Menelaus's Theorem (Vectors)

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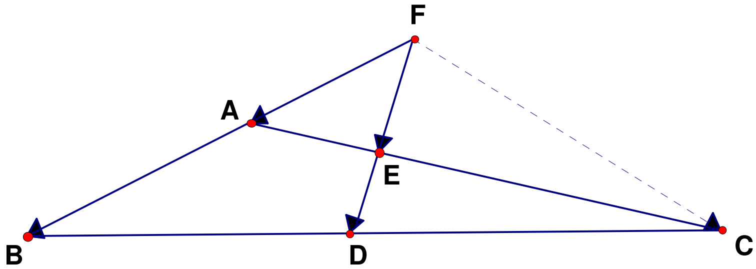
In ΔABC, suppose a line cuts BC at D, AC at E and AB at F, then $\frac{BD}{CD}\cdot\frac{CE}{AE}\cdot\frac{AF}{BF}=1$

A transversal cuts the sides BC, CA and AB of a triangle ABC in D, E and F respectively.

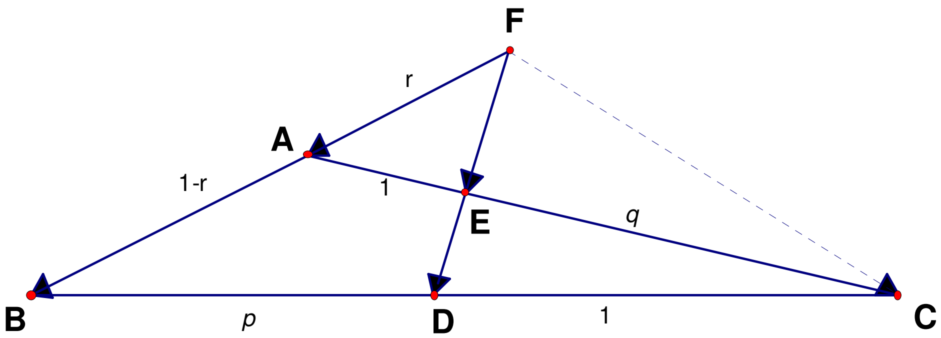
Let $\frac{BD}{DC}=p$, $\frac{CE}{EA}=q$, $\frac{AF}{FB}=r$, $\overrightarrow{FA}=\vec{a}$, $\overrightarrow{FC}=\vec{c}$.



(a) Express $\overrightarrow{FE}$ in terms of q, $\vec{a}$ and $\vec{c}$.

(b) Express $\overrightarrow{FD}$ in terms of p, r, $\vec{a}$ and $\vec{c}$.

(c) Hence prove that $pqr = 1$



(a) $\overrightarrow{FE} = \frac{q\vec{a}+\vec{c}}{1+q}$

(b) $\overrightarrow{FD} = \frac{\overrightarrow{FB}+p\vec{c}}{1+p}$

$= \frac{\frac{1}{r}\vec{a}+p\vec{c}}{1+p}$

$= \frac{\vec{a}+rp\vec{c}}{r+rp}$

(c) $\overrightarrow{FE} \mathbin{//} \overrightarrow{FD}$

$k\frac{q\vec{a}+\vec{c}}{1+q} = \frac{\vec{a}+rp\vec{c}}{r+rp}$

Compare coefficients of $\vec{a}$ and $\vec{c}$.

$$\begin{cases}\frac{1}{r+rp}=\frac{kq}{1+q}\cdots\cdots(1)\\ \frac{rp}{r+rp}=\frac{k}{1+q}\cdots\cdots(2)\end{cases}$$

$\frac{(1)}{(2)}: \frac{1}{rp}=q$

$pqr = 1$