Supplementary Notes on Vector Components

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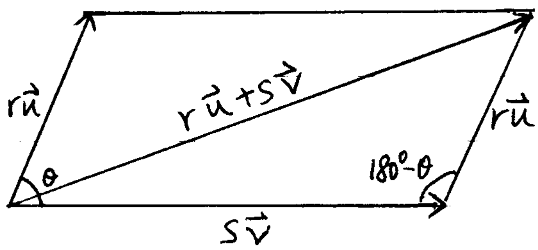
1. Given that $\vec{u}$ and $\vec{v}$ are 2 non-zero vectors which are not parallel.

If $r\vec{u} + s\vec{v} = \vec{0}$ then $r = s = 0$

Proof: In the figure, $r\vec{u} + s\vec{v}$ is the diagonal vector of the parallelogram.



Let θ be the angle between $r\vec{u}$ and $s\vec{v}$.

$\because r\vec{u} + s\vec{v} = \vec{0}$

$\therefore |r\vec{u} + s\vec{v}|^2 = |\vec{0}|^2$

By cosine rule, $|r\vec{u}|^2 + |s\vec{v}|^2 - 2|r\vec{u}|\,|s\vec{v}|\cos(180° - \theta) = |r\vec{u} + s\vec{v}|^2 = 0$

$|r\vec{u}|^2 + |s\vec{v}|^2 - 2|r\vec{u}|\,|s\vec{v}| + 2|r\vec{u}|\,|s\vec{v}|(1 + \cos\theta) = 0$

$(|r\vec{u}| - |s\vec{v}|)^2 + 2|r\vec{u}|\,|s\vec{v}|(1 + \cos\theta) = 0$ ....(*)

Note that the above equation (*) may be written as $a + b = 0$,

where $a = (|r\vec{u}| - |s\vec{v}|)^2$ and $b = 2|r\vec{u}|\,|s\vec{v}|(1 + \cos\theta)$

clearly $a \ge 0$ and $\cos\theta \ge -1 \Rightarrow 1 + \cos\theta \ge 0 \Rightarrow b \ge 0$,

which means that LHS = $a + b \ge 0$ but RHS = 0

This implies $a = 0$ and $b = 0$

$|r\vec{u}| - |s\vec{v}| = 0$ and $2|r\vec{u}|\,|s\vec{v}|(1 + \cos\theta) = 0$

$|r\vec{u}| = |s\vec{v}|$ and $|r\vec{u}| = 0$ or $|s\vec{v}| = 0$ ($\because \vec{u}$, $\vec{v}$ are not parallel, $1+\cos\theta \ne 0$)

$\because |\vec{u}| \ne 0$ and $|\vec{v}| \ne 0$

The above 2 equations gives $r = s = 0$

The method of comparing coefficients

2. Given that $\vec{u}$ and $\vec{v}$ are 2 non-zero vectors which are not parallel.

If $a\vec{u} + b\vec{v} = c\vec{u} + d\vec{v}$, then $a = c$ and $b = d$

Proof: $a\vec{u} + b\vec{v} = c\vec{u} + d\vec{v} \Rightarrow (a - c)\vec{u} + (b - d)\vec{v} = \vec{0}$

By the result of (*), $r = a - c = 0$ and $s = b - d = 0$

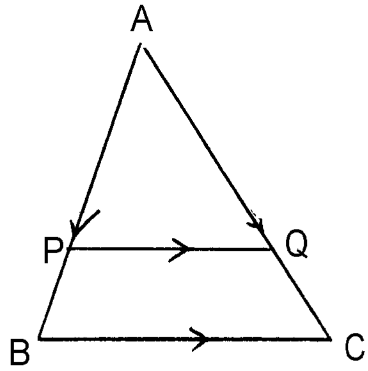
$\Rightarrow a = c$ and $b = d$

The proof is completed.

Example 1 In ΔABC, P and Q lie in AB and AC respectively such that BC // PQ, use vector method to show that $\frac{AP}{AB}=\frac{AQ}{AC}$.

(Remark: This result is known as the theorem of equal ratio)

Solution: Add the vectors on the right diagram.



$\because BC \,//\, PQ$

$\therefore \overrightarrow{PQ}=k\overrightarrow{BC}$, where k is a real constant.

$\overrightarrow{AQ}-\overrightarrow{AP} = k(\overrightarrow{AC}-\overrightarrow{AB})$ ....(*)

$\because \overrightarrow{AQ}$ and $\overrightarrow{AC}$ are in the same direction

$\therefore \overrightarrow{AQ} = m\overrightarrow{AC}$ ....(1)

$\because \overrightarrow{AP}$ and $\overrightarrow{AB}$ are in the same direction

$\therefore \overrightarrow{AP} = n\overrightarrow{AB}$ ....(2)

Sub. (1) and (2) into (*), $m\overrightarrow{AC} - n\overrightarrow{AB} = k\overrightarrow{AC}-k\overrightarrow{AB}$

Comparing coefficients of $\overrightarrow{AC}$ and $\overrightarrow{AB}$

We have $m = k$ and $n = k$

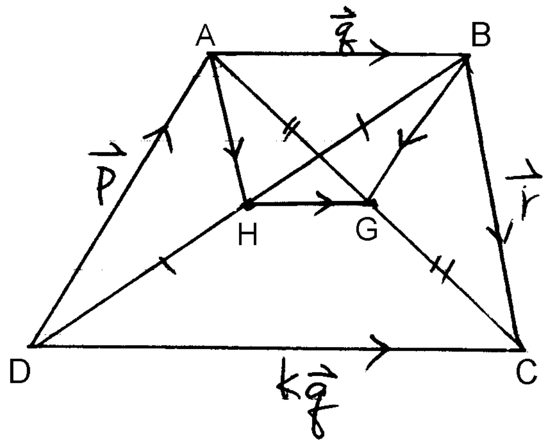
$\Rightarrow \frac{AQ}{AC}=k$ and $\frac{AP}{AB}=k$

$\Rightarrow \frac{AP}{AB}=\frac{AQ}{AC}$

The proof is completed.

