Exercise on Vector Space

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Let **V** be a non-empty set. **V** is a **VECTOR SPACE** over **R** if it satisfies the following properties:

Define the binary operations '+': $\mathbf{V}\times\mathbf{V} \to \mathbf{V}$ and '·': $\mathbf{R}\times\mathbf{V} \to \mathbf{V}$

[A1] For any $\vec{u}$, $\vec{v} \in \mathbf{V}$, $\vec{u}+\vec{v} \in \mathbf{V}$ (additive closure)

[A2] For any $\vec{u}$, $\vec{v}$, $\vec{w} \in \mathbf{V}$, $(\vec{u}+\vec{v})+\vec{w} = \vec{u}+(\vec{v}+\vec{w})$ (associative)

[A3] For any $\vec{u}$, $\vec{v} \in \mathbf{V}$, $\vec{u}+\vec{v} = \vec{v}+\vec{u}$ (additive commutative)

[A4] There is an element $\vec{0} \in \mathbf{V}$ such that for all $\vec{v} \in \mathbf{V}$, $\vec{0}+\vec{v} = \vec{v}$ (existence of zero)

[A5] For all $\vec{v} \in \mathbf{V}$, there is a vector $(-\vec{v}) \in \mathbf{V}$ such that $(-\vec{v}) + \vec{v} = \vec{0}$
(Existence of additive inverse)

[M1] For any $a \in \mathbf{R}$, and any $\vec{v} \in \mathbf{V}$, $a\vec{v} \in \mathbf{V}$ (multiplicative closure)

[M2] For any $a, b \in \mathbf{R}$, and any $\vec{v} \in \mathbf{V}$, $a(b\vec{v}) = (ab)\vec{v}$ (associative)

[M3] For any $a, b \in \mathbf{R}$, and $\vec{u}$, $\vec{v} \in \mathbf{V}$, $(a+b)\vec{v} = a\vec{v} + b\vec{v}$; $a(\vec{u}+\vec{v}) = a\vec{u} + a\vec{v}$ (distributive)

[M4] For all $\vec{v} \in \mathbf{V}$, $1 \in \mathbf{R}$ such that $1\vec{v} = \vec{v}$ (multiplication identity)

Elements in **V** are called **vectors**; elements in **R** are **scalars**.

Q1 Let $\mathbf{R}_+^2$ be the set of all order pair of positive numbers. Define addition and multiplication in $\mathbf{R}_+^2$ as follows: For all $(a_1, a_2), (b_1, b_2) \in \mathbf{R}_+^2$ and $t \in \mathbf{R}$,

$(a_1, a_2) + (b_1, b_2) = (a_1b_1, a_2b_2)$ and $t(a_1, a_2) = (a_1^t, a_2^t)$.

Show that $\mathbf{R}_+^2$ with these operations is a vector space over **R**.

Definitions: Let $\mathbf{R}^n$ be the vector space over **R**. If $\vec{u} = (u_1, u_2, \ldots, u_n)$ and $\vec{v} = (v_1, v_2, \ldots, v_n)$. Then the **Euclidean inner product** is $\vec{u}\cdot\vec{v} = u_1v_1 + u_2v_2 + \ldots + u_nv_n$

The **Euclidean norm** (or **Euclidean length**) of $\vec{u}$ is: $|\vec{u}| = \sqrt{\vec{u}\cdot\vec{u}} = \sqrt{u_1^2 + u_2^2 + \cdots + u_n^2}$

The **Euclidean distance** between $\vec{u}$ and $\vec{v}$ is: $\mathrm{D}(\vec{u}, \vec{v}) = |\vec{u}-\vec{v}|$

Q2 Let $\vec{u}_1 = (-1, 3, 2, 0)$, $\vec{u}_2 = (2, 0, 4, -1)$, $\vec{u}_3 = (7, 1, 1, 4)$ and $\vec{u}_4 = (6, 3, 1, 2)$. Find scalars $c_1\vec{u}_1 + c_2\vec{u}_2 + c_3\vec{u}_3 + c_4\vec{u}_4 = (0, 5, 6, -3)$.

Q3 Show that there do not exist scalars c_1, c_2, c_3 such that

$c_1(1, 0, -2, 1) + c_2(2, 0, 1, 2) + c_3(1, -2, 2, 3) = (1, 0, 1, 0)$.

Q4 Let $\vec{e}_1$, $\vec{e}_2$ be two perpendicular vectors of length 1 in R^3 and $\vec{V}$ is any vector.

Define $|\vec{V}|$ = the magnitude of the vector $\vec{V}$.

Prove that for any real numbers b_1 b_2:

$|\vec{V} - (\vec{V}\cdot\vec{e}_1)\vec{e}_1 - (\vec{V}\cdot\vec{e}_2)\vec{e}_2| \le |\vec{V} - b_1\vec{e}_1 - b_2\vec{e}_2|$

Q5 (a) Find two vectors in $\mathbf{R}^2$ with Euclidean norm 1 whose Euclidean inner products with $(-2, 4)$ are zero.

(b) Show that there are infinitely many vectors in $\mathbf{R}^3$ with Euclidean norm 1 whose inner product with $(-1, 7, 2)$ is zero.

Q6 Find the Euclidean distance between $\vec{u}$ and $\vec{v}$ when:

(a) $\vec{u} = (1, 1, -1)$, $\vec{v} = (2, 6, 0)$

(b) $\vec{u} = (6, 0, 1, 3, 0)$, $\vec{v} = (-1, 4, 2, 8, 3)$

Definition: Let **V** be a vector space over **R**. A subset **W** of **V** is called a subspace of **V** if **W** is itself a vector space over **R**. For a normal vector space **V**, there are at least two vector subspaces. $\{\vec{0}\}$ (zero subspace) and **V** itself.

Theorem: Let **W** be a non-empty subset of a vector space **V**, then **W** is a subspace of **V** if and only if the following conditions hold:

(a) If $\vec{u}$ and $\vec{v} \in$ **W**, then $\vec{u} + \vec{v} \in$ **W**.

(b) If $k \in$ **R** and $\vec{u} \in$ **W**, then $k\vec{u} \in$ **W**.

Q7 Show that the set **W** of all 2×2 matrices having zeros on the main diagonal is a subspace of the vector space $\mathbf{M}_{22}$ of all 2×2 matrices.

Q8 Consider a system of m linear equations in n unknowns:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n & = & b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n & = & b_2 \\ \cdots & \cdots & \cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n & = & b_m \end{cases}$$

If $b_1 = b_2 = \ldots = b_m = 0$, the system is known as a **homogeneous**.

Show that the set of solutions of a homogeneous system is a vector subspace of $\mathbf{R}^n$.

Q9 Let $\mathbf{V} = \mathbf{R}^3$. Determine which of the following are subspaces of $\mathbf{R}^3$.

(a) all vectors of the form $(a, 0, 0)$.

(b) all vectors of the form $(a, 1, 1)$.

(c) all vectors of the form (a, b, c) where $b = a + c$.

(d) all vectors of the form (a, b, c) where $b = a + c + 1$

Q10 Let $\mathbf{V} = \mathbf{M}_{22}$, the set of all 2×2 matrices. Determine which of the following are subspaces of $\mathbf{M}_{22}$.

(a) all matrices of the form $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, where a, b, c and d are integers.

(b) all matrices of the form $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, where $a + d = 0$.

(c) all 2×2 matrices A such that $A = A^t$. (the transpose of the matrix A)

(d) all 2×2 matrices A such that $\det(A) = 0$.

Definition: A vector $\vec{w}$ is called a linear combination of vectors $\vec{v}_1$, $\vec{v}_2$, … , $\vec{v}_r$ if it can be expressed in the form $\vec{w} = k_1\vec{v}_1 + k_2\vec{v}_2 + \dots + k_r\vec{v}_r$, where $k_1, k_2, \dots, k_r$ are scalars.

Q11 Express the followings as linear combinations of $\vec{p}_1 = 2+x+4x^2$, $\vec{p}_2 = 1-x+3x^2$, $\vec{p}_3 = 3+2x+5x^2$

(a) $5 + 9x + 5x^2$

(b) $2 + 6x^2$

(c) $2 + 2x + 3x^2$

Definitions: Let { $\vec{v}_1$, $\vec{v}_2$, … , $\vec{v}_r$ } be a set of vectors in **V**.

Let **W** = {all linear combination of $\vec{v}_1$, $\vec{v}_2$, … , $\vec{v}_r$ } = $\{k_1\vec{v}_1 + k_2\vec{v}_2 + \dots + k_r\vec{v}_r : k_1, k_2, \dots, k_r \in \mathbf{R}\}$

W is called the linear span of { $\vec{v}_1$, $\vec{v}_2$, … , $\vec{v}_r$ }. Denote **W** = $<\vec{v}_1, \vec{v}_2, \dots, \vec{v}_r>$

If **W** = **V**, then we called { $\vec{v}_1$, $\vec{v}_2$, … , $\vec{v}_r$ } spans **V**.

Q12 Determine whether $\vec{v}_1 = (1, 1, 2)$, $\vec{v}_2 = (1, 0, 1)$, $\vec{v}_3 = (2, 1, 3)$ span $\mathbf{R}^3$.

Q13 Determine which of the following lie in the space spanned by $\vec{f} = \cos^2 x$ and $\vec{g} = \sin^2 x$

(a) $\cos 2x$

(b) $3 + x^2$

(c) 1

(d) $\sin x$

Q14 Let $\mathbf{P}_2$ be the set of real polynomials of degree ≤ 2. Determine if the following polynomials spans $\mathbf{P}_2$:
$\vec{p}_1 = 1 + 2x - x^2$, $\vec{p}_2 = 3 + x^2$, $\vec{p}_3 = 5 + 4x - x^2$, $\vec{p}_4 = -2 + 2x - 2x^2$

Q15 Find an equation for the plane spanned by the vectors $\vec{u} = (1, 1, -1)$, $\vec{v} = (2, 3, 5)$.

Definition: If **S** = { $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_r$ } is a set of vectors, then the vector equation $k_1\vec{v}_1 + k_2\vec{v}_2 + \dots + k_r\vec{v}_r = 0$ has at least one solution, namely $k_1 = 0$, $k_2 = 0$, …, $k_r = 0$. If this is the only solution, then **S** is called a **linear independent set**. If there are other solutions, then **S** is called a **linear dependent set**.

