Property of Matrix Multiplication

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Advanced Level Pure Mathematics Algebra P.254 Theorem 8-4 by K.S. Ng, Y.K. Kwok.

Let k be a scalar and $A=\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22}\end{pmatrix}$, $B=\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22}\end{pmatrix}$, $C=\begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22}\end{pmatrix}$. Then

(1) $(AB)C = A(BC)$

Proof: $(AB)C=\begin{pmatrix} a_{11}b_{11}+a_{12}b_{21} & a_{11}b_{12}+a_{12}b_{22} \\ a_{21}b_{11}+a_{22}b_{21} & a_{21}b_{12}+a_{22}b_{22}\end{pmatrix}\begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22}\end{pmatrix}$

$$=\begin{pmatrix} a_{11}b_{11}c_{11}+a_{12}b_{21}c_{11}+a_{11}b_{12}c_{21}+a_{12}b_{22}c_{21} & a_{11}b_{11}c_{12}+a_{12}b_{21}c_{12}+a_{11}b_{12}c_{22}+a_{12}b_{22}c_{22} \\ a_{21}b_{11}c_{11}+a_{22}b_{21}c_{11}+a_{21}b_{12}c_{21}+a_{22}b_{22}c_{21} & a_{21}b_{11}c_{12}+a_{22}b_{21}c_{12}+a_{21}b_{12}c_{22}+a_{22}b_{22}c_{22}\end{pmatrix}$$

$$A(BC)=\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22}\end{pmatrix}\begin{pmatrix} b_{11}c_{11}+b_{12}c_{21} & b_{11}c_{12}+b_{12}c_{22} \\ b_{21}c_{11}+b_{22}c_{21} & b_{21}c_{12}+b_{22}c_{22}\end{pmatrix}$$

$$=\begin{pmatrix} a_{11}b_{11}c_{11}+a_{11}b_{12}c_{21}+a_{12}b_{21}c_{11}+a_{12}b_{22}c_{21} & a_{11}b_{11}c_{12}+a_{11}b_{12}c_{22}+a_{12}b_{21}c_{12}+a_{12}b_{22}c_{22} \\ a_{21}b_{11}c_{11}+a_{21}b_{12}c_{21}+a_{22}b_{21}c_{11}+a_{22}b_{22}c_{21} & a_{21}b_{11}c_{12}+a_{21}b_{12}c_{22}+a_{22}b_{21}c_{12}+a_{22}b_{22}c_{22}\end{pmatrix}$$

$\therefore\ (AB)C = A(BC)$

(2) $A(B + C) = AB + AC$

Proof: $A(B+C)=\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22}\end{pmatrix}\begin{pmatrix} b_{11}+c_{11} & b_{12}+c_{12} \\ b_{21}+c_{21} & b_{22}+c_{22}\end{pmatrix}$

$$=\begin{pmatrix} a_{11}b_{11}+a_{11}c_{11}+a_{12}b_{21}+a_{12}c_{21} & a_{11}b_{12}+a_{11}c_{12}+a_{12}b_{22}+a_{12}c_{22} \\ a_{21}b_{11}+a_{21}c_{11}+a_{22}b_{21}+a_{22}c_{21} & a_{21}b_{12}+a_{21}c_{12}+a_{22}b_{22}+a_{22}c_{22}\end{pmatrix}$$

$$AB+AC=\begin{pmatrix} a_{11}b_{11}+a_{12}b_{21} & a_{11}b_{12}+a_{12}b_{22} \\ a_{21}b_{11}+a_{22}b_{21} & a_{21}b_{12}+a_{22}b_{22}\end{pmatrix}+\begin{pmatrix} a_{11}c_{11}+a_{12}c_{21} & a_{11}c_{12}+a_{12}c_{22} \\ a_{21}c_{11}+a_{22}c_{21} & a_{21}c_{12}+a_{22}c_{22}\end{pmatrix}$$

