Exercise on Vectors

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The following questions are inter-related. Do these one by one.

1. Let $\overrightarrow{V_1} = (a, b, c)$

   $\overrightarrow{V_2} = (d, e, f)$

   Define the vector dot product by $\overrightarrow{V_1} \cdot \overrightarrow{V_2} = ad + be + cf$.

   By drawing suitable triangle, show that if $\overrightarrow{V_1} \perp \overrightarrow{V_2}$, then $\overrightarrow{V_1} \cdot \overrightarrow{V_2} = 0$.

   Show that the converse is also true.

2. Show that in general $\overrightarrow{V_1} \cdot \overrightarrow{V_2} = \left|\overrightarrow{V_1}\right| \cdot \left|\overrightarrow{V_2}\right| \cos \theta$, where $\left|\overrightarrow{V_i}\right| = \left(\overrightarrow{V_i} \cdot \overrightarrow{V_i}\right)^{\frac{1}{2}}$ for $i = 1, 2$;

   and θ = angle between $\overrightarrow{V_1}$ and $\overrightarrow{V_2}$.

3. Let $\vec{p}_0 = (x_0, y_0, z_0)$ and $\vec{p} = (x, y, z)$. Describe the set of all the points (x, y, z) for which $\left|\vec{p} - \vec{p}_0\right| = 1$.

4. Prove that $\left|\overrightarrow{V_1} + \overrightarrow{V_2}\right| \le \left|\overrightarrow{V_1}\right| + \left|\overrightarrow{V_2}\right|$ for any vectors $\overrightarrow{V_1}$ and $\overrightarrow{V_2}$.

5. Let $\vec{U} = (u_1, u_2, u_3)$, $\vec{V} = (v_1, v_2, v_3)$.
   (*a*) Find the orthogonal projection of $\vec{U}$ on $\vec{V}$ in terms of dot product.
   (*b*) Hence find two vectors of norm 1 that are orthogonal to (3, –2). (norm = length)
   (*c*) What is the orthogonal projection of each vectors on (3, –2)?

6. Use vectors to find the cosines of the interior angles of the triangle with vertices (–1, 0), (–2, 1), (1, 4).

7. Establish the identity $\left|\vec{U} + \vec{V}\right|^2 + \left|\vec{U} - \vec{V}\right|^2 = 2\left|\vec{U}\right|^2 + 2\left|\vec{V}\right|^2$

8. Establish the identity $\vec{U} \cdot \vec{V} = \frac{1}{4}\left|\vec{U} + \vec{V}\right|^2 - \frac{1}{4}\left|\vec{U} - \vec{V}\right|^2$

9. Find the angle between a diagonal of a cube and one of its faces.
   Find the angle between a diagonal of a cube and one of its edges.

10. Show that if $\vec{V}$ is orthogonal to $\overrightarrow{W_1}$ and $\overrightarrow{W_2}$, then $\vec{V}$ is orthogonal to $k_1\overrightarrow{W_1} + k_2\overrightarrow{W_2}$ for all scalars k_1 and k_2.

11. Let $\vec{U}$ and $\vec{V}$ be two non-zero vectors. If $k = \left|\vec{U}\right|$ and $\ell = \left|\vec{V}\right|$, show that the vector $\vec{W} = \frac{1}{k + \ell}\left(k\vec{V} + \ell\vec{U}\right)$ bisects the angle between $\vec{U}$ and $\vec{V}$.

12. Define the cross product $\vec{U}\times\vec{V}=\left(\begin{vmatrix}u_2 & u_3\\ v_2 & v_3\end{vmatrix},-\begin{vmatrix}u_1 & u_3\\ v_1 & v_3\end{vmatrix},\begin{vmatrix}u_1 & u_2\\ v_1 & v_2\end{vmatrix}\right)$.

    Prove that $\vec{U}\cdot(\vec{U}\times\vec{V}) = \vec{V}\cdot(\vec{U}\times\vec{V}) = 0$.

13. Prove the **Langrange's identity** $\left|\vec{U}\times\vec{V}\right|^2=\left|\vec{U}\right|^2\left|\vec{V}\right|^2-\left(\vec{U}\cdot\vec{V}\right)^2$.

14. Use Q1, Q12 and Q13 to prove that $\vec{U}\times\vec{V}=\left|\vec{U}\right|\left|\vec{V}\right|\sin\theta\,\hat{n}$, where $\hat{n}$ is the unit normal vector perpendicular to $\vec{U}$ and $\vec{V}$ determined by right hand rule.

15. Find the area of the triangle determined by the points $P(2, 2, 0)$, $Q(-1, 0, 2)$, $R(0, 4, 3)$.

16. Prove that $\vec{U}\cdot(\vec{V}\times\vec{W})=\begin{vmatrix}u_1 & u_2 & u_3\\ v_1 & v_2 & v_3\\ w_1 & w_2 & w_3\end{vmatrix}$.

    Hence compute $\vec{U}\cdot(\vec{V}\times\vec{W})$ when $\vec{U}=(-1, 4, 7)$, $\vec{V}=(6, -7, 3)$, $\vec{W}=(4, 0, 1)$.

17. Let $\vec{U}=(-1, 3, 2)$ and $\vec{W}=(1, 1, -1)$. Find all vectors $\vec{X}$ that satisfy $\vec{U}\times\vec{X}=\vec{W}$.