Q16 Which of the following sets of vectors in $\mathbf{R}^3$ are linear dependent?

(a) (2, –1, 4), (3, 6, 2), (2, 10, –4)

(b) (3, 1, 1), (2, –1, 5), (4, 0, –3)

(c) (1, 3, 3), (0, 1, 4), (5, 6, 3), (7, 2, –1)

Q17 Which of the following sets of vectors in $\mathbf{P}_2$ are linear dependent?

(a) $2 - x + 4x^2$, $3 + 6x + 2x^2$, $2 + 10x - 4x^2$.

(b) $1 + 3x + 3x^2$, $x + 4x^2$, $5 + 6x + 3x^2$, $7 + 2x - x^2$

Q18 Let **V** be the vector space of all real valued functions defined on the entire real line. Which of the following sets of vectors in **V** are linear dependent?

(a) $2, 4\sin^2 x, \cos^2 x$ (b) $x, \cos x$

(c) $1, \sin x, \sin 2x$ (d) $\cos 2x, \sin^2 x, \cos^2 x$

(e) $(1 + x)^2, x^2 + 2x, 3$ (f) $0, x, x^2$

Q19 In each part determine whether the three vectors lie in a plane pass through the origin.

(a) $\vec{v}_1 = (1, 0, -2), \vec{v}_2 = (3, 1, 2), \vec{v}_3 = (1, -1, 0)$

(b) $\vec{v}_1 = (2, -1, 4), \vec{v}_2 = (4, 2, 3), \vec{v}_3 = (2, 7, -6)$

Q20 In each part determine whether the three vectors lie on the same line.

(a) $\vec{v}_1 = (3, -6, 9),\ \vec{v}_2 = (2, -4, 6),\ \vec{v}_3 = (1, 1, 1)$

(b) $\vec{v}_1 = (2, -1, 4),\ \vec{v}_2 = (4, 2, 3),\ \vec{v}_3 = (2, 7, -6)$

(c) $\vec{v}_1 = (4, 6, 8),\ \vec{v}_2 = (2, 3, 4),\ \vec{v}_3 = (-2, -3, -4)$

Q21 For which real values of t do the following vectors form a linear dependent set in $\mathbf{R}^3$?

$$\vec{v}_1 = (t,\ -\frac{1}{2},\ -\frac{1}{2}),\ \vec{v}_2 = (-\frac{1}{2}, t,\ -\frac{1}{2}),\ \vec{v}_3 = (-\frac{1}{2},\ -\frac{1}{2}, t)$$

Q22 Let $\mathbf{S} = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_r\}$ be a set of vectors in a vector space $\mathbf{V}$. Show that if one of the vectors is zero, then $\mathbf{S}$ is linear dependent.

Q23 If $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is a linear independent set of vectors, show that the following sets are also linear independent:

(a) $\{\vec{v}_2\}$

(b) $\{\vec{v}_1, \vec{v}_3\}$

(c) $\{\vec{v}_1, \vec{v}_1+\vec{v}_2, \vec{v}_1+\vec{v}_2+\vec{v}_3\}$

Q24 If $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_r\}$ is a linear independent set of vectors, show that every subset of $\mathbf{S}$ with one or more vectors is also linear independent.

Q25 Prove that the space spanned by the two vectors in $\mathbf{R}^3$ is either a line through the origin, a plane through the origin, or the origin itself.

Q26 Let $\mathbf{V}$ be the vector space of real-valued functions defined on the entire real line. If $\vec{f}$, $\vec{g}$ and $\vec{h}$ are vectors in $\mathbf{V}$ that are twice differentiable, then the function $\vec{w} = w(x)$ defined by

$$w(x) = \begin{vmatrix} f(x) & g(x) & h(x) \\ f'(x) & g'(x) & h'(x) \\ f''(x) & g''(x) & h''(x) \end{vmatrix}$$

$\vec{w}$ is called the **Wronskian** of $\vec{f}$, $\vec{g}$ and $\vec{h}$. Prove that $\vec{f}$, $\vec{g}$ and $\vec{h}$ form a linear independent set if the **Wronskian** is not a zero vector in $\mathbf{V}$. (i.e. $w(x)$ is not identically zero.)

Q27 Use the **Wronskian** to show that the following sets of vector are linear independent.

(a) $1, x, e^x$

(b) $\sin x, \cos x, x \sin x$

(c) $e^x, x e^x, x^2 e^x$

(d) $1, x, x^2$

Q1 Let $\mathbf{R}_+^2$ be the set of all order pair of positive numbers. Define addition and multiplication in $\mathbf{R}_+^2$ as follows: For all $(a_1, a_2), (b_1, b_2) \in \mathbf{R}_+^2$ and $t \in \mathbf{R}$,

$(a_1, a_2) + (b_1, b_2) = (a_1b_1, a_2b_2)$ and $t(a_1, a_2) = (a_1^t, a_2^t)$.

Show that $\mathbf{R}_+^2$ with these operations is a vector space over $\mathbf{R}$.

[A1] Let $\vec{u} = (a_1, a_2)$, $\vec{v} = (b_1, b_2) \in \mathbf{R}_+^2$, where $a_1, a_2, b_1, b_2 > 0$

$\vec{u} + \vec{v} = (a_1, a_2) + (b_1, b_2) = (a_1b_1, a_2b_2)$, which is an order pair, $a_1b_1 > 0$ and $a_2b_2 > 0$

$\therefore\ \vec{u} + \vec{v} \in \mathbf{R}_+^2$

[A2] Let $\vec{w} = (c_1, c_2)$, where c_1, c_2 are positive real numbers.

$(\vec{u} + \vec{v}) + \vec{w} = [(a_1, a_2) + (b_1, b_2)] + (c_1, c_2)$

$= (a_1b_1, a_2b_2) + (c_1, c_2)$

$= (a_1b_1c_1, a_2b_2c_2)$

$\vec{u} + (\vec{v} + \vec{w}) = (a_1, a_2) + [(b_1, b_2) + (c_1, c_2)]$

$= (a_1, a_2) + (b_1c_1, b_2c_2)$

$= (a_1b_1c_1, a_2b_2c_2)$

$\therefore (\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$

[A3] $\vec{u} + \vec{v} = (a_1, a_2) + (b_1, b_2) = (a_1b_1, a_2b_2)$

$\vec{v} + \vec{u} = (b_1, b_2) + (a_1, a_2) = (b_1a_1, b_2a_2)$

For real numbers $a_1b_1 = b_1a_1$, $a_2b_2 = b_2a_2$

$\therefore\ \vec{u} + \vec{v} = \vec{v} + \vec{u}$

[A4] Let $\vec{0} = (1, 1)$

$\vec{0} + \vec{v} = (1, 1) + (b_1, b_2)$

$= (1\times b_1, 1\times b_2)$

$= (b_1, b_2) = \vec{v}$

[A5] $\forall \vec{v} = (b_1, b_2) \in \mathbf{R}_+^2$, b_1 and $b_2 \neq 0$

Let $(-\vec{v}) = \left(\frac{1}{b_1}, \frac{1}{b_2}\right) \in \mathbf{R}_+^2$

$(-\vec{v}) + \vec{v} = \left(\frac{1}{b_1}, \frac{1}{b_2}\right) + (b_1, b_2)$

$= \left(\frac{1}{b_1}\times b_1, \frac{1}{b_2}\times b_2\right)$

$= (1, 1)$

$= \vec{0}$

[M1] Let $a \in \mathbf{R}$, $a\vec{v} = a(b_1, b_2) = (b_1^a, b_2^a)$

$b_1, b_2 > 0 \Rightarrow b_1^a > 0, b_2^a > 0$

$\Rightarrow a\vec{v} \in \mathbf{R}_+^2$

[M2] $b \in \mathbf{R}$, $a(b\vec{v}) = a[b(b_1, b_2)] = a(b_1^b, b_2^b) = (b_1^{ba}, b_2^{ba})$

$(ab)\vec{v} = ab\,(b_1, b_2) = (b_1^{ab}, b_2^{ab})$

$\therefore a(b\vec{v}) = (ab)\vec{v}$

[M3] $(a+b)\vec{v} = (a+b)(b_1, b_2) = (b_1^{a+b}, b_2^{a+b})$

$a\vec{v} + b\vec{v} = a(b_1, b_2) + b(b_1, b_2)$

$= (b_1^a, b_2^a) + (b_1^b, b_2^b)$

$= (b_1^a \times b_1^b, b_2^a \times b_2^b)$

$= (b_1^{a+b}, b_2^{a+b})$

$\therefore (a+b)\vec{v} = a\vec{v} + b\vec{v}$

$a(\vec{u} + \vec{v}) = a[(a_1, a_2) + (b_1, b_2)]$

$= a(a_1 b_1, a_2 b_2)$

$= ((a_1 b_1)^a, (a_2 b_2)^a)$

$a\vec{u} + a\vec{v} = a(a_1, a_2) + a(b_1, b_2)$

$= (a_1^a, a_2^a) + (b_1^a, b_2^a)$

$= (a_1^a \times b_1^a, a_2^a \times b_2^a)$

$= ((a_1 b_1)^a, (a_2 b_2)^a)$

$\therefore a(\vec{u} + \vec{v}) = a\vec{u} + a\vec{v}$

[M4] $1\vec{v} = 1(b_1, b_2)$

$= (b_1^1, b_2^1) = \vec{v}$

$\therefore 1\vec{v} = \vec{v}$

$\therefore \mathbf{R}_+^2$ is a vector space under the special addition and under the special scalar multiplication.