$\therefore\ A(B + C) = AB + AC$

(3) $(A + B)C = AC + BC$

Proof: $(A+B)C=\begin{pmatrix} a_{11}+b_{11} & a_{12}+b_{12} \\ a_{21}+b_{21} & a_{22}+b_{22}\end{pmatrix}\begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22}\end{pmatrix}$

$$=\begin{pmatrix} a_{11}c_{11}+b_{11}c_{11}+a_{12}c_{21}+b_{12}c_{21} & a_{11}c_{12}+b_{11}c_{12}+a_{12}c_{22}+b_{12}c_{22} \\ a_{21}c_{11}+b_{21}c_{11}+a_{22}c_{21}+b_{22}c_{21} & a_{21}c_{12}+b_{21}c_{12}+a_{22}c_{22}+b_{22}c_{22}\end{pmatrix}$$

$$AC+BC=\begin{pmatrix} a_{11}c_{11}+a_{12}c_{21} & a_{11}c_{12}+a_{12}c_{22} \\ a_{21}c_{11}+a_{22}c_{21} & a_{21}c_{12}+a_{22}c_{22}\end{pmatrix}+\begin{pmatrix} b_{11}c_{11}+b_{12}c_{21} & b_{11}c_{12}+b_{12}c_{22} \\ b_{21}c_{11}+b_{22}c_{21} & b_{21}c_{12}+b_{22}c_{22}\end{pmatrix}$$

$\therefore\ (A + B)C = AC + BC$

(4) $A\underline{\mathbf{0}} = 0A = \underline{\mathbf{0}}$

Proof: The proof is easy, so is omitted.

(5) $k(AB) = (kA)B = A(kB)$

Proof: $k(AB)=k\begin{pmatrix} a_{11}b_{11}+a_{12}b_{21} & a_{11}b_{12}+a_{12}b_{22} \\ a_{21}b_{11}+a_{22}b_{21} & a_{21}b_{12}+a_{22}b_{22}\end{pmatrix}=\begin{pmatrix} ka_{11}b_{11}+ka_{12}b_{21} & ka_{11}b_{12}+ka_{12}b_{22} \\ ka_{21}b_{11}+ka_{22}b_{21} & ka_{21}b_{12}+ka_{22}b_{22}\end{pmatrix}$

$$(kA)B=\begin{pmatrix} ka_{11} & ka_{12} \\ ka_{21} & ka_{22}\end{pmatrix}\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22}\end{pmatrix}=\begin{pmatrix} ka_{11}b_{11}+ka_{12}b_{21} & ka_{11}b_{12}+ka_{12}b_{22} \\ ka_{21}b_{11}+ka_{22}b_{21} & ka_{21}b_{12}+ka_{22}b_{22}\end{pmatrix}$$

$$A(kB)=\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22}\end{pmatrix}\begin{pmatrix} kb_{11} & kb_{12} \\ kb_{21} & kb_{22}\end{pmatrix}=\begin{pmatrix} ka_{11}b_{11}+ka_{12}b_{21} & ka_{11}b_{12}+ka_{12}b_{22} \\ ka_{21}b_{11}+ka_{22}b_{21} & ka_{21}b_{12}+ka_{22}b_{22}\end{pmatrix}$$

$\therefore\ k(AB) = (kA)B = A(kB)$

Let $A = [a_{ij}]_{m\times n}$, then $A^t = [a_{ji}]_{n\times m}$, called the transpose of A.

(6) $(AB)^t = B^tA^t$

Proof: $(AB)^t = \begin{pmatrix} a_{11}b_{11}+a_{12}b_{21} & a_{11}b_{12}+a_{12}b_{22} \\ a_{21}b_{11}+a_{22}b_{21} & a_{21}b_{12}+a_{22}b_{22} \end{pmatrix}^t$

$B^tA^t = \begin{pmatrix} b_{11} & b_{21} \\ b_{12} & b_{22} \end{pmatrix}\begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{pmatrix} = \begin{pmatrix} a_{11}b_{11}+a_{12}b_{21} & a_{11}b_{12}+a_{12}b_{22} \\ a_{21}b_{11}+a_{22}b_{21} & a_{21}b_{12}+a_{22}b_{22} \end{pmatrix}^t$

$\therefore (AB)^t = B^tA^t$

If $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$, $B = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}$, then we can also prove that $(AB)^t = B^tA^t$

$$(AB)^t = \begin{pmatrix} a_{11}b_{11}+a_{12}b_{21}+a_{13}b_{31} & a_{11}b_{12}+a_{12}b_{22}+a_{13}b_{32} & a_{11}b_{13}+a_{12}b_{23}+a_{13}b_{33} \\ a_{21}b_{11}+a_{22}b_{21}+a_{23}b_{31} & a_{21}b_{12}+a_{22}b_{22}+a_{23}b_{32} & a_{21}b_{13}+a_{22}b_{23}+a_{23}b_{33} \\ a_{31}b_{11}+a_{32}b_{21}+a_{33}b_{31} & a_{31}b_{12}+a_{32}b_{22}+a_{33}b_{32} & a_{31}b_{13}+a_{32}b_{23}+a_{33}b_{33} \end{pmatrix}^t$$

