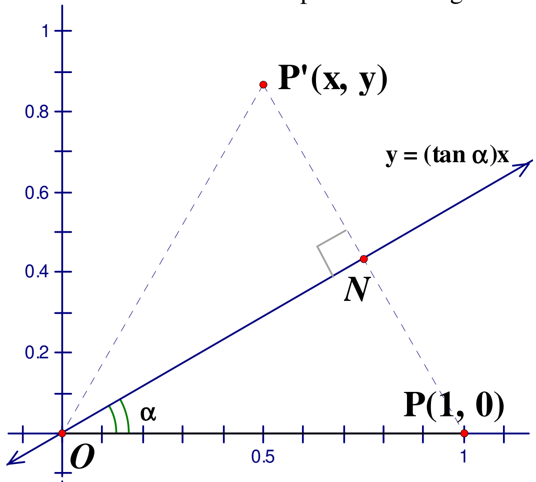
Reflection in the line $y = (\tan \alpha)\, x$

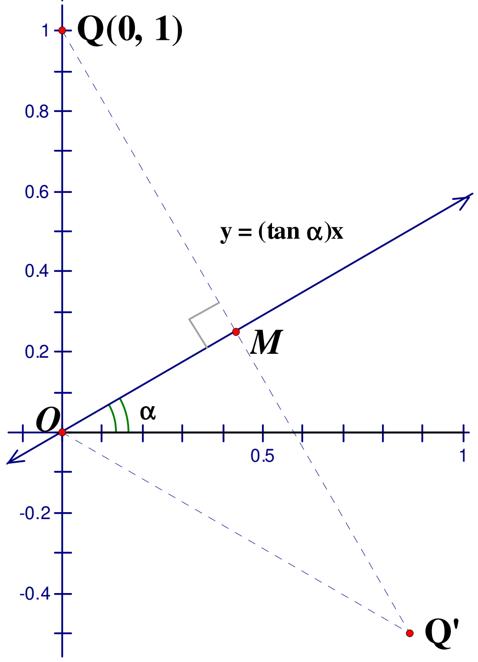
Created by Mr. Francis Hung on 20110420 Last updated: 07 August 2021

Suppose a point $P(1, 0)$ is transform to another point $P'(x, y)$ as shown in the figure.



Join PP', which intersects the line $y = (\tan \alpha)\, x$ at N.
By the property of reflection, $PP' \perp ON$ and $PN = P'N$
$\Delta OPN \cong \Delta OP'N$ (S.A.S.)
$\therefore\ OP' = OP = 1$ (corr. sides $\cong \Delta$'s)
$\angle P'ON = \alpha$ (corr. $\angle$s $\cong \Delta$'s)
$\angle P'OP = 2\alpha$
$\therefore\ P' = (\cos 2\alpha, \sin 2\alpha)$

Suppose a point $Q(0, 1)$ is transform to another point Q' as shown in the figure.



Join QQ', which intersects the line $y = (\tan \alpha)\, x$ at M.
By the property of reflection, $QQ' \perp OM$ and $QM = Q'M$
$\Delta OQM \cong \Delta OQ'M$ (S.A.S.)
$\therefore\ OQ' = OQ = 1$ (corr. sides $\cong \Delta$'s)

$\angle QOM = \frac{\pi}{2} - \alpha$

$\angle Q'OM = \frac{\pi}{2} - \alpha$ (corr. $\angle$s $\cong \Delta$'s)

$\angle Q'Ox\text{-axis} = \frac{\pi}{2} - 2\alpha$

$$\therefore\ Q' = \left(\cos\left(\frac{\pi}{2} - 2\alpha\right), -\sin\left(\frac{\pi}{2} - 2\alpha\right)\right)$$
$$= (\sin 2\alpha, -\cos 2\alpha)$$

$\therefore$ The matrix of reflection is $\begin{pmatrix} \cos 2\alpha & \sin 2\alpha \\ \sin 2\alpha & -\cos 2\alpha \end{pmatrix}$.