Q2 Let $\vec{u}_1 = (-1, 3, 2, 0)$, $\vec{u}_2 = (2, 0, 4, -1)$, $\vec{u}_3 = (7, 1, 1, 4)$ and $\vec{u}_4 = (6, 3, 1, 2)$. Find scalars $c_1\vec{u}_1 + c_2\vec{u}_2 + c_3\vec{u}_3 + c_4\vec{u}_4 = (0, 5, 6, -3)$.

$$c_1\begin{pmatrix}-1\\3\\2\\0\end{pmatrix} + c_2\begin{pmatrix}2\\0\\4\\-1\end{pmatrix} + c_3\begin{pmatrix}7\\1\\1\\4\end{pmatrix} + c_4\begin{pmatrix}6\\3\\1\\2\end{pmatrix} = \begin{pmatrix}0\\5\\6\\-3\end{pmatrix}$$

$$\left(\begin{array}{cccc|c}-1&2&7&6&0\\3&0&1&3&5\\2&4&1&1&6\\0&-1&4&2&-3\end{array}\right) \sim \begin{matrix}\\R_2+3R_1\\R_3+2R_1\\\\\end{matrix}\left(\begin{array}{cccc|c}-1&2&7&6&0\\0&6&22&21&5\\0&8&15&13&6\\0&-1&4&2&-3\end{array}\right) \sim \begin{matrix}\\R_4\\\\R_2\end{matrix}\left(\begin{array}{cccc|c}-1&2&7&6&0\\0&-1&4&2&-3\\0&8&15&13&6\\0&6&22&21&5\end{array}\right)$$

$$\sim \begin{matrix}\\\\R_3+8R_2\\R_4+6R_2\end{matrix}\left(\begin{array}{cccc|c}-1&2&7&6&0\\0&-1&4&2&-3\\0&0&47&29&-18\\0&0&46&33&-13\end{array}\right) \sim \begin{matrix}\\\\\\47R_4-46R_3\end{matrix}\left(\begin{array}{cccc|c}-1&2&7&6&0\\0&-1&4&2&-3\\0&0&47&29&-18\\0&0&0&217&217\end{array}\right)$$

$$\sim \begin{matrix}\\\\\\\frac{R_4}{217}\end{matrix}\left(\begin{array}{cccc|c}-1&2&7&6&0\\0&-1&4&2&-3\\0&0&47&29&-18\\0&0&0&1&1\end{array}\right) \sim \begin{matrix}R_1-6R_4\\R_2-2R_4\\R_3-29R_4\\\\\end{matrix}\left(\begin{array}{cccc|c}-1&2&7&0&-6\\0&-1&4&0&-5\\0&0&47&0&-47\\0&0&0&1&1\end{array}\right) \sim \begin{matrix}\\\\\frac{R_3}{47}\\\\\end{matrix}\left(\begin{array}{cccc|c}-1&2&7&0&-6\\0&-1&4&0&-5\\0&0&1&0&-1\\0&0&0&1&1\end{array}\right)$$

$$\sim \begin{matrix}7R_3-R_1\\4R_3-R_2\\\\\\\end{matrix}\left(\begin{array}{cccc|c}1&-2&0&0&-1\\0&1&0&0&1\\0&0&1&0&-1\\0&0&0&1&1\end{array}\right) \sim \begin{matrix}R_1+2R_2\\\\\\\\\end{matrix}\left(\begin{array}{cccc|c}1&0&0&0&1\\0&1&0&0&1\\0&0&1&0&-1\\0&0&0&1&1\end{array}\right).$$

$\therefore c_1 = 1, c_2 = 1, c_3 = -1, c_4 = 1$

Q3 Show that there do not exist scalars c_1, c_2, c_3 such that

$$c_1(1, 0, -2, 1) + c_2(2, 0, 1, 2) + c_3(1, -2, 2, 3) = (1, 0, 1, 0).$$

$$c_1\begin{pmatrix}1\\0\\-2\\1\end{pmatrix}+c_2\begin{pmatrix}2\\0\\1\\2\end{pmatrix}+c_3\begin{pmatrix}1\\-2\\2\\3\end{pmatrix}=\begin{pmatrix}1\\0\\1\\0\end{pmatrix}$$

$$\left(\begin{array}{ccc|c}1&2&1&1\\0&0&-2&0\\-2&1&2&1\\1&2&3&0\end{array}\right)\sim\begin{matrix}\\-\frac{R_2}{2}\\R_3+2R_1\\R_4-R_1\end{matrix}\left(\begin{array}{ccc|c}1&2&1&1\\0&0&1&0\\0&5&4&3\\0&0&2&-1\end{array}\right)\sim\begin{matrix}\\ \\ \\R_4-2R_2\end{matrix}\left(\begin{array}{ccc|c}1&2&1&1\\0&0&1&0\\0&5&4&3\\0&0&0&-1\end{array}\right)$$

Equation (iv) $0 + 0 + 0 = -1$, which is a contradiction. $\therefore$ no solution.

Q4 Let $\vec{e}_1$, $\vec{e}_2$ be two perpendicular vectors of length 1 in R^3 and $\vec{V}$ is any vector.

Define $|\vec{V}|$ = the length of the vector $\vec{V}$.

Prove that for any real numbers b_1 b_2:

$$|\vec{V}-(\vec{V}\cdot\vec{e}_1)\vec{e}_1-(\vec{V}\cdot\vec{e}_2)\vec{e}_2|\le|\vec{V}-b_1\vec{e}_1-b_2\vec{e}_2|$$

Proof: $|\vec{V}-b_1\vec{e}_1-b_2\vec{e}_2|^2-|\vec{V}-(\vec{V}\cdot\vec{e}_1)\vec{e}_1-(\vec{V}\cdot\vec{e}_2)\vec{e}_2|^2$

$= |\vec{V}|^2 + b_1^2 + b_2^2 - 2b_1(\vec{V}\cdot\vec{e}_1) - 2b_2(\vec{V}\cdot\vec{e}_2) + 2b_1b_2(\vec{e}_1\cdot\vec{e}_2)$

$- [|\vec{V}|^2 + (\vec{V}\cdot\vec{e}_1)^2 + (\vec{V}\cdot\vec{e}_2)^2 - 2(\vec{V}\cdot\vec{e}_1)(\vec{V}\cdot\vec{e}_1) - 2(\vec{V}\cdot\vec{e}_2)(\vec{V}\cdot\vec{e}_2) + 2(\vec{V}\cdot\vec{e}_1)(\vec{V}\cdot\vec{e}_2)(\vec{e}_1\cdot\vec{e}_2)]$

$= b_1^2 + b_2^2 - 2b_1(\vec{V}\cdot\vec{e}_1) - 2b_2(\vec{V}\cdot\vec{e}_2) + (\vec{V}\cdot\vec{e}_1)^2 + (\vec{V}\cdot\vec{e}_2)^2$

$= (b_1 - \vec{V}\cdot\vec{e}_1)^2 + (b_2 - \vec{V}\cdot\vec{e}_2)^2 \ge 0$

$\therefore |\vec{V}-(\vec{V}\cdot\vec{e}_1)\vec{e}_1-(\vec{V}\cdot\vec{e}_2)\vec{e}_2|\le|\vec{V}-b_1\vec{e}_1-b_2\vec{e}_2|$

Q5 (a) Find two vectors in $\mathbf{R}^2$ with Euclidean norm 1 whose Euclidean inner products with (–2, 4) are zero.

(b) Show that there are infinitely many vectors in $\mathbf{R}^3$ with Euclidean norm 1 whose inner product with (–1, 7, 2) is zero.

(a) Let $\vec{v} = (x, y)$, $\vec{u} = (-2, 4)$

$|\vec{v}| = 1$ and $\vec{v}\cdot\vec{u} = 0$

$\sqrt{x^2+y^2} = 1$ and $-2x + 4y = 0$

$x^2 + y^2 = 1$ and $x = 2y$

$4y^2 + y^2 = 1$

$y^2 = \frac{1}{5} \Rightarrow y = \pm\frac{1}{\sqrt{5}}, x = \pm\frac{2}{\sqrt{5}}$

$\vec{v} = \left(\frac{2}{\sqrt{5}},\frac{1}{\sqrt{5}}\right)$ or $\left(-\frac{2}{\sqrt{5}},-\frac{1}{\sqrt{5}}\right)$

(b) $\vec{u} = (-1, 7, 2)$, $\vec{v} = (x, y, z)$

$|\vec{v}| = 1$ and $\vec{u}\cdot\vec{v} = 0$

$\sqrt{x^2+y^2+z^2} = 1$ and $-x + 7y + 2z = 0$

$x^2 + y^2 + z^2 = 1$ and $x = 7y + 2z$

$(7y + 2z)^2 + y^2 + z^2 = 1$

$49y^2 + 28yz + 4z^2 + y^2 + z^2 = 1$

$50y^2 + 28yz + 5z^2 - 1 = 0$

$y = \dfrac{-14z \pm \sqrt{196z^2 - 50\cdot(5z^2-1)}}{50}$, let $z = t$, where $t \in \mathbf{R}$

$= \dfrac{-14z \pm \sqrt{50-54z^2}}{50}$ valid for $-\dfrac{5}{3\sqrt{3}} \le z \le \dfrac{5}{3\sqrt{3}}$

$x = 7y + 2z = \dfrac{-98z \pm 7\sqrt{50-54z^2}}{50} + 2z = \dfrac{2t \pm 7\sqrt{50-54t^2}}{50}$

$\therefore \begin{cases} x = \dfrac{2t \pm 7\sqrt{50-54t^2}}{50} \\ y = \dfrac{-14z \pm \sqrt{50-54z^2}}{50} \\ z = t \end{cases}$, where $t \in$ R. $\therefore$ There are infinitely many solution in t.

Q6 Find the Euclidean distance between $\vec{u}$ and $\vec{v}$ when:

(a) $\vec{u} = (1, 1, -1)$, $\vec{v} = (2, 6, 0)$

(b) $\vec{u} = (6, 0, 1, 3, 0)$, $\vec{v} = (-1, 4, 2, 8, 3)$

(a) $\vec{u} = (1, 1, -1)$, $\vec{v} = (2, 6, 0)$

$|\vec{u} - \vec{v}| = \sqrt{(2-1)^2 + (6-1)^2 + [0-(-1)]^2} = \sqrt{1^2+5^2+1^2} = \sqrt{27} = 3\sqrt{3}$

(b) $|\vec{u} - \vec{v}| = \sqrt{[6-(-1)]^2 + (0-4)^2 + (1-2)^2 + (3-8)^2 + (0-3)^2}$

$= \sqrt{7^2+4^2+1^2+5^2+3^2} = \sqrt{49+16+1+25+9} = \sqrt{100} = 10$

Q7 Show that the set **W** of all 2×2 matrices having zeros on the main diagonal is a subspace of the vector space $\mathbf{M}_{22}$ of all 2×2 matrices.