$$= \begin{pmatrix} a_{11}b_{11}+a_{12}b_{21}+a_{13}b_{31} & a_{21}b_{11}+a_{22}b_{21}+a_{23}b_{31} & a_{31}b_{11}+a_{32}b_{21}+a_{33}b_{31} \\ a_{11}b_{12}+a_{12}b_{22}+a_{13}b_{32} & a_{21}b_{12}+a_{22}b_{22}+a_{23}b_{32} & a_{31}b_{12}+a_{32}b_{22}+a_{33}b_{32} \\ a_{11}b_{13}+a_{12}b_{23}+a_{13}b_{33} & a_{21}b_{13}+a_{22}b_{23}+a_{23}b_{33} & a_{31}b_{13}+a_{32}b_{23}+a_{33}b_{33} \end{pmatrix}$$

$$B^tA^t = \begin{pmatrix} b_{11} & b_{21} & b_{31} \\ b_{12} & b_{22} & b_{32} \\ b_{13} & b_{23} & b_{33} \end{pmatrix}\begin{pmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{pmatrix}$$

$$= \begin{pmatrix} a_{11}b_{11}+a_{12}b_{21}+a_{13}b_{31} & a_{21}b_{11}+a_{22}b_{21}+a_{23}b_{31} & a_{31}b_{11}+a_{32}b_{21}+a_{33}b_{31} \\ a_{11}b_{12}+a_{12}b_{22}+a_{13}b_{32} & a_{21}b_{12}+a_{22}b_{22}+a_{23}b_{32} & a_{31}b_{12}+a_{32}b_{22}+a_{33}b_{32} \\ a_{11}b_{13}+a_{12}b_{23}+a_{13}b_{33} & a_{21}b_{13}+a_{22}b_{23}+a_{23}b_{33} & a_{31}b_{13}+a_{32}b_{23}+a_{33}b_{33} \end{pmatrix}$$

Hence result follows.

If $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}$, $B = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$, then we can also prove that $(AB)^t = B^tA^t$

The proof is left as an exercise.

If $A_1A_2 \dots A_n$ are well defined product of matrices, then we can use mathematical induction to prove that $\left(A_1A_2\cdots A_n\right)^t = A_n^tA_{n-1}^t\cdots A_2^tA_1^t$.

If A is a square matrix, then $(A^n)^t = (A^t)^n$, where n is a positive integer.

(7) Let A, B be square matrix of the same order (2 or 3). Then $\det(AB) = \det A \det B$.

Using the property of determinant: $\begin{vmatrix} a+x & b \\ c+y & d \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} + \begin{vmatrix} x & b \\ y & d \end{vmatrix}$.

Proof: $n = 2$. $AB = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} a_{11}b_{11}+a_{12}b_{21} & a_{11}b_{12}+a_{12}b_{22} \\ a_{21}b_{11}+a_{22}b_{21} & a_{21}b_{12}+a_{22}b_{22} \end{pmatrix}$

$$\det(AB) = \begin{vmatrix} a_{11}b_{11}+a_{12}b_{21} & a_{11}b_{12}+a_{12}b_{22} \\ a_{21}b_{11}+a_{22}b_{21} & a_{21}b_{12}+a_{22}b_{22} \end{vmatrix}$$

$$= \begin{vmatrix} a_{11}b_{11} & a_{11}b_{12} \\ a_{21}b_{11} & a_{21}b_{12} \end{vmatrix} + \begin{vmatrix} a_{11}b_{11} & a_{12}b_{22} \\ a_{21}b_{11} & a_{22}b_{22} \end{vmatrix} + \begin{vmatrix} a_{12}b_{21} & a_{11}b_{12} \\ a_{22}b_{21} & a_{21}b_{12} \end{vmatrix} + \begin{vmatrix} a_{12}b_{21} & a_{12}b_{22} \\ a_{22}b_{21} & a_{22}b_{22} \end{vmatrix}$$

$$= b_{11}b_{12}\begin{vmatrix} a_{11} & a_{11} \\ a_{21} & a_{21} \end{vmatrix} + b_{11}b_{22}\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} + b_{12}b_{21}\begin{vmatrix} a_{12} & a_{11} \\ a_{22} & a_{21} \end{vmatrix} + b_{21}b_{22}\begin{vmatrix} a_{12} & a_{12} \\ a_{22} & a_{22} \end{vmatrix}$$

