Determinant Note

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M2 Example 14 Page 31 Consider the 3 ×3 determinant $\begin{vmatrix} 1 & -1 & 0 \\ 2 & 2 & 5 \\ 3 & 4 & 1 \end{vmatrix}$.

Minors

$M_{11} = \begin{vmatrix} 2 & 5 \\ 4 & 1 \end{vmatrix} = 2–20 = –18$	$M_{12} = \begin{vmatrix} 2 & 5 \\ 3 & 1 \end{vmatrix} = 2–15 = –13$	$M_{13} = \begin{vmatrix} 2 & 2 \\ 3 & 4 \end{vmatrix} = 8–6 = 2$
$M_{21} = \begin{vmatrix} -1 & 0 \\ 4 & 1 \end{vmatrix} = –1–0 = –1$	$M_{22} = \begin{vmatrix} 1 & 0 \\ 3 & 1 \end{vmatrix} = 1–0 = 1$	$M_{23} = \begin{vmatrix} 1 & -1 \\ 3 & 4 \end{vmatrix} = 4–(–3) = 7$
$M_{31} = \begin{vmatrix} -1 & 0 \\ 2 & 5 \end{vmatrix} = –5–0 = –5$	$M_{32} = \begin{vmatrix} 1 & 0 \\ 2 & 5 \end{vmatrix} = 5–0 = 5$	$M_{33} = \begin{vmatrix} 1 & -1 \\ 2 & 2 \end{vmatrix} = 2–(–2) = 4$

Cofactors

$A_{11} = (–1)^{1+1}M_{11} = (–1)^2(2–20) = –18$	$A_{12} = (–1)^{1+2}M_{12} = (–1)^3(–13) = 13$	$A_{13} = (–1)^{1+3}M_{13} = (–1)^4\, 2 = 2$
$A_{21} = (–1)^{2+1}M_{21} = (–1)(–1) = 1$	$A_{22} = (–1)^{2+2}M_{22} = 1$	$A_{23} = (–1)^{2+3}M_{23} = (–1)\times 7 = –7$
$A_{31} = (–1)^{3+1}M_{31} = –5$	$A_{32} = (–1)^{3+2}M_{32} = –5$	$A_{33} = (–1)^{3+3}M_{33} = 4$

The signs (+/–) of cofactors and minors

$A_{11} = M_{11}$	$A_{12} = –M_{12}$	$A_{13} = M_{13}$
$A_{21} = –M_{21}$	$A_{22} = M_{22}$	$A_{23} = –M_{23}$
$A_{31} = M_{31}$	$A_{32} = –M_{32}$	$A_{33} = M_{33}$

In short form,

+	–	+
–	+	–
+	–	+

Consider the 2×2 determinant $\begin{vmatrix} -3 & 2 \\ -1 & 4 \end{vmatrix}$.

Minors

$M_{11} = \lvert 4 \rvert = 4$	$M_{12} = \lvert -1 \rvert = -1$
$M_{21} = \lvert 2 \rvert = 2$	$M_{22} = \lvert -3 \rvert = -3$

Cofactors

$A_{11} = (-1)^{1+1}M_{11} = 4$	$A_{12} = (-1)^{1+2}M_{12} = 1$
$A_{21} = (-1)^{2+1}M_{21} = -2$	$A_{22} = (-1)^{2+2}M_{22} = -3$

The signs (+/–) of cofactors and minors

$A_{11} = M_{11}$	$A_{12} = -M_{12}$
$A_{21} = -M_{21}$	$A_{22} = M_{22}$

In short form,

+	–
–	+

Consider the 1×1 determinant $\lvert -3 \rvert$.

M_{11} = undefined, A_{11} is also undefined.