$\mathbf{V} = \mathbf{M}_{22} = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \Re\right\}$. It is known that **V** is a vector space.

$\mathbf{W} = \left\{\begin{pmatrix} 0 & p \\ q & 0 \end{pmatrix} : p, q \in \Re\right\}$. Clearly $\mathbf{W} \subset \mathbf{V}$

Let $\vec{0} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \in \mathbf{W}$, $\therefore \mathbf{W} \ne \phi$ (**W** is a non-empty subset of **V**.)

Let $\vec{u} = \begin{pmatrix} 0 & p_1 \\ q_1 & 0 \end{pmatrix}$, $\vec{v} = \begin{pmatrix} 0 & p_2 \\ q_2 & 0 \end{pmatrix}$, $\vec{u}$, $\vec{v} \in \mathbf{W}$

$\vec{u} + \vec{v} = \begin{pmatrix} 0 & p_1 \\ q_1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & p_2 \\ q_2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & p_1 + p_2 \\ q_1 + q_2 & 0 \end{pmatrix} \in \mathbf{W}$ (close under addition)

$\forall k \in \mathbf{R}$, $k\vec{u} = k\begin{pmatrix} 0 & p_1 \\ q_1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & kp_1 \\ kq_1 & 0 \end{pmatrix} \in \mathbf{W}$ (close under scalar multiplication)

$\therefore$ **W** is a vector subspace of **V** (= $\mathbf{M}_{22}$)

Q8 Consider a system of m linear equations in n unknowns:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n & = & b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n & = & b_2 \\ \cdots & \cdots & \cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n & = & b_m \end{cases}$$

If $b_1 = b_2 = \ldots = b_m = 0$, the system is known as a **homogeneous**.

Show that the set of solutions of a homogeneous system is a vector subspace of $\mathbf{R}^n$.

$\mathbf{V} = \mathbf{R}^n$, clearly $\mathbf{V}$ is a vector space over $\mathbf{R}$.

Let $A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$, this is the coefficient matrix of the system of equations.

$\mathbf{W} = \{ \vec{v} \in \mathbf{R}^n : A\vec{v} = \vec{0} \}$

To show that $\mathbf{W}$ is a subspace of $\mathbf{R}^n$.

$\forall \vec{v} \in \mathbf{W} \Rightarrow \vec{v} \in \mathbf{R}^n$ and $A\vec{v} = \vec{0}$

$\therefore \mathbf{W} \subset \mathbf{R}^n$ ($\mathbf{W}$ is a subset of $\mathbf{R}^n$)

Let $\vec{0} = (0, 0, \ldots, 0)^t \in \mathbf{R}^n$, then $A\vec{0} = \vec{0}$

$\therefore \vec{0} \in \mathbf{W}$

$\mathbf{W} \neq \phi$ ($\mathbf{W}$ is a non-empty subset of $\mathbf{R}^n$)

Let $\vec{u}, \vec{v} \in \mathbf{W} \Rightarrow A\vec{u} = \vec{0}, A\vec{v} = \vec{0}$

$\vec{u} + \vec{v} \in \mathbf{R}^n$ and $A(\vec{u} + \vec{v}) = A\vec{u} + A\vec{v}$

$= \vec{0} + \vec{0}$

$= \vec{0}$

$\therefore \vec{u} + \vec{v} \in \mathbf{W}$ ($\mathbf{W}$ is closed under addition)

Let $k \in \mathbf{R}$, $A(k\vec{u}) = k(A\vec{u}) = k\vec{0} = \vec{0}$

$k\vec{u} \in \mathrm{R}^{\boldsymbol{n}}$ and $k\vec{u} \in \mathbf{W}$ ($\mathbf{W}$ is closed under scalar multiplication)

$\therefore \mathbf{W}$ is a vector subspace of $\mathbf{V}$ ($= \mathbf{R}^n$)

Q9 Let $\mathbf{V} = \mathbf{R}^3$. Determine which of the following are subspaces of $\mathbf{R}^3$.

(a) all vectors of the form $(a, 0, 0)$.

(b) all vectors of the form $(a, 1, 1)$.

(c) all vectors of the form (a, b, c) where $b = a + c$.

(d) all vectors of the form (a, b, c) where $b = a + c + 1$

(a) $\mathbf{W} = \langle (a, 0, 0) \rangle = \{(a, 0, 0)t : t \in \mathbf{R}^n\}$

$= \{(at, 0, 0) : t \in \mathbf{R}\}$

$\forall \vec{v} \in \mathbf{W} \Rightarrow \vec{v} = (at, 0, 0) \in \mathbf{R}$ $\therefore \mathbf{W} \subset \mathbf{R}^3$

Let $t = 0$, $\vec{0} = (0, 0, 0) = 0(a, 0, 0)$ $\therefore \mathbf{W} \neq \phi$

Let $\vec{u} = (at_1, 0, 0)$, $\vec{v} = (at_2, 0, 0)$, then $\vec{u}, \vec{v} \in \mathbf{R}^3$

$\vec{u} + \vec{v} = (at_1, 0, 0) + (at_2, 0, 0) = (at_1 + at_2, 0, 0) \in \mathbf{R}^3$

$= (a(t_1 + t_2), 0, 0) \in \mathbf{W}$

$\forall k \in \mathbf{R}$, $k\vec{u} = k(at, 0, 0) = (akt, 0, 0) \in \mathbf{W}$

$\therefore$ **W** is vector subspace of $\mathbf{R}^3$.

(b) $\mathbf{W} = \{(a, 1, 1): a \in \mathbf{R}\}$

Let $\vec{u} = (a, 1, 1)$, $\vec{v} = (b, 1, 1)$

$\vec{u} + \vec{v} = (a, 1, 1) + (b, 1, 1)$

$= (a + b, 2, 2)$

$\therefore$ $\vec{u} + \vec{v} \notin \mathbf{W}$

W is not a vector subspace of $\mathbf{R}^3$

(c) $\mathbf{W} = \{(a, b, c): b = a + c\}$

$\forall \vec{u} \in \mathbf{W} \Rightarrow \vec{u} = (a, b, c) \in \mathbf{R}^3$. $\therefore$ $\mathbf{W} \subset \mathbf{R}^3$

$\vec{0} = (0, 0, 0)$, $0 = 0 + 0$. $\therefore$ $\vec{0} \in \mathbf{W} \Rightarrow \mathbf{W} \neq \phi$

Let $\vec{u} = (a_1, b_1, c_1)$, where $b_1 = a_1 + c_1$ and $\vec{v} = (a_2, b_2, c_2)$, where $b_2 = a_2 + c_2$.

$\vec{u} + \vec{v} = (a_1 + a_2, b_1 + b_2, c_1 + c_2)$

$b_1 + b_2 = a_1 + c_1 + a_2 + c_2$

$= (a_1 + a_2) + (c_1 + c_2)$

$\therefore$ $\vec{u} + \vec{v} \in \mathbf{W}$

$\forall k \in \mathbf{R}$, $k\vec{u} = (ka_1, kb_1, kc_1)$

$kb_1 = k(a_1 + c_1) = ka_1 + kc_1$

$\therefore$ $k\vec{u} \in \mathbf{W}$

W is a vector subspace of $\mathbf{R}^3$

(d) $\mathbf{W} = \{(a, b, c): b = a + c + 1\}$

Let $\vec{u} = (a, b, c) \in \mathbf{W} \Rightarrow b = a + c + 1$

$2\vec{u} = (2a, 2b, 2c)$

$2b = 2(a + c + 1) = 2a + 2c + 1 + 1 \neq 2a + 2c + 1$

$\therefore$ $2\vec{u} \notin \mathbf{W}$

W is not a vector subspace of $\mathbf{R}^3$.

Q10 Let $\mathbf{V} = \mathbf{M}_{22}$, the set of all 2×2 matrices. Determine which of the following are subspaces of $\mathbf{M}_{22}$.

(a) all matrices of the form $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, where a, b, c and d are integers.

(b) all matrices of the form $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, where $a + d = 0$.

(c) all 2×2 matrices A such that $A = A^t$. (the transpose of the matrix A)

(d) all 2×2 matrices A such that $\det(A) = 0$.

$\mathbf{M}_{22} = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \Re\right\}$, given that $\mathbf{M}_{22}$ is a vector space.

(a) $\mathbf{W} = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathrm{Z}\right\}$

$\forall \vec{u} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbf{W} \Rightarrow a, b, c, d \in \mathbf{Z}$

$\Rightarrow$ $\vec{u} \in \mathbf{M}_{22}$ $\therefore$ $\mathbf{W} \subset \mathbf{M}_{22}$

Let $\vec{0}=\begin{pmatrix}0&0\\0&0\end{pmatrix}\in \mathbf{W}\subset \mathbf{M}_{22}$ $\therefore$ $\mathbf{W}\neq\phi$

Let $\vec{u}=\begin{pmatrix}a_1&b_1\\c_1&d_1\end{pmatrix}$, $\vec{v}=\begin{pmatrix}a_2&b_2\\c_2&d_2\end{pmatrix}$, where $a_1, a_2, b_1, b_2, c_1, c_2, d_1, d_2 \in \mathbf{Z}$

$\vec{u}+\vec{v}=\begin{pmatrix}a_1&b_1\\c_1&d_1\end{pmatrix}+\begin{pmatrix}a_2&b_2\\c_2&d_2\end{pmatrix}=\begin{pmatrix}a_1+a_2&b_1+b_2\\c_1+c_2&d_1+d_2\end{pmatrix}\in \mathbf{M}_{22}$

$a_1+a_2, b_1+b_2, c_1+c_2, d_1+d_2\in \mathbf{Z}$

$\therefore$ $\vec{u}+\vec{v}\in\mathbf{W}$

$\forall k\in\mathbf{R}$, $k\vec{u}=k\begin{pmatrix}a_1&b_1\\c_1&d_1\end{pmatrix}=\begin{pmatrix}ka_1&kb_1\\kc_1&kd_1\end{pmatrix}$, ka_1 may not be an integer.

e.g. $\vec{u}=\begin{pmatrix}1&0\\0&0\end{pmatrix}\in\mathbf{W}$, $k=\frac{1}{2}$

$k\vec{u}=\begin{pmatrix}\frac{1}{2}&0\\0&0\end{pmatrix}\notin\mathbf{W}$. $\therefore\mathbf{W}$ is <u>not</u> a vector subspace of $\mathbf{M}_{22}$.