$$= \quad 0 \quad + b_{11}b_{22} \det A \quad - b_{12}b_{21}\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} + \quad 0$$

$$= (b_{11}b_{22} - b_{12}b_{21})\det A = \det A \det B$$

$\therefore \det(AB) = \det A \det B$

Similarly, using the property of determinant: $\begin{vmatrix} a+x & b & c \\ d+y & e & f \\ g+z & h & i \end{vmatrix} = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} + \begin{vmatrix} x & b & c \\ y & e & f \\ z & h & i \end{vmatrix}$,

and so $\begin{vmatrix} a+x & b+t & c \\ d+y & e+u & f \\ g+z & h+w & i \end{vmatrix} = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} + \begin{vmatrix} x & b & c \\ y & e & f \\ z & h & i \end{vmatrix} + \begin{vmatrix} a & t & c \\ d & u & f \\ g & w & i \end{vmatrix} + \begin{vmatrix} x & t & c \\ y & u & f \\ z & w & i \end{vmatrix}$.

$$n=3.\ AB=\begin{pmatrix} a_{11}b_{11}+a_{12}b_{21}+a_{13}b_{31} & a_{11}b_{12}+a_{12}b_{22}+a_{13}b_{32} & a_{11}b_{13}+a_{12}b_{23}+a_{13}b_{33} \\ a_{21}b_{11}+a_{22}b_{21}+a_{23}b_{31} & a_{21}b_{12}+a_{22}b_{22}+a_{23}b_{32} & a_{21}b_{13}+a_{22}b_{23}+a_{23}b_{33} \\ a_{31}b_{11}+a_{32}b_{21}+a_{33}b_{31} & a_{31}b_{12}+a_{32}b_{22}+a_{33}b_{32} & a_{31}b_{13}+a_{32}b_{23}+a_{33}b_{33} \end{pmatrix}$$

$$\det(AB)=\begin{vmatrix} a_{11}b_{11} & a_{11}b_{12} & a_{11}b_{13} \\ a_{21}b_{11} & a_{21}b_{12} & a_{21}b_{13} \\ a_{31}b_{11} & a_{31}b_{12} & a_{31}b_{13} \end{vmatrix}+\begin{vmatrix} a_{11}b_{11} & a_{11}b_{12} & a_{12}b_{23} \\ a_{21}b_{11} & a_{21}b_{12} & a_{22}b_{23} \\ a_{31}b_{11} & a_{31}b_{12} & a_{32}b_{23} \end{vmatrix}+\begin{vmatrix} a_{11}b_{11} & a_{11}b_{12} & a_{13}b_{33} \\ a_{21}b_{11} & a_{21}b_{12} & a_{23}b_{33} \\ a_{31}b_{11} & a_{31}b_{12} & a_{33}b_{33} \end{vmatrix}$$

$$+\begin{vmatrix} a_{11}b_{11} & a_{12}b_{22} & a_{11}b_{13} \\ a_{21}b_{11} & a_{22}b_{22} & a_{21}b_{13} \\ a_{31}b_{11} & a_{32}b_{22} & a_{31}b_{13} \end{vmatrix}+\begin{vmatrix} a_{11}b_{11} & a_{12}b_{22} & a_{12}b_{23} \\ a_{21}b_{11} & a_{22}b_{22} & a_{22}b_{23} \\ a_{31}b_{11} & a_{32}b_{22} & a_{32}b_{23} \end{vmatrix}+\underline{\begin{vmatrix} a_{11}b_{11} & a_{12}b_{22} & a_{13}b_{33} \\ a_{21}b_{11} & a_{22}b_{22} & a_{23}b_{33} \\ a_{31}b_{11} & a_{32}b_{22} & a_{33}b_{33} \end{vmatrix}}$$