18. Let $\vec{U}$, $\vec{V}$, $\vec{W}$ be non-zero vectors in 3D-space, no two of which are collinear. Show that

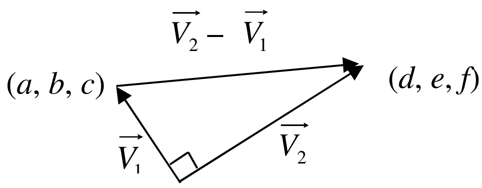
    (*a*) $\vec{U}\times(\vec{V}\times\vec{W})$ lies in the plane determined by $\vec{V}$ and $\vec{W}$.

    (*b*) $(\vec{U}\times\vec{V})\times\vec{W}$ lies in the plane determined by $\vec{U}$ and $\vec{V}$.

19. Prove that $\vec{x}\times(\vec{y}\times\vec{z})=(\vec{x}\cdot\vec{z})\,\vec{y}-(\vec{x}\cdot\vec{y})\,\vec{z}$.

20. Given the tetrahedron *OABC* with $OA = a$, $OB = b$, $OC = c$, $BC = d$, $AC = e$, $AB = f$. Find the volume in terms of the sides.

1. Given $\vec{V_1} \perp \vec{V_2}$, to prove $\vec{V_1} \cdot \vec{V_2} = 0$.



$\vec{V_2} - \vec{V_1} = (d - a, e - b, f - c)$

By Pythagoras' Theorem,

$\left|\vec{V_2} - \vec{V_1}\right|^2 = \left|\vec{V_1}\right|^2 + \left|\vec{V_2}\right|^2$

$(d - a)^2 + (e - b)^2 + (f - c)^2 = a^2 + b^2 + c^2 + d^2 + e^2 + f^2$

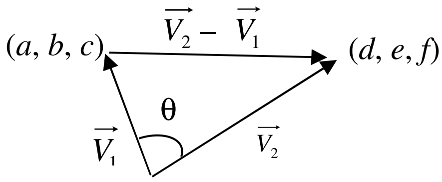
$a^2 + b^2 + c^2 + d^2 + e^2 + f^2 - 2ad - 2be - 2cf = a^2 + b^2 + c^2 + d^2 + e^2 + f^2$

$-2ad - 2be - 2cf = 0$

$ad + be + cf = 0$

$\vec{V_1} \cdot \vec{V_2} = 0$

To show that the converse is also true.



Given $\vec{V_1} \cdot \vec{V_2} = 0$, try to prove that $\vec{V_1} \perp \vec{V_2}$.

Let θ be the angle between $\vec{V_1}$ and $\vec{V_2}$.

By cosine law, $\left|\vec{V_2} - \vec{V_1}\right|^2 = \left|\vec{V_1}\right|^2 + \left|\vec{V_2}\right|^2 - 2\left|\vec{V_1}\right|\left|\vec{V_2}\right|\cos\theta$

$(d - a)^2 + (e - b)^2 + (f - c)^2 = a^2 + b^2 + c^2 + d^2 + e^2 + f^2 - 2\sqrt{a^2 + b^2 + c^2} \cdot \sqrt{d^2 + e^2 + f^2} \cos\theta$

After expansion and cancelling like terms,

$-2ad - 2be - 2cf = -2\sqrt{a^2 + b^2 + c^2} \cdot \sqrt{d^2 + e^2 + f^2} \cos\theta$

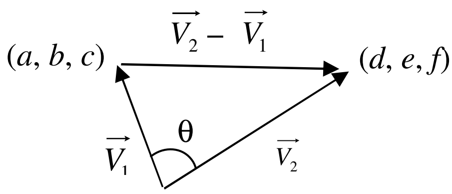
$\cos\theta = \dfrac{ad + be + cf}{\sqrt{a^2 + b^2 + c^2} \cdot \sqrt{d^2 + e^2 + f^2}}$

$\cos\theta = \dfrac{\vec{V_1} \cdot \vec{V_2}}{\sqrt{a^2 + b^2 + c^2} \cdot \sqrt{d^2 + e^2 + f^2}}$

$\cos\theta = 0$

$\Rightarrow \theta = 90°$

2. Show that $\vec{V_1} \cdot \vec{V_2} = \left|\vec{V_1}\right| \cdot \left|\vec{V_2}\right| \cos\theta$



By cosine law,

$(d - a)^2 + (e - b)^2 + (f - c)^2 = a^2 + b^2 + c^2 + d^2 + e^2 + f^2 - 2\sqrt{a^2 + b^2 + c^2} \cdot \sqrt{d^2 + e^2 + f^2} \cos\theta$

$-2ad - 2be - 2cf = -2\sqrt{a^2 + b^2 + c^2} \cdot \sqrt{d^2 + e^2 + f^2} \cos\theta$

$ad + be + cf = \sqrt{a^2 + b^2 + c^2} \cdot \sqrt{d^2 + e^2 + f^2} \cos\theta$

$\therefore \vec{V_1} \cdot \vec{V_2} = \left|\vec{V_1}\right| \cdot \left|\vec{V_2}\right| \cos\theta$

3. $\left|\vec{p} - \vec{p}_0\right| = 1$

$\left|(x - x_0, y - y_0, z - z_0)\right| = 1$

$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = 1$

$\therefore (x, y, z)$ are points on the surface with centre at (x_0, y_0, z_0) and radius = 1.