(b) $\mathbf{W}=\left\{\begin{pmatrix}a&b\\c&d\end{pmatrix}: a+d=0\right\}$

$\vec{u}=\begin{pmatrix}a&b\\c&d\end{pmatrix}\in\mathbf{W}\Rightarrow a+d=0$ and $\vec{u}\in\mathbf{M}_{22}$. $\therefore\mathbf{W}\subset\mathbf{M}_{22}$

$\vec{0}=\begin{pmatrix}0&0\\0&0\end{pmatrix}$, $0+0=0\Rightarrow$ $\vec{0}\in\mathbf{W}$ $\therefore\mathbf{W}\neq\phi$

$\vec{u}=\begin{pmatrix}a_1&b_1\\c_1&d_1\end{pmatrix}$, $\vec{v}=\begin{pmatrix}a_2&b_2\\c_2&d_2\end{pmatrix}$, where $a_1+d_1=0$, $a_2+d_2=0$

$\vec{u}+\vec{v}=\begin{pmatrix}a_1&b_1\\c_1&d_1\end{pmatrix}+\begin{pmatrix}a_2&b_2\\c_2&d_2\end{pmatrix}=\begin{pmatrix}a_1+a_2&b_1+b_2\\c_1+c_2&d_1+d_2\end{pmatrix}\in \mathbf{M}_{22}$

$a_1+a_2+d_1+d_2=(a_1+d_1)+(a_2+d_2)=0+0=0$

$\therefore$ $\vec{u}+\vec{v}\in\mathbf{W}$

$\forall k\in\mathbf{R}$, $k\vec{u}=k\begin{pmatrix}a_1&b_1\\c_1&d_1\end{pmatrix}=\begin{pmatrix}ka_1&kb_1\\kc_1&kd_1\end{pmatrix}$

$ka_1+kd_1=k(a_1+d_1)=0$

$\therefore$ $k\vec{u}\in\mathbf{W}$

$\therefore$ $\mathbf{W}$ is a vector subspace of $\mathbf{M}_{22}$.

(c) $\mathbf{W}=\{A\in\mathbf{M}_{22}: A=A^t\}$

$\forall\vec{u}\in\mathbf{W}\Rightarrow$ $\vec{u}=A\in\mathbf{M}_{22}$. $\therefore\mathbf{W}\subset\mathbf{M}_{22}$

Let $\vec{0}=\begin{pmatrix}0&0\\0&0\end{pmatrix}\in\mathbf{M}_{22}$, $\begin{pmatrix}0&0\\0&0\end{pmatrix}^t=\begin{pmatrix}0&0\\0&0\end{pmatrix}\Rightarrow$ $\vec{0}\in\mathbf{W}$, $\mathbf{W}\neq\phi$.

$\forall\vec{u}$, $\vec{v}\in\mathbf{W}\Rightarrow$ $\vec{u}=A$, $\vec{v}=B$, where $A=A^t$ and $B=B^t$.

$\vec{u}+\vec{v}=A+B\in\mathbf{M}_{22}$

$(A + B)^t = A^t + B^t = A + B$

$\therefore\ \vec{u}+\vec{v} \in \mathbf{W}$

$\forall k \in \mathbf{R},\ k\vec{u} = kA$

$(kA)^t = k\,A^t = kA$

$\therefore\ k\vec{u} \in \mathbf{W}$

$\therefore$ **W** is a vector subspace of $\mathbf{M}_{22}$.

(*d*) $\mathbf{W} = \{A \in \mathbf{M}_{22}: \det A = 0\}$

Let $\vec{u}=\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}$, $\vec{v}=\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}$

$\det\ \vec{u} = 0$, $\det\ \vec{v} = 0$, $\therefore \vec{u},\ \vec{v} \in \mathbf{W}$

$$\vec{u}+\vec{v}=\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}+\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$$

$\det(\vec{u}+\vec{v}) = 1 \neq 0$

$\therefore\ \vec{u}+\vec{v} \notin \mathbf{W}$

W is not a vector subspace of $\mathbf{M}_{22}$.

Q11 Express the followings as linear combinations of $\vec{p}_1=2+x+4x^2$, $\vec{p}_2=1-x+3x^2$, $\vec{p}_3= 3+2x+5x^2$

(a) $5 + 9x + 5x^2$

(b) $2 + 6x^2$

(c) $2 + 2x + 3x^2$

(a) $k_1\,\vec{p}_1 + k_2\,\vec{p}_2 + k_3\,\vec{p}_3 = 5 + 9x + 5x^2$

$$k_1\begin{pmatrix}2\\ x\\ 4x^2\end{pmatrix}+k_2\begin{pmatrix}1\\ -x\\ 3x^2\end{pmatrix}+k_3\begin{pmatrix}3\\ 2x\\ 5x^2\end{pmatrix}=\begin{pmatrix}5\\ 9x\\ 5x^2\end{pmatrix}$$

$$\begin{pmatrix}2 & 1 & 3\\ 1 & -1 & 2\\ 4 & 3 & 5\end{pmatrix}\begin{pmatrix}k_1\\ k_2\\ k_3\end{pmatrix}=\begin{pmatrix}5\\ 9\\ 5\end{pmatrix}$$

$$\text{Let } A=\begin{pmatrix}2 & 1 & 3\\ 1 & -1 & 2\\ 4 & 3 & 5\end{pmatrix}$$

$$\det A = 2\begin{vmatrix}-1 & 2\\ 3 & 5\end{vmatrix}-1\begin{vmatrix}1 & 2\\ 4 & 5\end{vmatrix}+3\begin{vmatrix}1 & -1\\ 4 & 3\end{vmatrix}$$

$= 2(-5 - 6) - (5 - 8) + 3(3 + 4) = -22 + 3 + 21 = 2 \neq 0 \Rightarrow A^{-1}$ exist.

$$A^{-1} = \frac{1}{|A|}adj(A)=\frac{1}{|A|}cof(A)^t$$

$$=\frac{1}{2}\begin{pmatrix} \begin{vmatrix}-1 & 2\\ 3 & 5\end{vmatrix} & -\begin{vmatrix}1 & 2\\ 4 & 5\end{vmatrix} & \begin{vmatrix}1 & -1\\ 4 & 3\end{vmatrix} \\ -\begin{vmatrix}1 & 3\\ 3 & 5\end{vmatrix} & \begin{vmatrix}2 & 3\\ 4 & 5\end{vmatrix} & -\begin{vmatrix}2 & 1\\ 4 & 3\end{vmatrix} \\ \begin{vmatrix}1 & 3\\ -1 & 2\end{vmatrix} & -\begin{vmatrix}2 & 3\\ 1 & 2\end{vmatrix} & \begin{vmatrix}2 & 1\\ 1 & -1\end{vmatrix}\end{pmatrix}^t = \frac{1}{2}\begin{pmatrix}-11 & 3 & 7\\ 4 & -2 & -2\\ 5 & -1 & -3\end{pmatrix}^t = \begin{pmatrix}-\frac{11}{2} & 2 & \frac{5}{2}\\ \frac{3}{2} & -1 & -\frac{1}{2}\\ \frac{7}{2} & -1 & -\frac{3}{2}\end{pmatrix}$$

$$\begin{pmatrix}k_1\\k_2\\k_3\end{pmatrix} = A^{-1}\begin{pmatrix}5\\9\\5\end{pmatrix} = \begin{pmatrix}-\frac{11}{2} & 2 & \frac{5}{2}\\ \frac{3}{2} & -1 & -\frac{1}{2}\\ \frac{7}{2} & -1 & -\frac{3}{2}\end{pmatrix}\begin{pmatrix}5\\9\\5\end{pmatrix} = \begin{pmatrix}3\\-4\\1\end{pmatrix}$$

$\therefore\ 5 + 9x + 5x^2 = 3\,\vec{p}_1 - 4\,\vec{p}_2 + \vec{p}_3$

(b) $$\begin{pmatrix}k_1\\k_2\\k_3\end{pmatrix} = A^{-1}\begin{pmatrix}2\\0\\6\end{pmatrix} = \begin{pmatrix}-\frac{11}{2} & 2 & \frac{5}{2}\\ \frac{3}{2} & -1 & -\frac{1}{2}\\ \frac{7}{2} & -1 & -\frac{3}{2}\end{pmatrix}\begin{pmatrix}2\\0\\6\end{pmatrix} = \begin{pmatrix}4\\0\\-2\end{pmatrix}$$

$\therefore\ 2 + 6x^2 = 4\,\vec{p}_1 - 2\,\vec{p}_3$

(c) $$\begin{pmatrix}k_1\\k_2\\k_3\end{pmatrix} = A^{-1}\begin{pmatrix}2\\2\\3\end{pmatrix} = \begin{pmatrix}-\frac{11}{2} & 2 & \frac{5}{2}\\ \frac{3}{2} & -1 & -\frac{1}{2}\\ \frac{7}{2} & -1 & -\frac{3}{2}\end{pmatrix}\begin{pmatrix}2\\2\\3\end{pmatrix} = \begin{pmatrix}\frac{1}{2}\\-\frac{1}{2}\\\frac{1}{2}\end{pmatrix}$$

$\therefore\ 2 + 2x + 3x^2 = \frac{1}{2}\,\vec{p}_1 - \frac{1}{2}\,\vec{p}_2 + \frac{1}{2}\,\vec{p}_3$

Q12 Determine whether $\vec{v}_1 = (1, 1, 2)$, $\vec{v}_2 = (1, 0, 1)$, $\vec{v}_3 = (2, 1, 3)$ span $\mathbf{R}^3$.