$$+\begin{vmatrix} a_{11}b_{11} & a_{13}b_{32} & a_{11}b_{13} \\ a_{21}b_{11} & a_{23}b_{32} & a_{21}b_{13} \\ a_{31}b_{11} & a_{33}b_{32} & a_{31}b_{13} \end{vmatrix}+\underline{\begin{vmatrix} a_{11}b_{11} & a_{13}b_{32} & a_{12}b_{23} \\ a_{21}b_{11} & a_{23}b_{32} & a_{22}b_{23} \\ a_{31}b_{11} & a_{33}b_{32} & a_{32}b_{23} \end{vmatrix}}+\begin{vmatrix} a_{11}b_{11} & a_{13}b_{32} & a_{13}b_{33} \\ a_{21}b_{11} & a_{23}b_{32} & a_{23}b_{33} \\ a_{31}b_{11} & a_{33}b_{32} & a_{33}b_{33} \end{vmatrix}$$

$$+\begin{vmatrix} a_{12}b_{21} & a_{11}b_{12} & a_{11}b_{13} \\ a_{22}b_{21} & a_{21}b_{12} & a_{21}b_{13} \\ a_{32}b_{21} & a_{31}b_{12} & a_{31}b_{13} \end{vmatrix}+\begin{vmatrix} a_{12}b_{21} & a_{11}b_{12} & a_{12}b_{23} \\ a_{22}b_{21} & a_{21}b_{12} & a_{22}b_{23} \\ a_{32}b_{21} & a_{31}b_{12} & a_{32}b_{23} \end{vmatrix}+\underline{\begin{vmatrix} a_{12}b_{21} & a_{11}b_{12} & a_{13}b_{33} \\ a_{22}b_{21} & a_{21}b_{12} & a_{23}b_{33} \\ a_{32}b_{21} & a_{31}b_{12} & a_{33}b_{33} \end{vmatrix}}$$

$$+\begin{vmatrix} a_{12}b_{21} & a_{12}b_{22} & a_{11}b_{13} \\ a_{22}b_{21} & a_{22}b_{22} & a_{21}b_{13} \\ a_{32}b_{21} & a_{32}b_{22} & a_{31}b_{13} \end{vmatrix}+\begin{vmatrix} a_{12}b_{21} & a_{12}b_{22} & a_{12}b_{23} \\ a_{22}b_{21} & a_{22}b_{22} & a_{22}b_{23} \\ a_{32}b_{21} & a_{32}b_{22} & a_{32}b_{23} \end{vmatrix}+\begin{vmatrix} a_{12}b_{21} & a_{12}b_{22} & a_{13}b_{33} \\ a_{22}b_{21} & a_{22}b_{22} & a_{23}b_{33} \\ a_{32}b_{21} & a_{32}b_{22} & a_{33}b_{33} \end{vmatrix}$$

$$+\underline{\begin{vmatrix} a_{12}b_{21} & a_{13}b_{32} & a_{11}b_{13} \\ a_{22}b_{21} & a_{23}b_{32} & a_{21}b_{13} \\ a_{32}b_{21} & a_{33}b_{32} & a_{31}b_{13} \end{vmatrix}}+\begin{vmatrix} a_{12}b_{21} & a_{13}b_{32} & a_{12}b_{23} \\ a_{22}b_{21} & a_{23}b_{32} & a_{22}b_{23} \\ a_{32}b_{21} & a_{33}b_{32} & a_{32}b_{23} \end{vmatrix}+\begin{vmatrix} a_{12}b_{21} & a_{13}b_{32} & a_{13}b_{33} \\ a_{22}b_{21} & a_{23}b_{32} & a_{23}b_{33} \\ a_{32}b_{21} & a_{33}b_{32} & a_{33}b_{33} \end{vmatrix}$$

$$+\begin{vmatrix} a_{13}b_{31} & a_{11}b_{12} & a_{11}b_{13} \\ a_{23}b_{31} & a_{21}b_{12} & a_{21}b_{13} \\ a_{33}b_{31} & a_{31}b_{12} & a_{31}b_{13} \end{vmatrix}+\underline{\begin{vmatrix} a_{13}b_{31} & a_{11}b_{12} & a_{12}b_{23} \\ a_{23}b_{31} & a_{21}b_{12} & a_{22}b_{23} \\ a_{33}b_{31} & a_{31}b_{12} & a_{32}b_{23} \end{vmatrix}}+\begin{vmatrix} a_{13}b_{31} & a_{11}b_{12} & a_{13}b_{33} \\ a_{23}b_{31} & a_{21}b_{12} & a_{23}b_{33} \\ a_{33}b_{31} & a_{31}b_{12} & a_{33}b_{33} \end{vmatrix}$$