4. To prove that $\left|\vec{V_1}+\vec{V_2}\right| \le \left|\vec{V_1}\right| + \left|\vec{V_2}\right|$.

$\left(\left|\vec{V_1}\right|+\left|\vec{V_2}\right|\right)^2 - \left|\vec{V_1}+\vec{V_2}\right|^2$

$= \left|\vec{V_1}\right|^2 + \left|\vec{V_2}\right|^2 + 2\left|\vec{V_1}\right|\left|\vec{V_2}\right| - \left(\vec{V_1}+\vec{V_2}\right)\cdot\left(\vec{V_1}+\vec{V_2}\right)$

$= \left|\vec{V_1}\right|^2 + \left|\vec{V_2}\right|^2 + 2\left|\vec{V_1}\right|\left|\vec{V_2}\right| - \left|\vec{V_1}\right|^2 - \left|\vec{V_2}\right|^2 - 2\vec{V_1}\cdot\vec{V_2}$

$= 2\left|\vec{V_1}\right|\left|\vec{V_2}\right| - 2\vec{V_1}\cdot\vec{V_2}$

$= 2\left|\vec{V_1}\right|\left|\vec{V_2}\right| - 2\left|\vec{V_1}\right|\left|\vec{V_2}\right|\cos\theta$

$= 2\left|\vec{V_1}\right|\left|\vec{V_2}\right|(1-\cos\theta)$

$\because -1 \le \cos\theta \le 1$

$\therefore 1 - \cos\theta \ge 0$

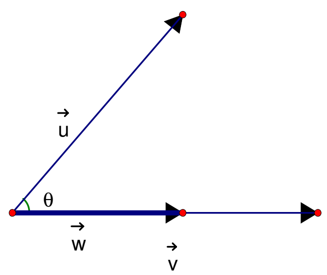
$\therefore \left(\left|\vec{V_1}\right|+\left|\vec{V_2}\right|\right)^2 - \left|\vec{V_1}+\vec{V_2}\right|^2 \ge 0$

$\left(\left|\vec{V_1}\right|+\left|\vec{V_2}\right|\right)^2 \ge \left|\vec{V_1}+\vec{V_2}\right|^2$

$\because$ All quantities are positive, take square root:

$\left|\vec{V_1}+\vec{V_2}\right| \le \left|\vec{V_1}\right| + \left|\vec{V_2}\right|$.

5. (*a*) Let $\vec{W}$ be the orthogonal projection of $\vec{V}$.



$\vec{U}\cdot\vec{V} = \left|\vec{U}\right|\left|\vec{V}\right|\cos\theta$

$\left|\vec{W}\right| = \left|\vec{U}\right|\cos\theta = \dfrac{\left|\vec{U}\right|\left|\vec{V}\right|\cos\theta}{\left|\vec{V}\right|}$

$\vec{W} = \dfrac{\left|\vec{U}\right|\left|\vec{V}\right|\cos\theta}{\left|\vec{V}\right|}\dfrac{\vec{V}}{\left|\vec{V}\right|}$, where $\dfrac{\vec{V}}{\left|\vec{V}\right|}$ is the unit vector in the direction of $\vec{V}$.

$\vec{W} = \dfrac{\vec{U}\cdot\vec{V}}{\vec{V}\cdot\vec{V}}\vec{V}$ (note that $\left|\vec{V}\right|^2 = \vec{V}\cdot\vec{V}$)

(*b*) $\vec{V} = (3, -2)$

$\left|\vec{V}\right| = \sqrt{3^2+(-2)^2} = \sqrt{13}$

Let $\vec{U} = (x, y)$

Norm of $\vec{U} = 1 \Rightarrow x^2 + y^2 = 1$ ……….(1)

$\vec{U} \perp \vec{V} \Rightarrow \vec{U}\cdot\vec{V} = 0 \Rightarrow 3x - 2y = 0$ ……..(2)

From (2), $x = \dfrac{2y}{3}$ ……….(3)

Sub. (3) into (1): $\left(\dfrac{2y}{3}\right)^2 + y^2 = 1$

$4y^2 + 9y^2 = 9$

$13y^2 = 9$

$y = \pm\frac{3}{\sqrt{13}} \Rightarrow x = \pm\frac{2}{\sqrt{13}}$

$\therefore\ \vec{U} = \left(\frac{2}{\sqrt{13}}, \frac{3}{\sqrt{13}}\right)$ or $\left(-\frac{2}{\sqrt{13}}, -\frac{3}{\sqrt{13}}\right)$

(c) $\because\ \vec{U} \perp \vec{V} \Rightarrow \vec{U}\cdot\vec{V} = 0$

From (a), The orthogonal projection on $\vec{U}$ on $\vec{V}$ is $\vec{W} = \frac{\vec{U}\cdot\vec{V}}{\vec{V}\cdot\vec{V}}\vec{V} = \vec{0}$

6. Let $A(-1, 0)$, $B(-2, 1)$, $C(1, 4)$

$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = (-2, 1) - (-1, 0) = (-1, 1)$

$\overrightarrow{AC} = (1, 4) - (-1, 0) = (2, 4)$

$\overrightarrow{BC} = (1, 4) - (-2, 1) = (3, 3)$

$\overrightarrow{AB}\cdot\overrightarrow{AC} = \left|\overrightarrow{AB}\right|\left|\overrightarrow{AC}\right|\cos A$