$\forall (a, b, c) \in \mathbf{R}^3$, $k_1\vec{v}_1 + k_2\vec{v}_2 + k_3\vec{v}_3 = (a, b, c)$, find k_1, k_2, k_3.

$$\begin{pmatrix}1 & 1 & 2\\ 1 & 0 & 1\\ 2 & 1 & 3\end{pmatrix}\begin{pmatrix}k_1\\k_2\\k_3\end{pmatrix} = \begin{pmatrix}a\\b\\c\end{pmatrix}, \text{ let A} = \begin{pmatrix}1 & 1 & 2\\ 1 & 0 & 1\\ 2 & 1 & 3\end{pmatrix}$$

$$\det A = \begin{vmatrix}1 & 1 & 2\\ 1 & 0 & 1\\ 2 & 1 & 3\end{vmatrix} = 1\begin{vmatrix}0 & 1\\ 1 & 3\end{vmatrix} - 1\begin{vmatrix}1 & 1\\ 2 & 3\end{vmatrix} + 2\begin{vmatrix}1 & 0\\ 2 & 1\end{vmatrix} = -1 - 1 + 2 = 0$$

$\therefore A^{-1}$ does not exist. $\vec{v}_1, \vec{v}_2, \vec{v}_3$ does not span $\mathbf{R}^3$.

Q13 Determine which of the following lie in the space spanned by $\vec{f} = \cos^2 x$ and $\vec{g} = \sin^2 x$

(a) $\cos 2x$

(b) $3 + x^2$

(c) 1

(d) $\sin x$

(a) $k_1 \cos^2 x + k_2 \sin^2 x \equiv \cos 2x$, find k_1, k_2

We can easily find that $k_1 = 1$, $k_2 = -1$

$\therefore$ $\cos 2x$ lies on $< \vec{f}, \vec{g} >$

(b) $k_1 \cos^2 x + k_2 \sin^2 x \equiv 3 + x^2$

Put $x = 0$, $k_1 = 3$

Put $x = \pi$, $k_1 = 3 + \pi^2$!!!

$\therefore$ no solution

$3 + x^2 \notin < \vec{f}, \vec{g} >$

(c) $k_1 \cos^2 x + k_2 \sin^2 x \equiv 1$

We can easily find that $k_1 = 1$, $k_2 = 1$

$\therefore$ $1 \in < \vec{f}, \vec{g} >$

(d) $k_1 \cos^2 x + k_2 \sin^2 x \equiv \sin x$

$x = \frac{\pi}{2}$, $k_2 = 1$

$x = -\frac{\pi}{2}$, $k_2 = -1$!!!

$\therefore$ no solution

$\sin x \notin < \vec{f}, \vec{g} >$

Q14 Let $\mathbf{P}_2$ be the set of real polynomials of degree ≤ 2. Determine if the following polynomials spans $\mathbf{P}_2$:

$\vec{p}_1 = 1 + 2x - x^2$, $\vec{p}_2 = 3 + x^2$, $\vec{p}_3 = 5 + 4x - x^2$, $\vec{p}_4 = -2 + 2x - 2x^2$

$\mathbf{P}_2 = \{a + bx + cx^2 : a, b, c \in \mathbf{R}\}$

$k_1 \vec{p}_1 + k_2 \vec{p}_2 + k_3 \vec{p}_3 + k_4 \vec{p}_4 = a + bx + cx^2$, find k_1, k_2, k_3, k_4.

$$k_1\begin{pmatrix}1\\2x\\-x^2\end{pmatrix}+k_2\begin{pmatrix}3\\0\\x^2\end{pmatrix}+k_3\begin{pmatrix}5\\4x\\-x^2\end{pmatrix}+k_4\begin{pmatrix}-2\\2x\\-2x^2\end{pmatrix}=\begin{pmatrix}a\\bx\\cx^2\end{pmatrix}$$

$$\left(\begin{array}{cccc|c}1&3&5&-2&a\\2&0&4&2&b\\-1&1&-1&-2&c\end{array}\right)\sim\begin{matrix}\\-R_2+2R_1\\R_1+R_3\end{matrix}\left(\begin{array}{cccc|c}1&3&5&-2&a\\0&6&6&-6&2a-b\\0&4&4&-4&a+c\end{array}\right)\sim\begin{matrix}\\\frac{R_2}{6}\\\frac{R_3}{4}\end{matrix}\left(\begin{array}{cccc|c}1&3&5&-2&a\\0&1&1&-1&\frac{2a-b}{6}\\0&1&1&-1&\frac{a+c}{4}\end{array}\right)$$

$$\sim\begin{matrix}\\\\R_3-R_2\end{matrix}\left(\begin{array}{cccc|c}1&3&5&-2&a\\0&1&1&-1&\frac{2a-b}{6}\\0&0&0&0&\frac{-a+2b+3c}{12}\end{array}\right)$$. The system of equation is valid for any $a, b, c \in \mathbf{R}$

In general, $\frac{-a+2b+3c}{12} \neq 0$. $\therefore$ no solution in k_1, k_2, k_3.

$< \vec{p}_1, \vec{p}_2, \vec{p}_3, \vec{p}_4 > \neq \mathbf{P}_2$.

Q15 Find an equation for the plane spanned by the vectors $\vec{u} = (1, 1, -1)$, $\vec{v} = (2, 3, 5)$.

The plane is given by $k_1\vec{u} + k_2\vec{v} = k_1(1, 1, -1) + k_2(2, 3, 5)$

$= (k_1 + 2k_2, k_1 + 3k_2, -k_1 + 5k_2)$, where $k_1, k_2 \in \mathbf{R}$

If $(x, y, z) = (k_1 + 2k_2, k_1 + 3k_2, -k_1 + 5k_2)$

$x = k_1 + 2k_2$ ....(1)

$y = k_1 + 3k_2$ ....(2)

$z = -k_1 + 5k_2$..................(3)

(2) + (3): $y + z = 8k_2$ ....(4)

(2) – (1): $y - x = k_2$ ....(5)

(4) – 8(5): $y + z - 8(y - x) = 0$

$8x - 7y + z = 0$

Q16 Which of the following sets of vectors in $\mathbf{R}^3$ are linear dependent?

(a) (2, –1, 4), (3, 6, 2), (2, 10, –4)

(b) (3, 1, 1), (2, –1, 5), (4, 0, –3)

(c) (1, 3, 3), (0, 1, 4), (5, 6, 3), (7, 2, –1)

(a) $k_1\begin{pmatrix}2\\-1\\4\end{pmatrix}+k_2\begin{pmatrix}3\\6\\2\end{pmatrix}+k_3\begin{pmatrix}2\\10\\-4\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}\Rightarrow\begin{pmatrix}2&3&2\\-1&6&10\\4&2&-4\end{pmatrix}\begin{pmatrix}k_1\\k_2\\k_3\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$

$$\det = \begin{vmatrix}2&3&2\\-1&6&10\\4&2&-4\end{vmatrix} = 2\begin{vmatrix}6&10\\2&-4\end{vmatrix} - 3\begin{vmatrix}-1&10\\4&-4\end{vmatrix} + 2\begin{vmatrix}-1&6\\4&2\end{vmatrix}$$

$= 2(-44) - 3(-36) + 2(-26) = -88 + 108 - 52 = -32 \ne 0$

$\therefore$ (2, –1, 4), (3, 6, 2), (2, 10, –4) are independent.

$\therefore$ (3, 1, 1), (2, –1, 5), (4, 0, –3) are independent.

$$\left(\begin{array}{cccc|c}1&0&5&7&0\\3&1&6&2&0\\3&4&3&-1&0\end{array}\right) \sim \begin{matrix}\\R_2-3R_1\\R_3-3R_1\end{matrix}\left(\begin{array}{cccc|c}1&0&5&7&0\\0&1&-9&-19&0\\0&4&-12&-22&0\end{array}\right) \sim \begin{matrix}\\ \\ \tfrac{1}{2}R_3-2R_2\end{matrix}\left(\begin{array}{cccc|c}1&0&5&7&0\\0&1&-9&-19&0\\0&0&12&27&0\end{array}\right)$$

Number of unknowns > number of equations

$\therefore$ There are infinitely many solutions.

We can find k_1, k_2, k_3, k_4 not all zero such that

$k_1(1, 3, 3) + k_2(0, 1, 4) + k_3(5, 6, 3) + k_4(7, 2, -1) = (0, 0, 0)$

$\therefore$ (1, 3, 3), (0, 1, 4), (5, 6, 3), (7, 2, –1) are dependent.

(b) $\begin{vmatrix}3&2&4\\1&-1&0\\1&5&-3\end{vmatrix} = 4\begin{vmatrix}1&-1\\1&5\end{vmatrix} - 3\begin{vmatrix}3&2\\1&-1\end{vmatrix} = 4(6) - 3(-5) \ne 0$

(c) $k_1\begin{pmatrix}1\\3\\3\end{pmatrix}+k_2\begin{pmatrix}0\\1\\4\end{pmatrix}+k_3\begin{pmatrix}5\\6\\3\end{pmatrix}+k_4\begin{pmatrix}7\\2\\-1\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$, find k_1, k_2, k_3, k_4.

Q17 Which of the following sets of vectors in $\mathbf{P}_2$ are linear dependent?

(a) $2 - x + 4x^2$, $3 + 6x + 2x^2$, $2 + 10x - 4x^2$.

(b) $1 + 3x + 3x^2$, $x + 4x^2$, $5 + 6x + 3x^2$, $7 + 2x - x^2$

(a) $$k_1\begin{pmatrix}2\\-x\\4x^2\end{pmatrix}+k_2\begin{pmatrix}3\\6x\\2x^2\end{pmatrix}+k_3\begin{pmatrix}2\\10x\\-4x^2\end{pmatrix}=\begin{pmatrix}0\\0x\\0x^2\end{pmatrix}$$

$$\det=\begin{vmatrix}2&3&2\\-1&6&10\\4&2&-4\end{vmatrix}=2\begin{vmatrix}6&10\\2&-4\end{vmatrix}-3\begin{vmatrix}-1&10\\4&-4\end{vmatrix}+2\begin{vmatrix}-1&6\\4&2\end{vmatrix}$$ (same as Q61(a))

$= 2(-44) - 3(-36) + 2(-26) = -88 + 108 - 52 = -32 \neq 0$

$2 - x + 4x^2$, $3 + 6x + 2x^2$, $2 + 10x - 4x^2$ are independent.

