are not coplanar.

Base area of the parallelepiped $= \left\|\begin{matrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -11 & 1 & -3 \\ -2 & 1 & 0 \end{matrix}\right\| = |3\mathbf{i} + 6\mathbf{j} - 9\mathbf{k}| = \sqrt{3^2 + 6^2 + 9^2} = 3\sqrt{14}$

Let the height of the parallelepiped be h.

$3\sqrt{14}\, h = 3$

$h = \dfrac{1}{\sqrt{14}}$

$\overrightarrow{AE} = \overrightarrow{OE} - \overrightarrow{OA} = \overrightarrow{AD} + \dfrac{1}{\sqrt{14}} \dfrac{\overrightarrow{AB} \times \overrightarrow{AC}}{\left|\overrightarrow{AB} \times \overrightarrow{AC}\right|}$, where E is the projection of D on ABC.

$\overrightarrow{OE} = (2, -1, 3) + (-2, -4, -3) + \dfrac{1}{\sqrt{14}} \dfrac{(3\mathbf{i} + 6\mathbf{j} - 9\mathbf{k})}{|3\mathbf{i} + 6\mathbf{j} - 9\mathbf{k}|}$

$= (0, -5, 0) + \dfrac{1}{\sqrt{14}} \dfrac{(3\mathbf{i} + 6\mathbf{j} - 9\mathbf{k})}{\sqrt{126}}$

$= -5\mathbf{j} + \dfrac{1}{14}(\mathbf{i} + 2\mathbf{j} - 3\mathbf{k})$

$= \dfrac{1}{14}\mathbf{i} - \dfrac{34}{7}\mathbf{j} - \dfrac{3}{14}\mathbf{k}$