Centres of triangle (vector)

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HKDSE Sample paper (Frame 30)

14.

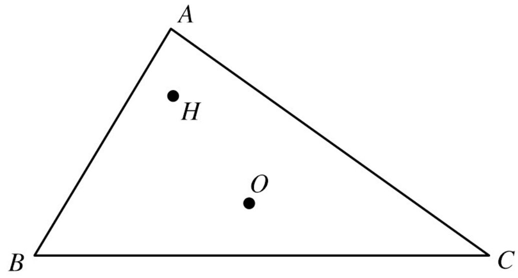


Figure 3

In Figure 3, ΔABC is an acute-angled triangle, where O and H are the circumcentre and orthocentre respectively. Let $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$, $\overrightarrow{OC} = \mathbf{c}$ and $\overrightarrow{OH} = \mathbf{h}$.

(*a*) Show that $(\mathbf{h} - \mathbf{a})//(\mathbf{b} + \mathbf{c})$.

(*b*) Let $\mathbf{h} - \mathbf{a} = t(\mathbf{b} + \mathbf{c})$, where t is a non-zero constant. Show that

(*i*) $t(\mathbf{b} + \mathbf{c}) + \mathbf{a} - \mathbf{b} = s(\mathbf{c} + \mathbf{a})$ for some scalar s,

(*ii*) $(t - 1)(\mathbf{b} - \mathbf{a})\cdot(\mathbf{c} - \mathbf{a}) = 0$.

(*c*) Express $\mathbf{h}$ in terms of $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$.

14. (*a*) $\overrightarrow{AH} = \overrightarrow{OH} - \overrightarrow{OA} = \mathbf{h} - \mathbf{a}$

$\overrightarrow{CB} = \overrightarrow{OB} - \overrightarrow{OC} = \mathbf{b} - \mathbf{c}$

$\because$ AH is the altitude

$\therefore AH \perp CB$

$(\mathbf{h} - \mathbf{a}) \perp (\mathbf{b} - \mathbf{c})$

$(\mathbf{b} + \mathbf{c})\cdot (\mathbf{b} - \mathbf{c}) = |\mathbf{b}|^2 - |\mathbf{c}|^2 = 0$ ($\because$ O is the circumcentre $\therefore$ $OA = OB = OC$)

$\therefore (\mathbf{b} + \mathbf{c}) \perp (\mathbf{b} - \mathbf{c})$

$(\mathbf{h} - \mathbf{a})//(\mathbf{b} + \mathbf{c})$.

(*b*) (*i*) $\mathbf{h} - \mathbf{a} = t(\mathbf{b} + \mathbf{c})$ ……(1) where t is a non-zero constant.

Using a similar method as in (*a*), $(\mathbf{h} - \mathbf{b})//(\mathbf{a} + \mathbf{c})$

$\mathbf{h} - \mathbf{b} = s(\mathbf{a} + \mathbf{c})$ ……(2) where s is a non-zero constant.

(2) – (1): $\mathbf{a} - \mathbf{b} = s(\mathbf{a} + \mathbf{c}) - t(\mathbf{b} + \mathbf{c})$

$t(\mathbf{b} + \mathbf{c}) + \mathbf{a} - \mathbf{b} = s(\mathbf{c} + \mathbf{a})$ for some scalar s.

(*ii*) $t\mathbf{b} + t\mathbf{c} + \mathbf{a} - \mathbf{b} = s(\mathbf{c} + \mathbf{a})$

$t\mathbf{b} - t\mathbf{a} + \mathbf{a} - \mathbf{b} = s(\mathbf{c} + \mathbf{a}) - t\mathbf{a} - t\mathbf{c}$

$(t - 1)(\mathbf{b} - \mathbf{a}) = (s - t)(\mathbf{a} + \mathbf{c})$

$(t - 1)(\mathbf{b} - \mathbf{a})\cdot(\mathbf{c} - \mathbf{a}) = (s - t)(\mathbf{a} + \mathbf{c})\cdot(\mathbf{c} - \mathbf{a})$