$$+\underline{\begin{vmatrix} a_{13}b_{31} & a_{12}b_{22} & a_{11}b_{13} \\ a_{23}b_{31} & a_{22}b_{22} & a_{21}b_{13} \\ a_{33}b_{31} & a_{32}b_{22} & a_{31}b_{13} \end{vmatrix}}+\begin{vmatrix} a_{13}b_{31} & a_{12}b_{22} & a_{12}b_{23} \\ a_{23}b_{31} & a_{22}b_{22} & a_{22}b_{23} \\ a_{33}b_{31} & a_{32}b_{22} & a_{32}b_{23} \end{vmatrix}+\begin{vmatrix} a_{13}b_{31} & a_{12}b_{22} & a_{13}b_{33} \\ a_{23}b_{31} & a_{22}b_{22} & a_{23}b_{33} \\ a_{33}b_{31} & a_{32}b_{22} & a_{33}b_{33} \end{vmatrix}$$

$$+\begin{vmatrix} a_{13}b_{31} & a_{13}b_{32} & a_{11}b_{13} \\ a_{23}b_{31} & a_{23}b_{32} & a_{21}b_{13} \\ a_{33}b_{31} & a_{33}b_{32} & a_{31}b_{13} \end{vmatrix}+\begin{vmatrix} a_{13}b_{31} & a_{13}b_{32} & a_{12}b_{23} \\ a_{23}b_{31} & a_{23}b_{32} & a_{22}b_{23} \\ a_{33}b_{31} & a_{33}b_{32} & a_{32}b_{23} \end{vmatrix}+\begin{vmatrix} a_{13}b_{31} & a_{13}b_{32} & a_{13}b_{33} \\ a_{23}b_{31} & a_{23}b_{32} & a_{23}b_{33} \\ a_{33}b_{31} & a_{33}b_{32} & a_{33}b_{33} \end{vmatrix}$$

(terms other than the underlying terms are zero)

$$=\begin{vmatrix} a_{11}b_{11} & a_{12}b_{22} & a_{13}b_{33} \\ a_{21}b_{11} & a_{22}b_{22} & a_{23}b_{33} \\ a_{31}b_{11} & a_{32}b_{22} & a_{33}b_{33} \end{vmatrix}+\begin{vmatrix} a_{11}b_{11} & a_{13}b_{32} & a_{12}b_{23} \\ a_{21}b_{11} & a_{23}b_{32} & a_{22}b_{23} \\ a_{31}b_{11} & a_{33}b_{32} & a_{32}b_{23} \end{vmatrix}+\begin{vmatrix} a_{12}b_{21} & a_{11}b_{12} & a_{13}b_{33} \\ a_{22}b_{21} & a_{21}b_{12} & a_{23}b_{33} \\ a_{32}b_{21} & a_{31}b_{12} & a_{33}b_{33} \end{vmatrix}$$

$$+\begin{vmatrix} a_{12}b_{21} & a_{13}b_{32} & a_{11}b_{13} \\ a_{22}b_{21} & a_{23}b_{32} & a_{21}b_{13} \\ a_{32}b_{21} & a_{33}b_{32} & a_{31}b_{13} \end{vmatrix}+\begin{vmatrix} a_{13}b_{31} & a_{11}b_{12} & a_{12}b_{23} \\ a_{23}b_{31} & a_{21}b_{12} & a_{22}b_{23} \\ a_{33}b_{31} & a_{31}b_{12} & a_{32}b_{23} \end{vmatrix}+\begin{vmatrix} a_{13}b_{31} & a_{12}b_{22} & a_{11}b_{13} \\ a_{23}b_{31} & a_{22}b_{22} & a_{21}b_{13} \\ a_{33}b_{31} & a_{32}b_{22} & a_{31}b_{13} \end{vmatrix}$$

$$=b_{11}b_{22}b_{33}\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}-b_{11}b_{23}b_{32}\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}-b_{12}b_{21}b_{33}\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$+b_{13}b_{21}b_{32}\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}+b_{12}b_{23}b_{31}\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}-b_{13}b_{22}b_{31}\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$= (b_{11}b_{22}b_{33} + b_{12}b_{23}b_{31} + b_{13}b_{21}b_{32} - b_{11}b_{23}b_{32} - b_{12}b_{21}b_{33} - b_{13}b_{22}b_{31})\det(A)$$

$$= \det A \det B$$

Let A be a square matrix of order 2 or 3.
Define the **cofactor matrix** of A as **cof(A)** = the matrix formed by the cofactors of A.
Define the **adjoint matrix** of A as **adj(A)** = cof(A)^t. (The transpose of the cofactor matrix A.)