$-1\times2 + 1\times4 = \sqrt{(-1)^2 + 1^2}\sqrt{2^2 + 4^2}\cos A$

$\cos A = \frac{2}{\sqrt{40}} = \frac{1}{\sqrt{10}}$

$\overrightarrow{AC}\cdot\overrightarrow{BC} = \left|\overrightarrow{AC}\right|\left|\overrightarrow{BC}\right|\cos C$

$2\times3 + 4\times3 = \sqrt{2^2 + 4^2}\sqrt{3^2 + 3^2}\cos C$

$\cos C = \frac{18}{\sqrt{360}} = \frac{3}{\sqrt{10}}$

$\overrightarrow{AB}\cdot\overrightarrow{CB} = \left|\overrightarrow{AB}\right|\left|\overrightarrow{BC}\right|\cos B$

$-[3\times(-1) + 1\times3] = \sqrt{(-1)^2 + 1^2}\sqrt{3^2 + 3^2}\cos B$

$\cos B = \frac{0}{\sqrt{36}} = 0$

7. $\left|\vec{U} + \vec{V}\right|^2 + \left|\vec{U} - \vec{V}\right|^2 = \left(\vec{U} + \vec{V}\right)\cdot\left(\vec{U} + \vec{V}\right) + \left(\vec{U} - \vec{V}\right)\cdot\left(\vec{U} - \vec{V}\right)$

$= \vec{U}\cdot\vec{U} + 2\vec{U}\cdot\vec{V} + \vec{V}\cdot\vec{V} + \vec{U}\cdot\vec{U} - 2\vec{U}\cdot\vec{V} + \vec{V}\cdot\vec{V}$

$= 2\vec{U}\cdot\vec{U} + 2\vec{V}\cdot\vec{V}$

$= 2\left|\vec{U}\right|^2 + 2\left|\vec{V}\right|^2$

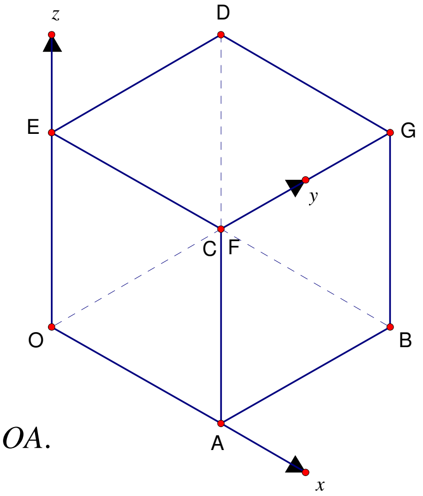
8. $\frac{1}{4}\left|\vec{U} + \vec{V}\right|^2 - \frac{1}{4}\left|\vec{U} - \vec{V}\right|^2 = \frac{1}{4}\left(\vec{U} + \vec{V}\right)\cdot\left(\vec{U} + \vec{V}\right) - \frac{1}{4}\left(\vec{U} - \vec{V}\right)\cdot\left(\vec{U} - \vec{V}\right)$

$= \frac{1}{4}\left(\vec{U}\cdot\vec{U} + 2\vec{U}\cdot\vec{V} + \vec{V}\cdot\vec{V} - \vec{U}\cdot\vec{U} + 2\vec{U}\cdot\vec{V} - \vec{V}\cdot\vec{V}\right)$

$= \frac{1}{4}\left(4\vec{U}\cdot\vec{V}\right)$

$= \vec{U}\cdot\vec{V}$

9. Let the cube be *OABCDEFG*. Then all edges have the same length.
Let OA = 1 unit, let OA = x-axis, OC = y-axis, OE = z-axis.
Then OG = diagonal = (1, 1, 1)

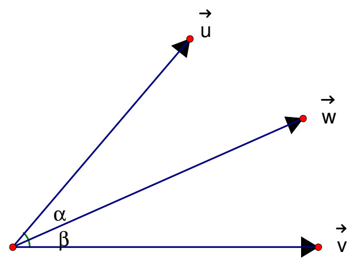


(*a*) To find the angle between OG and a face, say $OABC$.
The required angle = $\angle BOG$
$\overrightarrow{OB} = (1, 1, 0)$
$\overrightarrow{OB} \cdot \overrightarrow{OG} = \left|\overrightarrow{OB}\right|\left|\overrightarrow{OG}\right| \cos \angle BOG$
$1\times1 + 1\times1 + 0\times1 = \sqrt{1^2+1^2}\sqrt{1^2+1^2+1^2}\cos\angle BOG$
$\cos\angle BOG = \dfrac{2}{\sqrt{6}} = \sqrt{\dfrac{2}{3}}$, $\angle BOG = 35.3°$

(*b*) To find the angle between a diagonal (OG) and its edge, say OA.
The required angle = $\angle AOG$
$\overrightarrow{OA} = (1, 0, 0)$
$\overrightarrow{OA} \cdot \overrightarrow{OG} = \left|\overrightarrow{OA}\right|\left|\overrightarrow{OG}\right| \cos \angle AOG$
$1\times1 + 0\times1 + 0\times1 = 1\cdot\sqrt{1^2+1^2+1^2}\cos\angle AOG$
$\cos\angle AOG = \dfrac{1}{\sqrt{3}}$, $\angle AOG = 54.7°$