$= (s - t)(|\mathbf{c}|^2 - |\mathbf{a}|^2) = 0$

(*c*) $(\mathbf{b} - \mathbf{a})\cdot(\mathbf{c} - \mathbf{a}) = \overrightarrow{AB} \cdot \overrightarrow{AC}$

$\because$ ΔABC is an acute-angled triangle

$\therefore$ $\overrightarrow{AB} \cdot \overrightarrow{AC} \neq 0$

$(\mathbf{b} - \mathbf{a})\cdot(\mathbf{c} - \mathbf{a}) \neq 0$

By (*b*)(*ii*), $(t - 1)(\mathbf{b} - \mathbf{a})\cdot(\mathbf{c} - \mathbf{a}) = 0$

$\Rightarrow t = 1$

$\therefore$ $\mathbf{h} - \mathbf{a} = t(\mathbf{b} + \mathbf{c})$ by (*b*)(*i*)

$\mathbf{h} = \mathbf{a} + \mathbf{b} + \mathbf{c}$

Frame 24

Let G be the centroid and O be the circumcentre of ΔABC.

Let $\overrightarrow{OA} = \mathbf{a}, \overrightarrow{OB} = \mathbf{b}, \overrightarrow{OC} = \mathbf{c}, \quad \overrightarrow{OG} = \mathbf{g}$.

$\overrightarrow{OD} = \frac{1}{2}(\mathbf{b} + \mathbf{c})$ ($\because$ D is the mid-point of BC)

As G is the centroid, $AG : GD = 2 : 1$

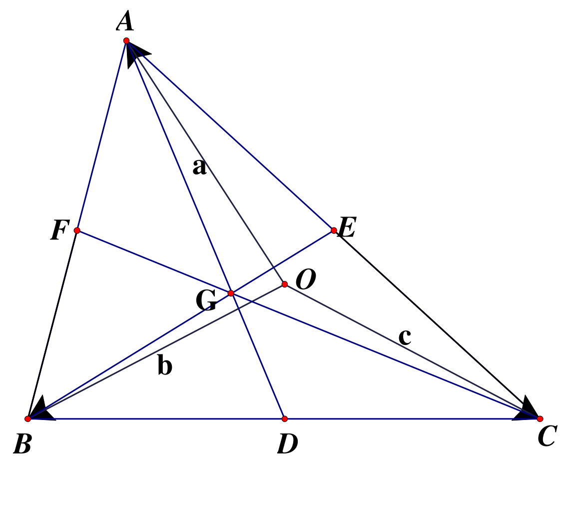
$\mathbf{g} = \frac{1}{3}\left(\mathbf{a} + 2\overrightarrow{OD}\right) = \frac{1}{3}(\mathbf{a} + \mathbf{b} + \mathbf{c})$ …… (**)

$\overrightarrow{GA} = \mathbf{a} - \mathbf{g}, \overrightarrow{GB} = \mathbf{b} - \mathbf{g}, \overrightarrow{GC} = \mathbf{c} - \mathbf{g}$

$\overrightarrow{GA} + \overrightarrow{GB} + \overrightarrow{GC} = \mathbf{a} - \mathbf{g} + \mathbf{b} - \mathbf{g} + \mathbf{c} - \mathbf{g}$

$= (\mathbf{a} + \mathbf{b} + \mathbf{c}) - 3\mathbf{g}$

$= 3\mathbf{g} - 3\mathbf{g} = \mathbf{0}$ by (**)



Frame 31

Let H be the orthocentre and O be the circumcentre of ΔABC.

Let $\overrightarrow{OA} = \mathbf{a}, \overrightarrow{OB} = \mathbf{b}, \overrightarrow{OC} = \mathbf{c}, \quad \overrightarrow{OH} = \mathbf{h}$.