Example 1 Let $A=\begin{pmatrix}-3 & 2\\ -1 & 4\end{pmatrix}$, then Cofactors

$A_{11}=(-1)^{1+1}M_{11}=4$	$A_{12}=(-1)^{1+2}M_{12}=1$
$A_{21}=(-1)^{2+1}M_{21}=-2$	$A_{22}=(-1)^{2+2}M_{22}=-3$

cof(A) $=\begin{pmatrix}4 & 1\\ -2 & -3\end{pmatrix}$, adj($A$) =cof($A$)^t $=\begin{pmatrix}4 & -2\\ 1 & -3\end{pmatrix}$.

Example 2 Let $A=\begin{pmatrix}1 & -1 & 0\\ 2 & 2 & 5\\ 3 & 4 & 1\end{pmatrix}$, then Cofactors

$A_{11}=(-1)^{1+1}M_{11}=(-1)^2(2-20)=-18$	$A_{12}=(-1)^{1+2}M_{12}=(-1)^3(-13)=13$	$A_{13}=(-1)^{1+3}M_{13}=(-1)^4\,2=2$
$A_{21}=(-1)^{2+1}M_{21}=(-1)(-1)=1$	$A_{22}=(-1)^{2+2}M_{22}=1$	$A_{23}=(-1)^{2+3}M_{23}=(-1)\times 7=-7$
$A_{31}=(-1)^{3+1}M_{31}=-5$	$A_{32}=(-1)^{3+2}M_{32}=-5$	$A_{33}=(-1)^{3+3}M_{33}=4$

cof(A) $=\begin{pmatrix}-18 & 13 & 2\\ 1 & 1 & -7\\ -5 & -5 & 4\end{pmatrix}$, adj($A$) =cof($A$)^t $=\begin{pmatrix}-18 & 1 & -5\\ 13 & 1 & -5\\ 2 & -7 & 4\end{pmatrix}$.

Theorem $A\,\text{adj}(A)=\text{adj}(A)A=\det(A)I$

Proof: $n=2$. Let $A=\begin{pmatrix}a & b\\ c & d\end{pmatrix}$, $\det(A)=ad-bc$. cof(A) $=\begin{pmatrix}d & -c\\ -b & a\end{pmatrix}$, adj($A$) $=\begin{pmatrix}d & -b\\ -c & a\end{pmatrix}$.

$$A\,\text{adj}(A)=\begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}d & -b\\ -c & a\end{pmatrix}=\begin{pmatrix}ad-bc & 0\\ 0 & ad-bc\end{pmatrix}=(ad-bc)\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\det(A)I$$

$$\text{adj}(A)A=\begin{pmatrix}d & -b\\ -c & a\end{pmatrix}\begin{pmatrix}a & b\\ c & d\end{pmatrix}=\begin{pmatrix}ad-bc & 0\\ 0 & ad-bc\end{pmatrix}=(ad-bc)\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}=\det(A)I$$

$n=3$. Let $A=\begin{pmatrix}a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{pmatrix}$, cof($A$) $=\begin{pmatrix}A_{11} & A_{12} & A_{13}\\ A_{21} & A_{22} & A_{23}\\ A_{31} & A_{32} & A_{33}\end{pmatrix}$, adj($A$) $=\begin{pmatrix}A_{11} & A_{21} & A_{31}\\ A_{12} & A_{22} & A_{32}\\ A_{13} & A_{23} & A_{33}\end{pmatrix}$

$$A\,\text{adj}(A)=\begin{pmatrix}a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{pmatrix}\begin{pmatrix}A_{11} & A_{21} & A_{31}\\ A_{12} & A_{22} & A_{32}\\ A_{13} & A_{23} & A_{33}\end{pmatrix}$$

$$=\begin{pmatrix}a_{11}A_{11}+a_{12}A_{12}+a_{13}A_{13} & a_{11}A_{21}+a_{12}A_{22}+a_{13}A_{23} & a_{11}A_{31}+a_{12}A_{32}+a_{13}A_{33}\\ a_{21}A_{11}+a_{22}A_{12}+a_{23}A_{33} & a_{21}A_{21}+a_{22}A_{22}+a_{23}A_{23} & a_{21}A_{31}+a_{22}A_{32}+a_{23}A_{33}\\ a_{31}A_{11}+a_{32}A_{12}+a_{33}A_{13} & a_{31}A_{21}+a_{32}A_{22}+a_{33}A_{23} & a_{31}A_{31}+a_{32}A_{32}+a_{33}A_{33}\end{pmatrix}$$

