Solve for integral solutions of $\frac{1}{a^2}+\frac{1}{b^2}=\frac{1}{c^2}$.

Created by Mr. Francis Hung on 29 August, 2012. Last updated: 1 July 2023.

Find also the sum of all possible $0 < a \le 100$.

Clearly a, b and c are non-zero.

Multiply both sides by $a^2b^2c^2$.

$(bc)^2 + (ac)^2 = (ab)^2$

Let $x = bc$, $y = ac$, $z = ab$, then $x^2 + y^2 = z^2$, where x, y and z are non-zero integers.

$\because$ (x, y, z) is a Pythagoreans' triple.

$\therefore$ The solutions are $x = bc = 2kpq \cdots\cdots (1)$, $y = ac = k(p^2 - q^2) \cdots\cdots (2)$, $z = ab = k(p^2 + q^2) \cdots (3)$

where k is an integer, $p > q$, p and q are relatively prime positive integers.

$$\frac{(1)\times(2)}{(3)}: c^2=\frac{2pq(p^2-q^2)k}{p^2+q^2} \Rightarrow c=\pm\sqrt{\frac{2pq(p^2-q^2)k}{p^2+q^2}}$$

$$\frac{(1)\times(3)}{(2)}: b^2=\frac{2pq(p^2+q^2)k}{p^2-q^2} \Rightarrow b=\pm\sqrt{\frac{2pq(p^2+q^2)k}{p^2-q^2}}$$

$$\frac{(2)\times(3)}{(1)}: a^2=\frac{(p^2-q^2)(p^2+q^2)k}{2pq} \Rightarrow a=\pm\sqrt{\frac{(p^2-q^2)(p^2+q^2)k}{2pq}}$$

Let $t=\sqrt{\frac{k}{2pq(p^2-q^2)(p^2+q^2)}}$, then $a = \pm(p^2 - q^2)(p^2 + q^2)t$, $b = \pm 2pq(p^2 + q^2)t$, $c = \pm 2pq(p^2 - q^2)t$

t	p	q	a	b	c
1	2	1	15	20	12
2	2	1	30	40	24
3	2	1	45	60	36
4	2	1	60	80	48
5	2	1	75	100	60
6	2	1	90	120	72
1	3	1	80	60	48
1	3	2	65	156	60

The highlight triples are repeated. So one of them is rejected.

There is no other triples for which a is less than or equal to 100.

Note that a and b can be interchanged and '60' appeared twice.

Sum of all possible a = 15 + 30 + 45 + 60 + 75 + 90 + 65 + 20 + 40 + 80 + 100 = 620