(1) Cofactor expansion of 3×3 determinant

$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13}$ (cofactor expansion along the first row) ⋯⋯ (1)

$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{21}A_{21} + a_{22}A_{22} + a_{23}A_{23}$ (cofactor expansion along the second row) ⋯⋯ (2)

$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{31}A_{31} + a_{32}A_{32} + a_{33}A_{33}$ (cofactor expansion along the third row) ⋯⋯ (3)

$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}A_{11} + a_{21}A_{21} + a_{31}A_{31}$ (cofactor expansion along the first column) ⋯⋯ (4)

$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{12}A_{12} + a_{22}A_{22} + a_{32}A_{32}$ (cofactor expansion along the second column) ⋯⋯ (5)

$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{13}A_{13} + a_{23}A_{23} + a_{33}A_{33}$ (cofactor expansion along the third column) ⋯⋯ (6)

We shall prove the equality of (3):

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - (a_{13}a_{22}a_{31} + a_{11}a_{23}a_{32} + a_{12}a_{21}a_{33})$$

$$\text{R.H.S.} = a_{31}\begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} + a_{32}(-1)^{3+2}\begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} + a_{33}\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

$$= a_{31}(a_{12}a_{23} - a_{22}a_{13}) - a_{32}(a_{11}a_{23} - a_{21}a_{13}) + a_{33}(a_{11}a_{22} - a_{21}a_{12})$$

$$= a_{31}a_{12}a_{23} + a_{32}a_{21}a_{13} + a_{33}a_{11}a_{22} - (a_{31}a_{22}a_{13} + a_{32}a_{11}a_{23} + a_{33}\,a_{21}a_{12})$$

∴ L.H.S. = R.H.S.

Equality (3) holds

For the other 5 equalities, please prove them by yourself.

Consider a 3×3 determinant $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$.

(2) If a row or a column consists of entirely zeros, then the value of the determiant is **zero**.

Proof: Suppose the second row consists of entirely zero, then

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ 0 & 0 & 0 \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = 0A_{21} + 0A_{22} + 0A_{23} = 0 \text{ (Cofactor expansion along the second row)}$$

Suppose the third column consists of entirely zero, then

$$\begin{vmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & 0 \end{vmatrix} = 0A_{13} + 0A_{23} + 0A_{33} = 0 \text{ (Cofactor expansion along the third column)}$$

(3) If two rows or two columns are identical, then the value of the determiant is **zero**.

Proof: Suppose the second row is identical to the third row, then

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{21} & a_{22} & a_{23} \end{vmatrix} = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{22} & a_{23} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{21} & a_{23} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{21} & a_{22} \end{vmatrix} = 0 \text{ (Cofactor expansion along the first row)}$$

Suppose the first column is identical to the third column, then

$$\begin{vmatrix} a_{11} & a_{12} & a_{11} \\ a_{21} & a_{22} & a_{21} \\ a_{31} & a_{32} & a_{31} \end{vmatrix} = a_{12}\begin{vmatrix} a_{21} & a_{21} \\ a_{31} & a_{31} \end{vmatrix} - a_{22}\begin{vmatrix} a_{11} & a_{11} \\ a_{31} & a_{31} \end{vmatrix} + a_{32}\begin{vmatrix} a_{11} & a_{11} \\ a_{21} & a_{22} \end{vmatrix} = 0 \text{ (Cofactor expansion along the second column)}$$

In particular, $a_{11}A_{21} + a_{12}A_{22} + a_{13}A_{23} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{vmatrix} = 0$

$a_{11}A_{31} + a_{12}A_{32} + a_{13}A_{33} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \end{vmatrix} = 0$

More generally, $a_{i1}A_{j1} + a_{i2}A_{j2} + a_{i3}A_{j3} = \begin{cases} 0 & \text{for } i \neq j \\ \det A & \text{for } i = j \end{cases}$.