10. Note that the meaning of "orthogonal" = "perpendicular".
Given that $\vec{V} \perp \overrightarrow{W_1}$ and $\vec{V} \perp \overrightarrow{W_2}$, to prove that $\vec{V} \perp (k_1\overrightarrow{W_1} + k_2\overrightarrow{W_2})$.
$\vec{V} \perp \overrightarrow{W_1} \Rightarrow \vec{V}\cdot\overrightarrow{W_1} = 0$
$\vec{V} \perp \overrightarrow{W_2} \Rightarrow \vec{V}\cdot\overrightarrow{W_2} = 0$
$\vec{V}\cdot(k_1\overrightarrow{W_1} + k_2\overrightarrow{W_2}) = k_1\vec{V}\cdot\overrightarrow{W_1} + k_2\vec{V}\cdot\overrightarrow{W_2}$
$= 0 + 0 = 0$
$\therefore\ \vec{V} \perp (k_1\overrightarrow{W_1} + k_2\overrightarrow{W_2})$

11. If $k = \left|\vec{U}\right|$ and $\ell = \left|\vec{V}\right|$, to show $\vec{W} = \dfrac{1}{k+\ell}\left(k\vec{V} + \ell\vec{U}\right)$ bisects the angle between $\vec{U}$ and $\vec{V}$.
Let α and β be the angles as shown.



$\vec{U}\cdot\vec{W} = \vec{U}\cdot\dfrac{1}{k+\ell}\left(k\vec{V} + \ell\vec{U}\right)$
$\left|\vec{U}\right|\left|\vec{W}\right|\cos\alpha = \dfrac{1}{k+\ell}\left(k\vec{U}\cdot\vec{V} + \ell\vec{U}\cdot\vec{U}\right)$
$\left|\vec{U}\right|\left|\vec{W}\right|\cos\alpha = \dfrac{1}{k+\ell}\left[k\left|\vec{U}\right|\left|\vec{V}\right|\cos(\alpha+\beta) + \ell\left|\vec{U}\right|^2\right]$
$k\left|\vec{W}\right|\cos\alpha = \dfrac{1}{k+\ell}\left[k^2\ell\cos(\alpha+\beta) + \ell k^2\right]$
$\left|\vec{W}\right|\cos\alpha = \dfrac{k\ell}{k+\ell}\left[\cos(\alpha+\beta) + 1\right]$............(1)
On the other hand,
$\vec{V}\cdot\vec{W} = \vec{V}\cdot\dfrac{1}{k+\ell}\left(k\vec{V} + \ell\vec{U}\right)$
$\left|\vec{V}\right|\left|\vec{W}\right|\cos\alpha = \dfrac{1}{k+\ell}\left(k\vec{V}\cdot\vec{V} + \ell\vec{V}\cdot\vec{U}\right)$

$$|\vec{V}||\vec{W}|\cos\alpha = \frac{1}{k+\ell}\left[k|\vec{V}||\vec{V}| + \ell|\vec{U}||\vec{V}|\cos(\alpha+\beta)\right]$$

$$\ell|\vec{W}|\cos\alpha = \frac{1}{k+\ell}\left[k\ell^2 + \ell^2 k\cos(\alpha+\beta)\right]$$

$$|\vec{W}|\cos\alpha = \frac{k\ell}{k+\ell}\left[\cos(\alpha+\beta)+1\right] \text{................(2)}$$

(1) = (2) $\Rightarrow \cos\alpha = \cos\beta$

$\alpha = \beta$

12. $$\vec{U}\times\vec{V} = \left(\begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix}, -\begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix}, \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}\right) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

$$\vec{U}\cdot(\vec{U}\times\vec{V}) = (u_1, u_2, u_3)\cdot\left(\begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix}, -\begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix}, \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}\right)$$

$$= u_1\begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} - u_2\begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} + u_3\begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}$$

= cofactor expansion of a determinant

$$= \begin{vmatrix} u_1 & u_2 & u_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = 0 \ (\because \text{ first row is identical to second row})$$

$$\vec{V}\cdot(\vec{U}\times\vec{V}) = (v_1, v_2, v_3)\cdot\left(\begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix}, -\begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix}, \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}\right)$$

$$= \begin{vmatrix} v_1 & v_2 & v_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

= 0 ($\because$ first row is identical to third row)

13. LHS $= \left|\vec{U}\times\vec{V}\right|^2$

$$= \sqrt{\begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix}^2 + \left(-\begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix}\right)^2 + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}^2}^{\,2}$$

$$= (u_2v_3 - u_3v_2)^2 + (u_1v_3 - u_3v_1)^2 + (u_1v_2 - u_2v_1)^2$$

$$= u_2^2v_3^2 + u_3^2v_2^2 + u_1^2v_3^2 + u_3^2v_1^2 + u_1^2v_2^2 + u_2^2v_1^2 - 2u_2u_3v_2v_3 - 2u_1u_3v_1v_3 - 2u_1u_2v_1v_2$$