(b) $$k_1\begin{pmatrix}1\\3\\3\end{pmatrix}+k_2\begin{pmatrix}0\\1\\4\end{pmatrix}+k_3\begin{pmatrix}5\\6\\3\end{pmatrix}+k_4\begin{pmatrix}7\\2\\-1\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$$, find k_1, k_2, k_3, k_4. (same as Q16 (c))

Number of unknowns > number of equations

$\therefore$ There are infinitely many solutions.

$\therefore$ $1 + 3x + 3x^2$, $x + 4x^2$, $5 + 6x + 3x^2$, $7 + 2x - x^2$ are dependent.

Q18 Let $\mathbf{V}$ be the vector space of all real valued functions defined on the entire real line. Which of the following sets of vectors in $\mathbf{V}$ are linear dependent?

(a) 2, $4\sin^2 x$, $\cos^2 x$ (b) x, $\cos x$

(c) 1, $\sin x$, $\sin 2x$ (d) $\cos 2x$, $\sin^2 x$, $\cos^2 x$

(e) $(1 + x)^2$, $x^2 + 2x$, 3 (f) 0, x, x^2

(a) $k_1\times 2 + k_2\times 4\sin^2 x + k_3\cos^2 x \equiv 0$, find k_1, k_2, k_3.

It can be easily found that $k_1 = -\frac{1}{2}$, $k_2 = \frac{1}{4}$, $k_3 = \frac{1}{4}$

$\therefore$ 2, $4\sin^2 x$, $\cos^2 x$ are dependent.

(b) $k_1x + k_2\cos x \equiv 0$ ....(1), find k_1, k_2.

Differentiate twice with respect to x: $-k_2\cos x \equiv 0$

Put $x = 0 \Rightarrow k_2 = 0$.

Put $k_2 = 0$ into (1): $k_1x \equiv 0 \Rightarrow k_1 = 0$

$\therefore$ x, $\cos x$ are independent.

(c) $k_1 + k_2\sin x + k_3\sin 2x \equiv 0$

Put $x = 0$, $k_1 = 0$

Put $x = \frac{\pi}{2}$, $k_2 = 0$

Sub. $k_1 = 0$, $k_2 = 0$ into the given equation: $k_3\sin 2x \equiv 0$

$\Rightarrow k_3 = 0$

$\therefore$ 1, $\sin x$, $\sin 2x$ are independent.

(d) $k_1\cos 2x + k_2\sin^2 x + k_3\cos^2 x \equiv 0$

It can be easily found that $k_1 = 1$, $k_2 = 1$, $k_3 = -1$

$\therefore$ $\cos 2x$, $\sin^2 x$, $\cos^2 x$ are dependent.

(e) $k_1(1 + x)^2 + k_2(x^2 + 2x) + 3k_3 \equiv 0$

It can be easily found that $k_1 = 1$, $k_2 = -1$, $k_3 = -\frac{1}{3}$.

$\therefore$ $(1 + x)^2$, $x^2 + 2x$, 3 are dependent.

(f) $0k_1 + k_2x + k_3x^2 \equiv 0$

It can be easily found that $k_1 = 1$, $k_2 = 0$, $k_3 = 0$

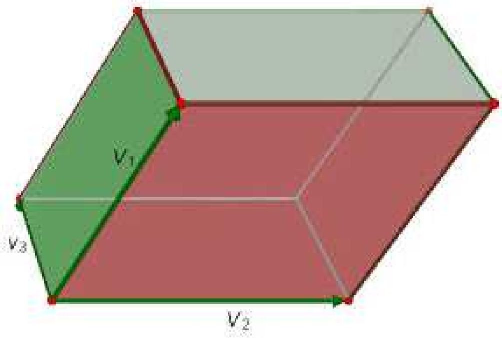
$\therefore$ 0, x, x^2 are dependent.

Q19 In each part determine whether the three vectors lie in a plane pass through the origin.

(a) $\vec{v}_1 = (1, 0, -2)$, $\vec{v}_2 = (3, 1, 2)$, $\vec{v}_3 = (1, -1, 0)$

(b) $\vec{v}_1 = (2, -1, 4)$, $\vec{v}_2 = (4, 2, 3)$, $\vec{v}_3 = (2, 7, -6)$

(a) We find the volume of the parallelepiped formed by $\vec{v}_1, \vec{v}_2, \vec{v}_3$.



$$\text{Volume} = \vec{v}_1 \cdot (\vec{v}_2 \times \vec{v}_3)$$

$$= \begin{vmatrix} 1 & 0 & -2 \\ 3 & 1 & 2 \\ 1 & -1 & 0 \end{vmatrix} = 1\begin{vmatrix} 1 & 2 \\ -1 & 0 \end{vmatrix} - 2\begin{vmatrix} 3 & 1 \\ 1 & -1 \end{vmatrix}$$

$$= 2 - 2(-4) \neq 0$$

$\therefore$ $\vec{v}_1, \vec{v}_2, \vec{v}_3$ do not lies on a plane.

(b) $$\text{Volume} = \vec{v}_1 \cdot (\vec{v}_2 \times \vec{v}_3) = \begin{vmatrix} 2 & -1 & 4 \\ 4 & 2 & 3 \\ 2 & 7 & -6 \end{vmatrix} = 2\begin{vmatrix} 2 & 3 \\ 7 & -6 \end{vmatrix} + \begin{vmatrix} 4 & 3 \\ 2 & -6 \end{vmatrix} + 4\begin{vmatrix} 4 & 2 \\ 2 & 7 \end{vmatrix}$$

$$= 2(-33) - 30 + 4(24) = 0$$

$\therefore$ $\vec{v}_1, \vec{v}_2, \vec{v}_3$ lies on a plane passing through the origin.

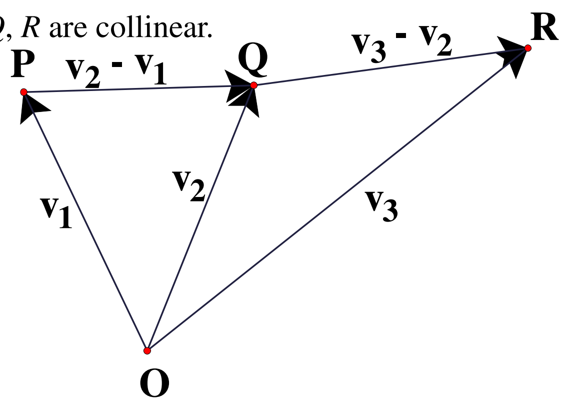
Q20 In each part determine whether the three vectors lie on the same line.

(a) $\vec{v}_1 = (3, -6, 9), \vec{v}_2 = (2, -4, 6), \vec{v}_3 = (1, 1, 1)$

(b) $\vec{v}_1 = (2, -1, 4), \vec{v}_2 = (4, 2, 3), \vec{v}_3 = (2, 7, -6)$

(c) $\vec{v}_1 = (4, 6, 8), \vec{v}_2 = (2, 3, 4), \vec{v}_3 = (-2, -3, -4)$

(a) $\vec{v}_1, \vec{v}_2, \vec{v}_3$ lies on a straight line if and only if P, Q, R are collinear.



$\Leftrightarrow$ $\vec{v}_2 - \vec{v}_1$ is parallel to $\vec{v}_3 - \vec{v}_2$

$\vec{v}_2 - \vec{v}_1 = (2, -4, 6) - (3, -6, 9) = (-1, 2, -3)$

$\vec{v}_3 - \vec{v}_2 = (1, 1, 1) - (2, -4, 6) = (-1, 5, -5)$

$\because$ $\vec{v}_2 - \vec{v}_1 \neq k(\vec{v}_3 - \vec{v}_2)$, i.e. not parallel.

$\therefore$ $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are not collinear.

(b) $\vec{v}_2 - \vec{v}_1 = (4, 2, 3) - (2, -1, 4) = (2, 3, -1)$

$\vec{v}_3 - \vec{v}_2 = (2, 7, -6) - (4, 2, 3) = (2, 5, -9)$

$\because$ $\vec{v}_2 - \vec{v}_1 \neq k(\vec{v}_3 - \vec{v}_2)$, i.e. not parallel.

$\therefore$ $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are not collinear.

(c) $\vec{v}_2-\vec{v}_1=(2, 3, 4)-(4, 6, 8)=(-2, -3, -4)$

$\vec{v}_3-\vec{v}_2=(-2, -3, -4)-(2, 3, 4)=(-4, -6, -8)$

$\because 2(\vec{v}_2-\vec{v}_1)=\vec{v}_3-\vec{v}_2$, i.e. $\vec{v}_2-\vec{v}_1$ is parallel to $\vec{v}_3-\vec{v}_2$.

$\therefore\ \vec{v}_1, \vec{v}_2, \vec{v}_3$ are collinear.

Q21 For which real values of t do the following vectors form a linear dependent set in $\mathbf{R}^3$?

$$\vec{v}_1=(t, -\frac{1}{2}, -\frac{1}{2}),\ \vec{v}_2=(-\frac{1}{2}, t, -\frac{1}{2}),\ \vec{v}_3=(-\frac{1}{2}, -\frac{1}{2}, t)$$

$$\begin{vmatrix} t & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & t & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & t \end{vmatrix}=0$$

$$t\begin{vmatrix} t & -\frac{1}{2} \\ -\frac{1}{2} & t \end{vmatrix}+\frac{1}{2}\begin{vmatrix} -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & t \end{vmatrix}-\frac{1}{2}\begin{vmatrix} -\frac{1}{2} & t \\ -\frac{1}{2} & -\frac{1}{2} \end{vmatrix}=0$$

$$t(t^2-\frac{1}{4})+\frac{1}{2}(-\frac{t}{2}-\frac{1}{4})-\frac{1}{2}(\frac{1}{4}+\frac{t}{2})=0$$

$$t^3-\frac{t}{4}-\frac{t}{4}-\frac{1}{8}-\frac{1}{8}-\frac{t}{4}=0$$

$$t^3-\frac{3t}{4}-\frac{1}{4}=0$$

$4t^3-3t-1=0$

Put $f(t)=4t^3-3t-1$

$f(1)=4-3-1=0 \Rightarrow t-1$ is a factor

By division,

$$\begin{array}{r|l} & 4t^2+4t+1 \\ t-1 & 4t^3 \qquad -3t-1 \\ & 4t^3-4t^2 \\ & \quad 4t^2-3t-1 \\ & \quad 4t^2-4t \\ & \qquad t-1 \\ & \qquad \underline{t-1} \end{array}$$

$\Rightarrow f(t)=(t-1)(4t^2+4t+1)=(t-1)(2t+1)^2=0$

$t=1$ or $-\frac{1}{2}$ (repeated roots)

Q22 Let $\mathbf{S}=\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_r\}$ be a set of vectors in a vector space $\mathbf{V}$. Show that if one of the vectors is zero, then $\mathbf{S}$ is linear dependent.