Example 2 Given that *ABCD* is a trapezium with *AB* // *CD*. If *H* and *G* are the mid-points of the diagonals *BD* and *AC* respectively, show that *HG* // *AB* // *CD* by vector method.

Solution: Add the vectors $\vec{p}$, $\vec{q}$ and $\vec{r}$ as shown.



$\because AB \,//\, CD$

$\therefore \overrightarrow{AB} = \vec{q}$ and $\overrightarrow{DC} = k\vec{q} = \vec{p} + \vec{q} + \vec{r}$

$\because$ *H* is the mid-point of *BD*

$\therefore \overrightarrow{AH} = \frac{1}{2}(\vec{q} - \vec{p})$

$\because$ *G* is the mid-point of *AC*

$\therefore \overrightarrow{BG} = \frac{1}{2}(\vec{r} - \vec{q})$

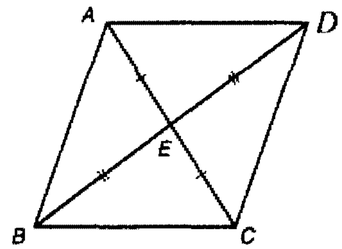
$$\begin{aligned}\overrightarrow{HG} &= -\overrightarrow{AH} + \vec{q} + \overrightarrow{BG} \\ &= -\frac{1}{2}(\vec{q} - \vec{p}) + \vec{q} + \frac{1}{2}(\vec{r} - \vec{q}) \\ &= \frac{1}{2}(\vec{p} + \vec{r}) \\ &= \frac{1}{2}(\vec{p} + \vec{q} + \vec{r} - \vec{q}) \\ &= \frac{1}{2}(k\vec{q} - \vec{q}) \\ &= \frac{1}{2}(k-1)\vec{q}\end{aligned}$$

$\therefore \overrightarrow{HG}$ is a scalar multiple of $\overrightarrow{AB}$.

$\Rightarrow HG \,//\, AB \,//\, CD$

The proof is completed.

Example 3 Given that *ABCD* is a parallelogram, prove that the diagonals *AC* and *BD* bisect each other at *E* by vector method.



Solution: **Method 1** $\because ABCD$ is a parallelogram

$\therefore \overrightarrow{AD} = \overrightarrow{BC}$

$\overrightarrow{ED} - \overrightarrow{EA} = \overrightarrow{EC} - \overrightarrow{EB}$

$\overrightarrow{EB} + \overrightarrow{ED} = \overrightarrow{EA} + \overrightarrow{EC}$ ....(1)

$\because B, E, D$ are collinear and A, E, C are collinear

$\therefore \overrightarrow{EB} = -m\overrightarrow{ED}$, $\overrightarrow{EC} = -n\overrightarrow{EA}$, where m and n are real constants

Sub. into (1), $-m\overrightarrow{ED} + \overrightarrow{ED} = \overrightarrow{EA} - n\overrightarrow{EA}$

$(1 - m)\overrightarrow{ED} = (1 - n)\overrightarrow{EA}$

$\because \overrightarrow{EA}$ and $\overrightarrow{ED}$ are non-zero vectors which are not parallel

By the theorem of comparing coefficients, $1 - m = 0$ and $1 - n = 0$

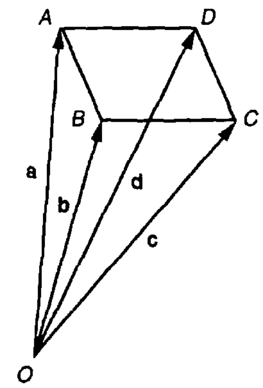
$m = n$

$\overrightarrow{EB} = -\overrightarrow{ED}$, $\overrightarrow{EC} = -\overrightarrow{EA}$

$\therefore$ the diagonals *AC* and *BD* bisect each other at *E*.

Method 2

Let O be the reference point and $\vec{a}$, $\vec{b}$, $\vec{c}$, $\vec{d}$ be the corresponding position vectors of *A*, *B*, *C* and *D* respectively.



$\because ABCD$ is a parallelogram

$\therefore \overrightarrow{AD} = \overrightarrow{BC}$

$\vec{d} - \vec{a} = \vec{c} - \vec{b}$

$\vec{a} + \vec{c} = \vec{b} + \vec{d}$

$\frac{1}{2}(\vec{a} + \vec{c}) = \frac{1}{2}(\vec{b} + \vec{d})$

which means that the mid-point of *AC* = the mid-point of *BD*

$\therefore$ the diagonals *AC* and *BD* bisect each other at *E*.