RHS $= \left|\vec{U}\right|^2\left|\vec{V}\right|^2 - \left(\vec{U}\cdot\vec{V}\right)^2$

$$= \sqrt{u_1^2 + u_2^2 + u_3^2}^{\,2}\sqrt{v_1^2 + v_2^2 + v_3^2}^{\,2} - (u_1v_1 + u_2v_2 + u_3v_3)^2$$

$$= (u_1^2 + u_2^2 + u_3^2)(v_1^2 + v_2^2 + v_3^2) - (u_1v_1 + u_2v_2 + u_3v_3)^2$$

$$= u_1^2v_1^2 + u_2^2v_1^2 + u_3^2v_1^2 + u_1^2v_2^2 + u_2^2v_2^2 + u_3^2v_2^2 + u_1^2v_3^2 + u_2^2v_3^2 + u_3^2v_3^2$$
$$- u_1^2v_1^2 - u_2^2v_2^2 - u_3^2v_3^2 - 2u_1u_2v_1v_2 - 2u_1u_3v_1v_3 - 2u_2u_3v_2v_3$$

$$= u_2^2v_3^2 + u_3^2v_2^2 + u_1^2v_3^2 + u_3^2v_1^2 + u_1^2v_2^2 + u_2^2v_1^2 - 2u_2u_3v_2v_3 - 2u_1u_3v_1v_3 - 2u_1u_2v_1v_2$$

$\therefore$ LHS = RHS

14. By Q13, $\left|\vec{U}\times\vec{V}\right|^2=\left|\vec{U}\right|^2\left|\vec{V}\right|^2-\left(\vec{U}\cdot\vec{V}\right)^2$

$\therefore\ \left|\vec{U}\times\vec{V}\right|^2=\left|\vec{U}\right|^2\left|\vec{V}\right|^2-\left(\left|\vec{U}\right|\left|\vec{V}\right|\cos\theta\right)^2$

$=\left|\vec{U}\right|^2\left|\vec{V}\right|^2\left(1-\cos^2\theta\right)$

$\left|\vec{U}\times\vec{V}\right|^2=\left|\vec{U}\right|^2\left|\vec{V}\right|^2\sin^2\theta$

Take positive square root: $\left|\vec{U}\times\vec{V}\right|=\left|\vec{U}\right|\left|\vec{V}\right|\sin\theta$ ....(1)

By Q12, $\vec{U}\cdot(\vec{U}\times\vec{V})=\vec{V}\cdot(\vec{U}\times\vec{V})=0$

$\Rightarrow\ \vec{U}\times\vec{V}\perp\vec{U}$ and $\vec{U}\times\vec{V}\perp\vec{V}$

$\therefore\ \vec{U}\times\vec{V}=k\hat{n}$, where $\hat{n}$ is the unit normal vector perpendicular to $\vec{U}$ and $\vec{V}$, and k is a non-negative constant.

$\left|\vec{U}\times\vec{V}\right|=k\left|\hat{n}\right|$

$\Rightarrow\ \left|\vec{U}\right|\left|\vec{V}\right|\sin\theta=k\times1=k$ by (1)

$\therefore\ \vec{U}\times\vec{V}=\left|\vec{U}\right|\left|\vec{V}\right|\sin\theta\,\hat{n}$

15. $P(2, 2, 0)$, $Q(-1, 0, 2)$, $R(0, 4, 3)$

$\overrightarrow{PR}=\overrightarrow{OR}-\overrightarrow{OP}$

$= (0, 4, 3) - (2, 2, 0) = (-2, 2, 3)$

$\overrightarrow{PQ}=\overrightarrow{OQ}-\overrightarrow{OP}$

$= (-1, 0, 2) - (2, 2, 0) = (-3, -2, 2)$

$\text{Area}=\frac{1}{2}\left|\overrightarrow{PR}\right|\left|\overrightarrow{PQ}\right|\sin\angle QPR$

$=\frac{1}{2}\left|\overrightarrow{PR}\times\overrightarrow{PQ}\right|$

$=\frac{1}{2}\left|\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ -2 & 2 & 3\\ -3 & -2 & 2\end{vmatrix}\right|$

$=\frac{1}{2}\left|10\vec{i}-5\vec{j}+10\vec{k}\right|$

$=\frac{1}{2}\sqrt{10^2+(-5)^2+10^2}$

= 7.5 sq. units.

16. $\vec{U}\cdot(\vec{V}\times\vec{W}) = (u_1, u_2, u_3)\cdot\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$

$= u_1\begin{vmatrix} v_2 & v_3 \\ w_2 & w_3 \end{vmatrix} - u_2\begin{vmatrix} v_1 & v_3 \\ w_1 & w_3 \end{vmatrix} + u_3\begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix}$

= cofactor expansion of a determinant

$= \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$

$\vec{U} = (-1, 4, 7)$, $\vec{V} = (6, -7, 3)$, $\vec{W} = (4, 0, 1)$

$\vec{U}\cdot(\vec{V}\times\vec{W}) = \begin{vmatrix} -1 & 4 & 7 \\ 6 & -7 & 3 \\ 4 & 0 & 1 \end{vmatrix} = 227$

17. $\vec{U} = (-1, 3, 2)$, $\vec{W} = (1, 1, -1)$; $\vec{U}\times\vec{X} = \vec{W}$

Let $\vec{X} = (x_1, x_2, x_3)$

$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -1 & 3 & 2 \\ x_1 & x_2 & x_3 \end{vmatrix} = (1, 1, -1)$

