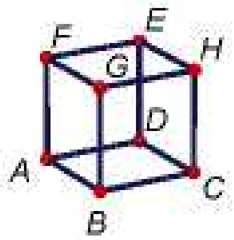
Vectors Exercise 2

Created by Mr. Francis Hung Last updated: 21 April 2011

1. Describe the surface whose equation is given by $x^2 + y^2 + z^2 - y = 0$.
   By using the method of completing the square,
   $$x^2 + \left(y - \frac{1}{2}\right)^2 + z^2 = \left(\frac{1}{2}\right)^2$$
   $$\{(x, y, z): x^2 + \left(y - \frac{1}{2}\right)^2 + z^2 = \left(\frac{1}{2}\right)^2\}$$
   The solution set represent the surface of a sphere whose centre is $\left(0, \frac{1}{2}, 0\right)$ and radius $\frac{1}{2}$.

2. Find the acute angle formed by two diagonals of a cube.
   Let the cube be $ABCDEFGH$ as shown.
   Without loss of generality, assume the length of side = 1.
   Introduce a rectangular coordinate system with
   AB as x-axis, AD as y-axis and AF as z-axis.



   $A(0, 0, 0)$, $B(1, 0, 0)$, $C(1, 1, 0)$, $D(0, 1, 0)$, $E(0, 1, 1)$, $F(0, 0, 1)$, $G(1, 0, 1)$, $H(1, 1, 1)$
   The two diagonals are $\overrightarrow{AH}$ and $\overrightarrow{CF}$. Let the angle between them be θ.
   $\overrightarrow{AH} = (1, 1, 1)$ and $\overrightarrow{CF} = (0, 0, 1) - (1, 1, 0) = (-1, -1, 1)$
   $$\overrightarrow{AH} \cdot \overrightarrow{CF} = \left\|\overrightarrow{AH}\right\| \cdot \left\|\overrightarrow{CF}\right\| \cos\theta$$
   $$(1, 1, 1)\cdot(-1, -1, 1) = \|(1,1,1)\|\|(-1,-1,1)\| \cos\theta$$
   $$-1 - 1 + 1 = \sqrt{3}\cdot\sqrt{3}\cos\theta$$
   $$\cos\theta = -\frac{1}{3}$$
   $\theta = 109.5°$
   The acute angle between them is $180° - 109.5° = 70.5°$

3. Determine if it is true that for any vectors a, b, c such that $a \neq 0$ and $a\cdot b = a\cdot c$, then $b = c$.
   $a \neq 0$ and $a\cdot b = a\cdot c$
   $\Leftrightarrow a\cdot(b - c) = 0$
   $\Leftrightarrow a \perp b - c$
   It is not true in general that $a \neq 0$ and $a\cdot b = a\cdot c$, then $b = c$.
   Counter-example: $a = (1, 0, 0)$, $b = (0, 0, 0)$ and $c = (0, 1, 0)$
   Then $a\cdot b = 0 = a\cdot c$ but $b \neq c$.

4. Determine if it is true that for any three vectors a, b, c, $a\times(b\times c) = (a\times b)\times c$.
   $b\times c \perp b$ and $b\times c \perp c$
   $\therefore a\times(b\times c)\perp a$ and $a\times(b\times c)\perp b\times c$
   $\Rightarrow a\times(b\times c)\perp b\times c$ and $b\times c \perp b$ and $b\times c \perp c$
   $\Rightarrow a\times(b\times c)\perp$ lies in the plane determined by b and c.
   $a\times b \perp a$ and $a\times b \perp b$
   $\therefore (a\times b)\times c \perp c$ and $(a\times b)\times c \perp a\times b$
   $\Rightarrow (a\times b)\times c \perp a\times b$ and $a\times b \perp a$ and $a\times b \perp c$
   $\Rightarrow (a\times b)\times c \perp$ lies in the plane determined by a and b.
   It is not true in general that $a\times(b\times c) = (a\times b)\times c$.
   Counter-example: $i\times[j\times(i + j)] = i\times(-k) = j$
   $(i\times j)\times(i + j) = k \times (i + j) = j - i$
   $\therefore i\times[j\times(i + j)] \neq (i\times j)\times(i + j)$

5. Show that in the 3-space the distance d from a point P to the line through points A and B can be expressed as $d=\dfrac{\left\|\overrightarrow{AP}\times\overrightarrow{AB}\right\|}{\left\|\overrightarrow{AB}\right\|}$.

$\dfrac{\left\|\overrightarrow{AP}\times\overrightarrow{AB}\right\|}{\left\|\overrightarrow{AB}\right\|}=\dfrac{\left\|\overrightarrow{AP}\right\|\left\|\overrightarrow{AB}\right\|\sin\theta}{\left\|\overrightarrow{AB}\right\|}$, where θ is the angle between $\overrightarrow{AP}$ and $\overrightarrow{AB}$.

$=\left\|\overrightarrow{AP}\right\|\sin\theta$

$=d$

6. Find the parametric equation of the line through the origin that is parallel to the line given by $x = 2t$, $y = -1 + t$, $z = 2$.

$x = 2t$, $y = t$, $z = 2$

7. Let L be the line that passes through the point (x_0, y_0, z_0) and is parallel to $v = (a, b, c)$, where a, b, c are nonzero. Show that a point (x, y, z) lies on the line L if and only if

$$\frac{x-x_0}{a}=\frac{y-y_0}{b}=\frac{z-z_0}{c}.$$

These equations, which are called the symmetric equations of L, provides a nonparametric representation of L.

Let $A = (x_0, y_0, z_0)$, $B = (x, y, z)$

If B lies on the line L,

$\overrightarrow{BA}$ is parallel to $\vec{v}$ $\Rightarrow (x-x_0)\vec{i}+(y-y_0)\vec{j}+(z-z_0)\vec{k}=m\vec{v}$ for some real constant m

$(x - x_0, y - y_0, z - z_0) = m(a, b, c)$

$m=\dfrac{x-x_0}{a}=\dfrac{y-y_0}{b}=\dfrac{z-z_0}{c}$.

Conversely, if $\dfrac{x-x_0}{a}=\dfrac{y-y_0}{b}=\dfrac{z-z_0}{c}$

Let $m=\dfrac{x-x_0}{a}=\dfrac{y-y_0}{b}=\dfrac{z-z_0}{c}$ $\Rightarrow (x - x_0, y - y_0, z - z_0) = m(a, b, c)$

$(x-x_0)\vec{i}+(y-y_0)\vec{j}+(z-z_0)\vec{k}=m\vec{v}$

∴ The vector $\overrightarrow{BA}$ is parallel to $\vec{v}$

the point (x_0, y_0, z_0) lies on L.