$$=\begin{pmatrix}\det A & 0 & 0\\ 0 & \det A & 0\\ 0 & 0 & \det A\end{pmatrix}=(\det A)\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}=\det(A)I$$

$$\text{adj}(A)A=\begin{pmatrix}A_{11} & A_{21} & A_{31}\\ A_{12} & A_{22} & A_{32}\\ A_{13} & A_{23} & A_{33}\end{pmatrix}\begin{pmatrix}a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{pmatrix}$$

$$= \begin{pmatrix} a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13} & a_{12}A_{11} + a_{22}A_{21} + a_{32}A_{31} & a_{13}A_{11} + a_{23}A_{21} + a_{33}A_{31} \\ a_{11}A_{12} + a_{21}A_{22} + a_{31}A_{32} & a_{12}A_{12} + a_{22}A_{22} + a_{32}A_{32} & a_{13}A_{12} + a_{23}A_{22} + a_{33}A_{32} \\ a_{11}A_{13} + a_{21}A_{23} + a_{31}A_{33} & a_{12}A_{13} + a_{22}A_{23} + a_{32}A_{33} & a_{13}A_{13} + a_{23}A_{23} + a_{33}A_{33} \end{pmatrix}$$

$$= \begin{pmatrix} \det A & 0 & 0 \\ 0 & \det A & 0 \\ 0 & 0 & \det A \end{pmatrix} = (\det A)\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \det(A)I$$

If $\det A \neq 0$, then $A^{-1} = \dfrac{1}{\det A}\operatorname{adj}(A)$.

Prove that $(A^t)^{-1} = (A^{-1})^t$

Prove that $(AB)^{-1} = B^{-1} A^{-1}$.

If $A_1A_2 \ldots A_n$ are well defined product of non-singular matrices, then we can use mathematical induction to prove that $(A_1A_2 \cdots A_n)^{-1} = A_n^{-1}A_{n-1}^{-1} \cdots A_2^{-1}A_1^{-1}$.

If A is a non-singular matrix, then $(A^n)^{-1} = (A^{-1})^n$, where n is a positive integer.

Let $A=\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$.

(a) Find the value of $A^2 - 4A$.
(b) Use (a) to find A^{-1}.
(c) Find A^3.
(d) Find the remainder when $x^{101} - 1$ is divided by $x^2 - 4x + 3$.
(e) Find the matrix $A^{101} - I$.

(*a*) $\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}^2 - 4\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 5 & -4 \\ -4 & 5 \end{pmatrix} - \begin{pmatrix} 8 & -4 \\ -4 & 8 \end{pmatrix} = \begin{pmatrix} -3 & 0 \\ 0 & -3 \end{pmatrix} = -3I$

(*b*) By (*a*), $A^2 - 4A = -3I$

$-\frac{1}{3}A(A-4I) = I$

$A^{-1} = \frac{1}{3}(4I - A) = \frac{1}{3}\left[\begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix} - \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}\right] = \frac{1}{3}\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} \end{pmatrix}$

(*c*) $A(A^2 - 4A) = -3AI$

$A^3 = 4A^2 - 3A = 4(4A - 3I) - 3A = 13A - 12I = 13\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} - 12\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 14 & -13 \\ -13 & 14 \end{pmatrix}$

(*d*) $x^{101} - 1 = (x^2 - 4x + 3)Q(x) + ax + b$

$x^{101} - 1 = (x - 1)(x - 3)Q(x) + ax + b$

Put $x = 1$, $0 = a + b$ …… (1)

Put $x = 3$, $3^{101} - 1 = 3a + b$ …… (2)

$[(2) - (1)] \div 2$: $a = \frac{1}{2}\left(3^{101} - 1\right)$

$b = \frac{1}{2}\left(1 - 3^{101}\right)$

The remainder $= \frac{1}{2}\left(3^{101} - 1\right)x + \frac{1}{2}\left(1 - 3^{101}\right)$.

(*e*) $A^{101} - I = \frac{1}{2}\left(3^{101} - 1\right)A + \frac{1}{2}\left(1 - 3^{101}\right)I$

$= \frac{1}{2}\left(3^{101} - 1\right)\left\{\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right\}$

$= \frac{1}{2}\left(3^{101} - 1\right)\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$