Triangle Theorem (Vectors)

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In ΔABC, AD, BF and CE are concurrent at G.

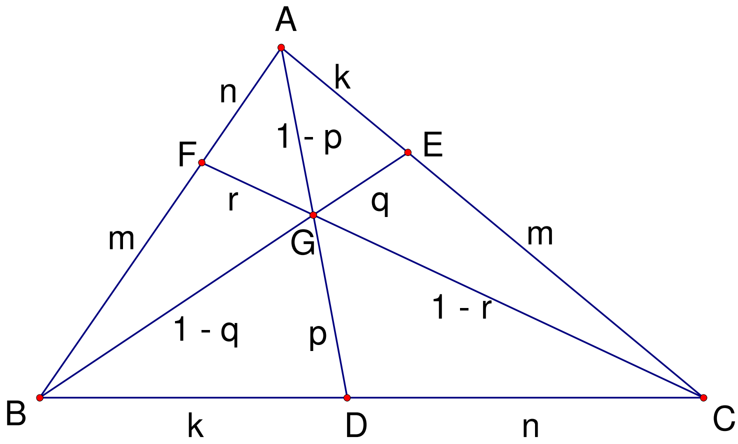
By Ceva's theorem, $\frac{BD}{DC}\cdot\frac{CE}{EA}\cdot\frac{AF}{FB}=1$

Let $BD : DC = k : n$, $CE : EA = m : k$, $AF : FB = n : m$

Let $AG : GD = 1 - p : p$, $BG : GE = 1 - q : q$, $CG : GF = 1 - r : r$

Then $p + q + r = 1$

Proof: method 1 (vector method)



Let $\overrightarrow{BC}=\vec{c}$, $\overrightarrow{BA}=\vec{a}$, $\overrightarrow{BD}=\frac{k}{k+n}\vec{c}$

$\overrightarrow{BG}=p\overrightarrow{BA}+(1-p)\overrightarrow{BD}=p\,\vec{a}+\frac{(1-p)k\vec{c}}{k+n}$

$\overrightarrow{BE}=\frac{m\vec{a}+k\vec{c}}{m+k}$

$\because \overrightarrow{BG}\,//\,\overrightarrow{BE}$, $\overrightarrow{BG}=s\,\overrightarrow{BE}$

$p\,\vec{a}+\frac{(1-p)k\vec{c}}{k+n}=\frac{sm\vec{a}}{m+k}+\frac{sk\vec{c}}{m+k}$

Compare coefficients,

$$\therefore \begin{cases}\frac{sm}{m+k}=p\cdots\cdots\cdots\cdots(1)\\ \frac{sk}{m+k}=\frac{(1-p)k}{k+n}\cdots(2)\end{cases}$$

$\frac{(2)}{(1)}:\ \frac{k}{m}=\frac{(1-p)k}{p(k+n)}$

$pk + pn = m - mp$

$p(k + m + n) = m$

$p=\frac{m}{k+m+n}$

similarly $q=\frac{n}{k+m+n}$, $r=\frac{k}{k+m+n}$

$\therefore p + q + r = 1$