120° triangle

Created by Mr. Francis Hung Last updated: April 18, 2024

Given a triangle with one angle is 120°. If all sides are integers, find all possible solutions.

This document was created by Mr. Francis Hung, last updated: April 18, 2024.

$c^2 = a^2 + b^2 - 2ab \cos 120°$

$c^2 = a^2 + b^2 + ab$

$c^2 = (a + b)^2 - ab$

$ab = (a + b)^2 - c^2$

$ab = (a + b + c)(a + b - c)$

$\frac{a+b+c}{a} = \frac{b}{a+b-c} = k$, where k is a positive constant.

$a + b + c = ak$; $b = (a + b - c)k$

$$\Rightarrow \begin{cases} a(1-k)+b+c = 0 \cdots\cdots(1) \\ ak + b(k-1) - ck = 0 \cdots\cdots(2) \end{cases}$$

From (1): $c = a(k - 1) - b$ ....(3)

Sub. (3) into (2): $ak + b(k - 1) - a(k^2 - k) + bk = 0$

$b(2k - 1) = a(k^2 - 2k)$

Let $a = (2k - 1)p$, $b = (k^2 - 2k)p$, then $c = (k^2 - k + 1)p$; where p is a positive integer.

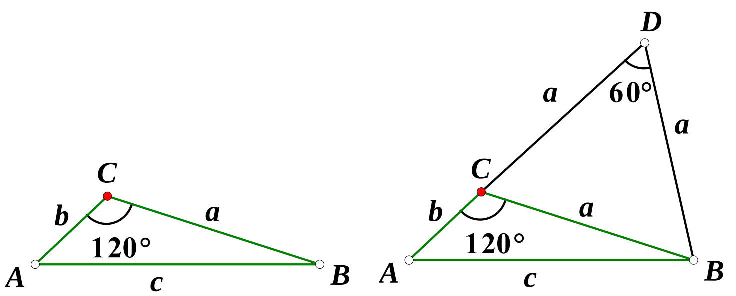
$a : b : c = (2k - 1) : (k^2 - 2k) : (k^2 - k + 1)$

Let $a = (2k - 1)p$, $b = k(k - 2)p$, $c = (k^2 - k + 1)p$; where p is a positive integer. Let $p = 1$.

k	a	b	c
3	5	3	7
4	7	8	13
5	9	15	21
6	11	24	31

Given a triangle with one angle is 60°. If all sides are integers, find all possible solution.

Given the above triangle with $\angle C = 120°$, we can construct another triangle ABD with $\angle D = 60°$



So, if (a, b, c) is a solution to a 120° triangle, then $(a, a+b, c)$ or $(a+b, b, c)$ is a solution to a 60° Δ.

The general solution are: $((2k - 1)p, (k^2 - 1)p, (k^2 - k + 1)p)$ or $((k^2 - 1)p, (k^2 - 2k)p, (k^2 - k + 1)p)$. Let $p = 1$.

k	a	$a + b$	c	$a + b$	b	c
2	3	3	3			
3	5	8	7	8	3	7
4	7	15	13	15	8	13
5	9	24	21	24	15	21
6	11	35	31	35	24	31