By the above result, $\mathbf{h} = \mathbf{a} + \mathbf{b} + \mathbf{c}$

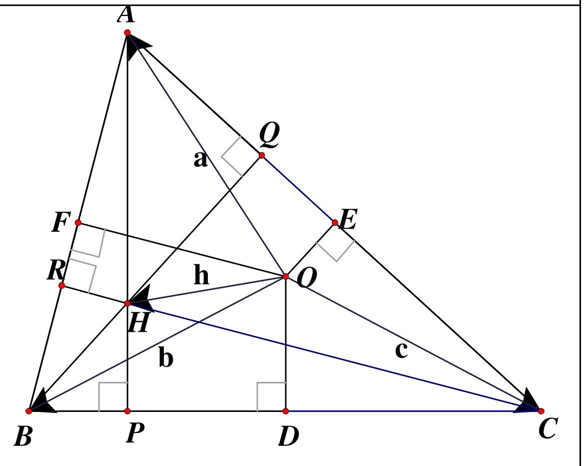
$\overrightarrow{HA} = \mathbf{a} - \mathbf{h}, \overrightarrow{HB} = \mathbf{b} - \mathbf{h}, \overrightarrow{HC} = \mathbf{c} - \mathbf{h}$

$\overrightarrow{HA} + \overrightarrow{HB} + \overrightarrow{HC} = \mathbf{a} - \mathbf{h} + \mathbf{b} - \mathbf{h} + \mathbf{c} - \mathbf{h}$

$= (\mathbf{a} + \mathbf{b} + \mathbf{c}) - 3\mathbf{h}$

$= -2\mathbf{h}$

$= 2\overrightarrow{HO}$



Frame 32

When $\mathbf{a} + \mathbf{b} + \mathbf{c} = \mathbf{h} = \mathbf{0}$, the orthocentre will overlap with the circumcentre.

$\because$ O is the circumcentre

$\therefore$ $OA = OB = OC \Rightarrow |\mathbf{a}| = |\mathbf{b}| = |\mathbf{c}|$

$\mathbf{c} = -(\mathbf{a} + \mathbf{b})$

$|\mathbf{c}|^2 = (\mathbf{a} + \mathbf{b})\cdot(\mathbf{a} + \mathbf{b}) = |\mathbf{a}|^2 + |\mathbf{b}|^2 + 2\mathbf{a}\cdot\mathbf{b}$

$2\mathbf{a}\cdot\mathbf{b} = -|\mathbf{a}|^2 \Rightarrow 2\mathbf{a}\cdot\mathbf{b} = -|\mathbf{c}|^2$ …… (*)

In this case, $\overrightarrow{AB} = \mathbf{b} - \mathbf{a}, \overrightarrow{BC} = \mathbf{c} - \mathbf{b}, \overrightarrow{CA} = \mathbf{a} - \mathbf{c}$

$|\overrightarrow{AB}|^2 = (\mathbf{b} - \mathbf{a})\cdot(\mathbf{b} - \mathbf{a}) = |\mathbf{b}|^2 + |\mathbf{a}|^2 - 2\mathbf{a}\cdot\mathbf{b}$

$= |\mathbf{a}|^2 + |\mathbf{b}|^2 + |\mathbf{c}|^2$ by (*)

Similarly, $|\overrightarrow{BC}|^2 = |\mathbf{a}|^2 + |\mathbf{b}|^2 + |\mathbf{c}|^2$

and $|\overrightarrow{CA}|^2 = |\mathbf{a}|^2 + |\mathbf{b}|^2 + |\mathbf{c}|^2$

$\therefore$ ΔABC is an equilateral triangle.