If $\vec{v}_1=\vec{0}$, $k_1\vec{v}_1+k_2\vec{v}_2+\ldots+k_r\vec{v}_r=\vec{0}$, find $k_1, k_2, \ldots, k_r$.

Let $k_1=3, k_2=0, \ldots, k_r=0$

$\therefore \mathbf{S}$ is linear dependent.

Q23 If { $\vec{v}_1$, $\vec{v}_2$, $\vec{v}_3$ } is a linear independent set of vectors, show that the following sets are also linear independent: (a) { $\vec{v}_2$ }
(b) { $\vec{v}_1$, $\vec{v}_3$ }
(c) { $\vec{v}_1$, $\vec{v}_1+\vec{v}_2$, $\vec{v}_1+\vec{v}_2+\vec{v}_3$ }

(a) $k_2\vec{v}_2=\vec{0}$, find k_2.
$0\vec{v}_1+k_2\vec{v}_2+0\vec{v}_3=\vec{0} \Rightarrow k_2=0$ (given that $\vec{v}_1$, $\vec{v}_2$, $\vec{v}_3$ are linear independent.)
$\therefore$ { $\vec{v}_2$ } is independent.

(b) $k_1\vec{v}_1+k_3\vec{v}_3=\vec{0}$, find k_1, k_3.
$k_1\vec{v}_1+0\vec{v}_2+k_3\vec{v}_3=\vec{0} \Rightarrow k_1=k_3=0$ (given that $\vec{v}_1$, $\vec{v}_2$, $\vec{v}_3$ are linear independent.)
$\therefore$ { $\vec{v}_1, \vec{v}_3$ } is independent.

(c) $k_1\vec{v}_1+k_2(\vec{v}_1+\vec{v}_2)+k_3((\vec{v}_1+\vec{v}_2+\vec{v}_3)=\vec{0}$, find k_1, k_2, k_3.
$(k_1+k_2+k_3)\vec{v}_1+(k_2+k_3)\vec{v}_2+k_3\vec{v}_3=\vec{0}$
$k_1+k_2+k_3=0$, $k_2+k_3=0$, $k_3=0$ (given that $\vec{v}_1$, $\vec{v}_2$, $\vec{v}_3$ are linear independent.)
After solving, $k_1=k_2=k_3=0$
$\therefore$ { $\vec{v}_1, \vec{v}_2, \vec{v}_3$ } is independent.

Q24 If { $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_r$ } is a linear independent set of vectors, show that every subset of **S** with one or more vectors is also linear independent.
Let **S** = { $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_r$ }, **W** ={ $\vec{v}_{i_1}, \vec{v}_{i_2}, \ldots, \vec{v}_{i_n}$ }, **W** $\subset$ **S**, **W** $\neq \phi$ (i.e. $1 \le n \le r$)
$k_1\vec{v}_{i_1}+k_2\vec{v}_{i_1}+\ldots+k_n\vec{v}_{i_n}=\vec{0}$, find $k_1, k_2, \ldots, k_n$.
$k_1\vec{v}_{i_1}+k_2\vec{v}_{i_1}+\ldots+k_n\vec{v}_{i_n}+0\vec{v}_{i_{n+1}}+\ldots+k_n\vec{v}_{i_r}=\vec{0}$ .... (1)
where $\vec{v}_{i_1}, \vec{v}_{i_1}, \vec{v}_{i_n}, \vec{v}_{i_{n+1}}, \ldots, \vec{v}_{i_r}$ are rearrangement of $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_r$.
$\because$ **S** is independent, (1) $\Rightarrow k_1=0, k_2=0, \ldots, k_n=0$
$\therefore$ **W** is also independent.

Q25 Prove that the space spanned by the two vectors in $\mathbf{R}^3$ is either a line through the origin, a plane through the origin, or the origin itself.
Let $\vec{u}$, $\vec{v} \in \mathbf{R}^3$. To find $<\vec{u}, \vec{v}> = \{k_1\vec{u}+k_2\vec{v} : k_1, k_2 \in \mathbf{R}\}$
Case 1 $\vec{u}=\vec{0}, \vec{v}=\vec{0}$
$<\vec{u}, \vec{v}> = \{k_1\vec{0}+k_2\vec{0} : k_1, k_2 \in \mathbf{R}\} = \{\vec{0}\}$
It is the origin itself.
Case 2 $\vec{u}=\vec{0}, \vec{v}\neq\vec{0}$
$<\vec{u}, \vec{v}> = \{k_2\vec{v} : k_2 \in \mathbf{R}\}$
which is a line passes through the origin and parallel to $\vec{v}$.
Case 3 $\vec{u}\neq\vec{0}, \vec{v}=\vec{0}$, similar to case 2.
Case 4 $\vec{u}\neq\vec{0}, \vec{v}\neq\vec{0}$, $\vec{u} // \vec{v}$, i.e. $\vec{v}=k\vec{u}$, $k\neq 0$
$<\vec{u}, \vec{v}> = \{k_1\vec{u}+k_2\vec{v} : k_1, k_2 \in \mathbf{R}\}$
$= \{k_1\vec{u}+k_2k\vec{u} : k_1, k_2 \in \mathbf{R}\} = \{(k_1+k_2k)\vec{u} : k_1, k_2 \in \mathbf{R}\} = <\vec{u}>$
It is a line passes through the origin and parallel to $\vec{u}$.
Case 5 $\vec{u}\neq\vec{0}, \vec{v}\neq\vec{0}$, $\vec{u} \not\parallel \vec{v}$,
$<\vec{u}, \vec{v}> = \{k_1\vec{u}+k_2\vec{v} : k_1, k_2 \in \mathbf{R}\}$
This is a plane equation through the origin.

Q26 Let **V** be the vector space of real-valued functions defined on the entire real line. If $\vec{f}$, $\vec{g}$ and $\vec{h}$ are vectors in **V** that are twice differentiable, then the function $\vec{w} = w(x)$ defined by

$$w(x) = \begin{vmatrix} f(x) & g(x) & h(x) \\ f'(x) & g'(x) & h'(x) \\ f''(x) & g''(x) & h''(x) \end{vmatrix}$$

$\vec{w}$ is called the **Wronskian** of $\vec{f}$, $\vec{g}$ and $\vec{h}$. Prove that $\vec{f}$, $\vec{g}$ and $\vec{h}$ form a linear independent set if the **Wronskian** is not a zero vector in **V**. (i.e. $w(x)$ is not identically zero.)

$k_1f(x) + k_2g(x) + k_3h(x) \equiv 0$ ....(1), find k_1, k_2, k_3.

Differentiate once: $k_1f'(x) + k_2g'(x) + k_3h'(x) \equiv 0$ ....(2)

Differentiate twice: $k_1f''(x) + k_2g''(x) + k_3h''(x) \equiv 0$ ....(3)

Let $A = \begin{pmatrix} f(x) & g(x) & h(x) \\ f'(x) & g'(x) & h'(x) \\ f''(x) & g''(x) & h''(x) \end{pmatrix}$, (1), (2), (3) is equivalent to $A(k_1, k_2, k_3)^t = (0, 0, 0)^t$.

Given that $\det A \neq 0$, A^{-1} exists and so $(k_1, k_2, k_3)^t = A^{-1}(0, 0, 0)^t = (0, 0, 0)^t$

$\therefore f(x), g(x), h(x)$ are independent.

Q27 Use the **Wronskian** to show that the following sets of vector are linear independent.

(a) $1, x, e^x$

(b) $\sin x, \cos x, x \sin x$

(c) $e^x, x e^x, x^2 e^x$

(d) $1, x, x^2$

(a) $w(x) = \begin{vmatrix} 1 & x & e^x \\ 0 & 1 & e^x \\ 0 & 0 & e^x \end{vmatrix} = e^x \begin{vmatrix} 1 & x & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{vmatrix} = e^x \neq 0$. $\therefore \{1, x, e^x\}$ is linear independent

(b) $w(x) = \begin{vmatrix} \sin x & \cos x & x\sin x \\ \cos x & -\sin x & \sin x + x\cos x \\ -\sin x & -\cos x & 2\cos x - x\sin x \end{vmatrix} \underset{R_1+R_3}{=} \begin{vmatrix} \sin x & \cos x & x\sin x \\ \cos x & -\sin x & \sin x + x\cos x \\ 0 & 0 & 2\cos x \end{vmatrix}$

$= 2 \cos x(-\sin^2 x - \cos^2 x) = -2 \cos x \neq 0$. $\therefore \{\sin x, \cos x, x \sin x\}$ is linear independent.

(c) $w(x) = \begin{vmatrix} e^x & xe^x & x^2e^x \\ e^x & e^x + xe^x & 2xe^x + x^2e^x \\ e^x & 2e^x + xe^x & 2 + 4xe^x + x^2e^x \end{vmatrix} = e^{3x} \begin{vmatrix} 1 & x & x^2 \\ 1 & 1+x & 2x+x^2 \\ 1 & 2+x & 2+4x+x^2 \end{vmatrix}$

$\underset{\substack{R_2 - R_1 \\ R_3 - R_1}}{=} e^{3x} \begin{vmatrix} 1 & x & x^2 \\ 0 & 1 & 2x \\ 0 & 2 & 2+4x \end{vmatrix} = 2e^{3x}$. $\therefore \{e^x, x e^x, x^2 e^x\}$ is linear independent.

(d) $w(x) = \begin{vmatrix} 1 & x & x^2 \\ 0 & 1 & 2x \\ 0 & 0 & 2 \end{vmatrix} = 2$. $\therefore \{1, x, x^2\}$ is linear independent.