$(3x_3 - 2x_2, 2x_1 + x_3, -x_2 - x_3) = (1, 1, -1)$

$$\begin{cases} \quad -2x_2 + 3x_3 = 1 \\ 2x_1 + x_3 = 1 \\ -3x_1 - x_2 = -1 \end{cases}$$

$$\sim\left(\begin{array}{ccc|c} 0 & -2 & 3 & 1 \\ 2 & 0 & 1 & 1 \\ -3 & -1 & 0 & -1 \end{array}\right)$$

$$\sim\left(\begin{array}{ccc|c} 0 & -2 & 3 & 1 \\ 2 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right) \quad (R_1 - 3R_2 - 2R_3 \to R_3)$$

$$\Rightarrow\begin{cases} -2x_2 + 3x_3 = 1 \text{.....(1)} \\ 2x_1 + x_3 = 1 \text{..........(2)} \end{cases}$$

Let $x_3 = t$

Sub. into (1): $x_2 = \frac{3}{2}t - \frac{1}{2}$

Sub. into (2): $x_1 = -\frac{1}{2}t + \frac{1}{2}$

$$\therefore\ \vec{X} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} \\ \frac{3}{2} \\ 1 \end{pmatrix}t + \begin{pmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ 0 \end{pmatrix}, \text{ where } t \in \mathbf{R}$$

18. (*a*) By Q14, $\vec{V}\times\vec{W}\perp\vec{V}$ and $\vec{V}\times\vec{W}\perp\vec{W}$

$\therefore\ \vec{U}\times(\vec{V}\times\vec{W})\perp\vec{U}$ and $\vec{U}\times(\vec{V}\times\vec{W})\perp(\vec{V}\times\vec{W})$

$\Rightarrow\vec{U}\times(\vec{V}\times\vec{W})\perp(\vec{V}\times\vec{W})$ and $\vec{V}\times\vec{W}\perp\vec{V}$ and $\vec{V}\times\vec{W}\perp\vec{W}$

$\Rightarrow\vec{U}\times(\vec{V}\times\vec{W})$ lies in the plane determined by $\vec{V}$ and $\vec{W}$.

(*b*) $\vec{U}\times\vec{V}\perp\vec{U}$ and $\vec{U}\times\vec{V}\perp\vec{V}$

$\therefore\ (\vec{U}\times\vec{V})\times\vec{W}\perp\vec{W}$ and $(\vec{U}\times\vec{V})\times\vec{W}\perp(\vec{U}\times\vec{V})$

$\Rightarrow(\vec{U}\times\vec{V})\times\vec{W}\perp(\vec{U}\times\vec{V})$ and $\vec{U}\times\vec{V}\perp\vec{U}$ and $\vec{U}\times\vec{V}\perp\vec{V}$

$\Rightarrow(\vec{U}\times\vec{V})\times\vec{W}$ lies in the plane determined by $\vec{U}$ and $\vec{V}$.

19. $\vec{x}\times(\vec{y}\times\vec{z}) = (x_1, x_2, x_3)\times(y_2 z_3 - y_3 z_2, y_3 z_1 - y_1 z_3, y_1 z_2 - y_2 z_1)$

$$=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ x_1 & x_2 & x_3\\ y_2z_3-y_3z_2 & y_3z_1-y_1z_3 & y_1z_2-y_2z_1\end{vmatrix}$$

$$= [x_2(y_1 z_2 - y_2 z_1) - x_3(y_3 z_1 - y_1 z_3)]\vec{i} + [x_3(y_2 z_3 - y_3 z_2) - x_1(y_1 z_2 - y_2 z_1)]\vec{j} + [x_1(y_3 z_1 - y_1 z_3) - x_2(y_2 z_3 - y_3 z_2)]\vec{k}$$

$$(\vec{x}\cdot\vec{z})\vec{y} - (\vec{x}\cdot\vec{y})\vec{z} = (x_1 z_1 + x_2 z_2 + x_3 z_3)(y_1, y_2, y_3) - (x_1 y_1 + x_2 y_2 + x_3 y_3)(z_1, z_2, z_3)$$

$$= [(x_1 z_1 + x_2 z_2 + x_3 z_3)y_1 - (x_1 y_1 + x_2 y_2 + x_3 y_3)z_1]\vec{i} + [(x_1 z_1 + x_2 z_2 + x_3 z_3)y_2 - (x_1 y_1 + x_2 y_2 + x_3 y_3)z_2]\vec{j} + [(x_1 z_1 + x_2 z_2 + x_3 z_3)y_3 - (x_1 y_1 + x_2 y_2 + x_3 y_3)z_3]\vec{k}$$

$\therefore\ \vec{x}\times(\vec{y}\times\vec{z}) = (\vec{x}\cdot\vec{z})\vec{y} - (\vec{x}\cdot\vec{y})\vec{z}$

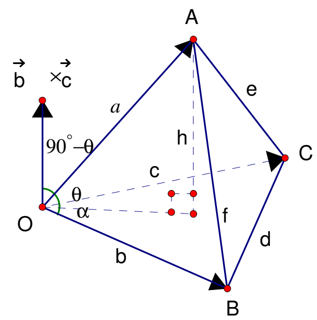
20. As shown in the figure, let $\angle BOC = \alpha$, $\overrightarrow{OA}=\vec{a}$, $\overrightarrow{OB}=\vec{b}$, $\overrightarrow{OC}=\vec{c}$.

Suppose OA makes an angle θ with the plane OBC.

The height of the tetrahedron from A onto the plane OBC is h.

$\vec{b}\times\vec{c}$ is perpendicular to the plane OBC.