Frame 34

Let X be any point. $\overrightarrow{OX} = \mathbf{x}$

$\overrightarrow{XA} + \overrightarrow{XB} + \overrightarrow{XC} = \mathbf{a} - \mathbf{x} + \mathbf{b} - \mathbf{x} + \mathbf{c} - \mathbf{x}$

$= (\mathbf{a} + \mathbf{b} + \mathbf{c}) - 3\mathbf{x}$

$= 3\left[\frac{1}{3}(\mathbf{a} + \mathbf{b} + \mathbf{c}) - \mathbf{x}\right]$

$= 3\left[\overrightarrow{OG} - \overrightarrow{OX}\right]$

$= 3\overrightarrow{XG}$

Frame 36: (2006 Q18)

$\because \overrightarrow{OA}+\overrightarrow{OB}+\overrightarrow{OC}=\overrightarrow{OH}$ and $\overrightarrow{OG}=\frac{1}{3}(\mathbf{a}+\mathbf{b}+\mathbf{c})$

$\therefore \overrightarrow{OH}=3\overrightarrow{OG}$

O, G, H are collinear and $OG : GH = 1 : 2$

The line segment O, G, H is called the Euler line.

The 9-point circle.

Let K, L, M, N be the mid-points of HA, HB, HC, OH respectively. $\overrightarrow{ON}=\frac{1}{2}\mathbf{h}$

$\overrightarrow{NK}=\frac{1}{2}\mathbf{a}$ (mid point theorem)

$\overrightarrow{NL}=\frac{1}{2}\mathbf{b}$ (mid point theorem)

$\overrightarrow{NM}=\frac{1}{2}\mathbf{c}$ (mid point theorem)

$\overrightarrow{ND}=\frac{1}{2}(\mathbf{b}+\mathbf{c})-\frac{1}{2}\mathbf{h}=-\frac{1}{2}\mathbf{a}$

$\overrightarrow{NE}=\frac{1}{2}(\mathbf{c}+\mathbf{a})-\frac{1}{2}\mathbf{h}=-\frac{1}{2}\mathbf{b}$

$\overrightarrow{NF}=\frac{1}{2}(\mathbf{a}+\mathbf{b})-\frac{1}{2}\mathbf{h}=-\frac{1}{2}\mathbf{c}$

$\overrightarrow{PK}=\overrightarrow{NK}-\overrightarrow{NP}=\frac{1}{2}\mathbf{a}-\overrightarrow{NP}$

$\overrightarrow{PD}=\overrightarrow{ND}-\overrightarrow{NP}=-\frac{1}{2}\mathbf{a}-\overrightarrow{NP}$

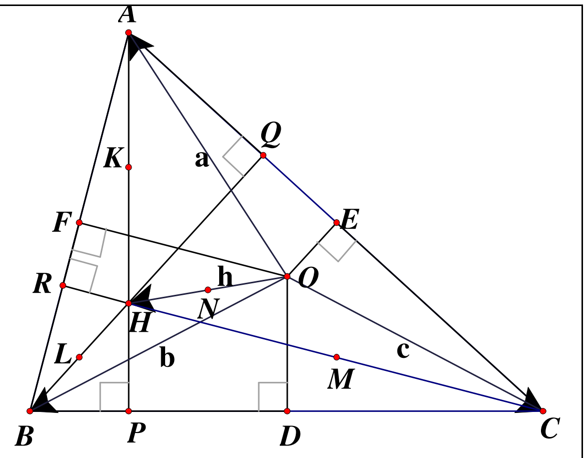
$\overrightarrow{PK}\cdot\overrightarrow{PD}=(\frac{1}{2}\mathbf{a}-\overrightarrow{NP})\cdot(-\frac{1}{2}\mathbf{a}-\overrightarrow{NP})=0$

$|\overrightarrow{NP}|^2=\left|\frac{1}{2}\mathbf{a}\right|^2$

Similarly, $|\overrightarrow{NQ}|^2=\left|\frac{1}{2}\mathbf{b}\right|^2$ and $|\overrightarrow{NR}|^2=\left|\frac{1}{2}\mathbf{c}\right|^2$

$\therefore$ We can use N as centre, radius $=\frac{1}{2}|\mathbf{a}|$ to draw a circle to pass through D E, F, K, L, M, P, Q, R.

This is the nine-point circle.



$\because KP \perp PD$