Also, $a_{12}A_{11} + a_{22}A_{21} + a_{32}A_{31} = \begin{vmatrix} a_{12} & a_{12} & a_{13} \\ a_{22} & a_{22} & a_{23} \\ a_{32} & a_{32} & a_{33} \end{vmatrix} = 0$

More generally, $a_{1i}A_{1j} + a_{2i}A_{2j} + a_{3i}A_{3j} = \begin{cases} 0 & \text{for } i \neq j \\ \det A & \text{for } i = j \end{cases}$.

(4) Let k be a constant, then $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ ka_{31} & ka_{32} & ka_{33} \end{vmatrix} = k\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$, the same results apply to all rows.

Also, $\begin{vmatrix} a_{11} & ka_{12} & a_{13} \\ a_{21} & ka_{22} & a_{23} \\ a_{31} & ka_{32} & a_{33} \end{vmatrix} = k\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$, the same results apply to all columns.

Proof: $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ ka_{31} & ka_{32} & ka_{33} \end{vmatrix} = ka_{31}A_{31} + ka_{32}A_{32} + ka_{33}A_{33}$ (cofactor expansion along the third row)

$= k(a_{31}A_{31} + a_{32}A_{32} + a_{33}A_{33})$

$= k\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

(5) Let b_1, b_2, b_3 be any numbers, then $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} b_1 & b_2 & b_3 \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11}+b_1 & a_{12}+b_2 & a_{13}+b_3 \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$.

The same results apply to all rows and all columns.

Proof: $\begin{vmatrix} a_{11}+b_1 & a_{12}+b_2 & a_{13}+b_3 \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = (a_{11}+b_1)A_{11} + (a_{12}+b_2)A_{12} + (a_{13}+b_3)A_{13}$ (cofactor expansion along the first row)

$= (a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13}) + (b_1A_{11} + b_2A_{12} + b_3A_{13})$

$= \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} b_1 & b_2 & b_3 \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

(6) If one row is a scalar multiple of the other row, then the value of the determinant is **zero**.
If one column is a scalar multiple of the other column, then the value of the determinant is **zero**.

Proof: Suppose the third column is a scalar multiple of the second column, then

$\begin{vmatrix} a_{11} & a_{12} & ka_{12} \\ a_{21} & a_{22} & ka_{22} \\ a_{31} & a_{32} & ka_{32} \end{vmatrix} = k\begin{vmatrix} a_{11} & a_{12} & a_{12} \\ a_{21} & a_{22} & a_{22} \\ a_{31} & a_{32} & a_{32} \end{vmatrix}$ (by property (4))

$= 0$ (by property (3))

(7) Interchange any two rows or any two columns gives **the negative value of the determinant**.

i.e. $\begin{vmatrix} a_{31} & a_{32} & a_{33} \\ a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \end{vmatrix} = -\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$ and $\begin{vmatrix} a_{11} & a_{13} & a_{12} \\ a_{21} & a_{23} & a_{22} \\ a_{31} & a_{33} & a_{32} \end{vmatrix} = -\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

Proof: $\begin{vmatrix} a_{31} & a_{32} & a_{33} \\ a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \end{vmatrix} = -a_{21}\begin{vmatrix} a_{32} & a_{33} \\ a_{12} & a_{13} \end{vmatrix} + a_{22}\begin{vmatrix} a_{31} & a_{33} \\ a_{11} & a_{13} \end{vmatrix} - a_{23}\begin{vmatrix} a_{31} & a_{32} \\ a_{11} & a_{12} \end{vmatrix}$ (cofactor expansion along the second row)

$= a_{21}\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} - a_{22}\begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} + a_{23}\begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix}$

$= -\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

The proof of the second identity is similar and will be left to you as an exercise.

(8) The determinant of a square matrix is equal to the determinant of its transpose.

i.e. $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{vmatrix}$

Proof: $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$ (cofactor expansion along the first row)

$$= a_{11}\begin{vmatrix} a_{22} & a_{32} \\ a_{23} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{31} \\ a_{23} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{31} \\ a_{22} & a_{32} \end{vmatrix}$$

$$= \begin{vmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{vmatrix}$$