Also, $\vec{b}\times\vec{c}$ makes an angle $90° - \theta$ with OA.



volume $V=\frac{1}{3}\text{base area}\times\text{height}$

$$V=\frac{1}{3}\cdot\frac{1}{2}bc\sin\alpha\times h$$

$$=\frac{1}{6}\cdot\left|\vec{b}\times\vec{c}\right|\times h$$

$$=\frac{1}{6}\left|\vec{b}\times\vec{c}\right|\times h$$

$$=\frac{1}{6}\left|\vec{b}\times\vec{c}\right|\times a\cos\left(90^\circ-\theta\right)$$

$$=\frac{1}{6}\vec{a}\cdot\left(\vec{b}\times\vec{c}\right)$$

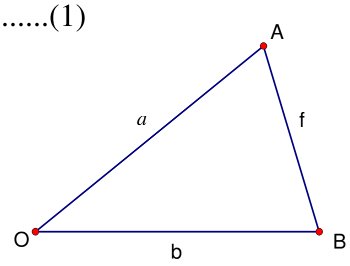
$$=\frac{1}{6}\begin{vmatrix}a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\\ c_1 & c_2 & c_3\end{vmatrix}$$, where $\vec{a} = (a_1, a_2, a_3)$, $\vec{b} = (b_1, b_2, b_3)$, $\vec{c} = (c_1, c_2, c_3)$

$$V^2 = \frac{1}{36}\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}\cdot\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$= \frac{1}{36}\begin{vmatrix} a_1^2+a_2^2+a_3^2 & a_1b_1+a_2b_2+a_3b_3 & a_1c_1+a_2c_2+a_3c_3 \\ b_1a_1+b_2a_2+b_3a_3 & b_1^2+b_2^2+b_3^2 & b_1c_1+b_2c_2+b_3c_3 \\ a_1c_1+a_2c_2+a_3c_3 & b_1c_1+b_2c_2+b_3c_3 & c_1^2+c_2^2+c_3^2 \end{vmatrix}$$

$$= \frac{1}{36}\begin{vmatrix} \vec{a}\cdot\vec{a} & \vec{a}\cdot\vec{b} & \vec{a}\cdot\vec{c} \\ \vec{a}\cdot\vec{b} & \vec{b}\cdot\vec{b} & \vec{b}\cdot\vec{c} \\ \vec{a}\cdot\vec{c} & \vec{b}\cdot\vec{c} & \vec{c}\cdot\vec{c} \end{vmatrix}$$

$$= \frac{1}{36}\begin{vmatrix} a^2 & ab\cos\angle AOB & ac\cos\angle AOC \\ ab\cos\angle AOB & b^2 & bc\cos\angle BOC \\ ac\cos\angle AOC & bc\cos\angle BOC & c^2 \end{vmatrix} \quad \text{................(1)}$$



Using cosine law, $f^2 = a^2 + b^2 - 2ab\cos\angle AOB$

$\Rightarrow ab\cos\angle AOB = \frac{1}{2}(a^2 + b^2 - f^2)$

Similarly, $ac\cos\angle AOC = \frac{1}{2}(a^2 + c^2 - e^2)$

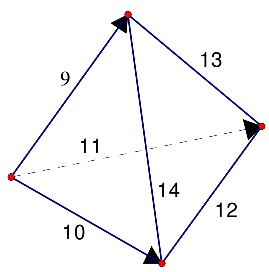
and $bc\cos\angle BOC = \frac{1}{2}(b^2 + c^2 - d^2)$

Sub. into (1): $V^2 = \frac{1}{36}\begin{vmatrix} a^2 & \frac{1}{2}(a^2+b^2-f^2) & \frac{1}{2}(a^2+c^2-e^2) \\ \frac{1}{2}(a^2+b^2-f^2) & b^2 & \frac{1}{2}(b^2+c^2-d^2) \\ \frac{1}{2}(a^2+c^2-e^2) & \frac{1}{2}(b^2+c^2-d^2) & c^2 \end{vmatrix}$

$$V^2 = \frac{1}{288}\begin{vmatrix} 2a^2 & a^2+b^2-f^2 & a^2+c^2-e^2 \\ a^2+b^2-f^2 & 2b^2 & b^2+c^2-d^2 \\ a^2+c^2-e^2 & b^2+c^2-d^2 & 2c^2 \end{vmatrix}$$

$$V = \frac{1}{12\sqrt{2}}\sqrt{\begin{vmatrix} 2a^2 & a^2+b^2-f^2 & a^2+c^2-e^2 \\ a^2+b^2-f^2 & 2b^2 & b^2+c^2-d^2 \\ a^2+c^2-e^2 & b^2+c^2-d^2 & 2c^2 \end{vmatrix}}$$

As an example, $a = 9$, $b = 10$, $c = 11$, $d = 12$, $e = 13$, $f = 14$,



then $V = \frac{1}{12\sqrt{2}}\sqrt{\begin{vmatrix} 2\times9^2 & 9^2+10^2-14^2 & 9^2+11^2-13^2 \\ 9^2+10^2-14^2 & 2\times10^2 & 10^2+11^2-12^2 \\ 9^2+11^2-13^2 & 10^2+11^2-12^2 & 2\times11^2 \end{vmatrix}}$

$$= \frac{1}{12\sqrt{2}}\sqrt{\begin{vmatrix} 162 & -15 & 33 \\ -15 & 200 & 77 \\ 33 & 77 & 242 \end{vmatrix}} = \frac{1}{12\sqrt{2}}\sqrt{6531822} = 150.6 \text{ cubic units.}$$