Chinese Remainder Theorem

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Question: N is a positive integer. When N is divided by 3, the remainder is 1. When N is divided by 5, the remainder is 2. When N is divided by 7, the remainder is 5. Find the least positive integral value of N.

Step 1: When N is divided by 3, the remainder is 1. Let $n_1 = 1$

When N is divided by 5, the remainder is 2. Let $n_2 = 2$

When N is divided by 7, the remainder is 5. Let $n_3 = 5$

$5\times7 = 35$, $3\times5 = 15$, $3\times7 = 21$, LCM of 3, 5, 7 is 105

Step 2: Solve $35m - 3k = 1$ for positive integral value of m gives $m = 2$, $k = 23$

Let $a_1 = 35\times m = 70$

Step 3: Solve $21m - 5k = 1$ for positive integral value of m gives $m = 1$, $k = 4$

Let $a_2 = 21\times m = 21$

Step 4: Solve $15m - 7k = 1$ for positive integral value of m gives $m = 1$, $k = 2$

Let $a_3 = 15\times m = 15$

Step 5: Let $N = a_1\times n_1 + a_2\times n_2 + a_3\times n_3 = 70\times1 + 21\times2 + 15\times5 = 187$

The minimum positive integral value of N is the remainder when 187 divided by 105, which is 82.

Please refer to the following website:
http://episte.math.ntu.edu.tw/articles/sm/sm_01_01_2/page6.html

有一組連續的四個正整數，從小到大依次排列，第一個是 5 的倍數；第二個是 7 的倍數；第三個是 9 的倍數；第四個是 11 的倍數。試求最小的可能整數。

Step 1:

Let the four numbers be n, $n + 1$, $n + 2$, $n + 3$

When n is divided by 5, the remainder is 0. Let $n_1 = 0$

$n + 1$ is divisible by 7; when n is divided by 7, the remainder is 6. Let $n_2 = 6$

$n + 2$ is divisible by 9; when n is divided by 9, the remainder is 7. Let $n_3 = 7$

$n + 3$ is divisible by 11, when n is divided by 11, the remainder is 8. Let $n_4 = 8$

$7\times9\times11 = 693$, $5\times9\times11 = 495$, $5\times7\times11 = 385$, $5\times7\times9 = 315$, LCM of 5, 7, 9,11 is 3465

Step 2: Solve $5m + 693k = 1$ for positive integral value gives $m = -277$, $k = 2$

Let $a_1 = 693\times2 = 1386$

Step 3: Solve $7m + 495k = 1$ for positive integral value gives $m = -212$, $k = 3$

Let $a_2 = 495\times3 = 1485$

Step 4: Solve $9m + 385k = 1$ for positive integral value gives $m = -171$, $k = 4$

Let $a_3 = 385\times4 = 1540$

Step 5: Solve $11m + 315k = 1$ for positive integral value gives $m = 84$, $k = -3$

Let $a_4 = 315\times(-3) = -945$

Step 6: Let $N = a_1\times n_1 + a_2\times n_2 + a_3\times n_3 + a_4\times n_4 = 1386\times0+1485\times6+1540\times7-945\times8=12130$

The minimum positive integral value of N is the remainder when 12130 divided by 3465, which is 1735.

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If x is a 4-digit positive integer satisfying $\begin{cases} 3x \equiv 17 \pmod{19} \\ 5x \equiv 27 \pmod{29} \\ 7x \equiv 2 \pmod{11} \end{cases}$, find the largest possible value of x.

$3x - 17 = 19a$ …… (1)

$5x - 27 = 29b$ …… (2)

$7x - 2 = 11a$ …… (3)

From (2), $5x - 29b = 27$

We solve $5x - 29b = 1$. A particular solution is $5\times6 - 29\times1 = 1$.

Multiply the whole equation by 27: $5\times162 - 29\times27 = 27$.

A particular solution to $5x - 29b = 27$ is $(x, b) = (162, 27)$.

The general solution is $(x, b) = (162 + 29k, 27 + 5k)$.

Sub. $x = 162 + 29k$ into (1): $3(162 + 29k) - 17 = 19a$

$19a - 87k = 469$

We solve $19a - 87k = 1$.

$87 = 19\times4 + 11$

$19 = 11 + 8$

$11 = 8 + 3$

$8 = 3\times2 + 2$

$3 = 2 + 1$

$1 = 3 - 2$

$2 = 8 - 3\times2$

$3 = 11 - 8$

$8 = 19 - 11$

$11 = 87 - 19\times4$

$1 = 3 - 2 = 3 - (8 - 3\times2) = 3\times3 - 8$

$1 = (11 - 8)\times3 - 8 = 11\times3 - 8\times4$

$1 = 11\times3 - (19 - 11)\times4 = 11\times7 - 19\times4$

$1 = (87 - 19\times4)\times7 - 19\times4 = 87\times7 - 19\times32$

Multiply the whole equation by 469: $19\times(-15008) - 87\times(-3283) = 469$

A particular solution to $19a - 87k = 469$ is $(a, k) = (-15008, -3283)$.

The general solution is $a = -15008 + 87s$, $k = -3283 + 19s$.

$x = 162 + 29k = 162+ 29(-3283 + 19s) = 551s - 95045$

$95045 = 551\times172 + 273$

Let $s = 173 + t$

$x = 551(173 + t) - 95045 = 278 + 551t$

Sub. $x = 278 + 551t$ into (3): $7(278 + 551t) - 2 = 11a$

$11a - 3857t = 1944$

We solve $11a - 3857t = 1$

$3857 = 11\times350 + 7$

$11 = 7 + 4$

$7 = 4 + 3$

$4 = 3 + 1$

$1 = 4 - 3$

$3 = 7 - 4$

$4 = 11 - 7$

$7 = 3857 - 11\times350$

$1 = 4 - 3 = 4 - (7 - 4) = 4\times2 - 7$

$1 = (11 - 7)\times2 - 7 = 11\times2 - 7\times3$

$1 = 11\times2 - 7\times3 = 11\times2 - (3857 - 11\times350)\times3 = 11\times1052 - 3857\times3$

Multiply the whole equation by 1944: $11\times2045088 - 3857\times5832$

A particular solution to $11a - 3857t = 1944$ is $(a, t) = (2045088, 5832)$.

The general solution is $(a, t) = (2045088 + 3857m, 5832 + 11m)$.

$x = 278 + 551t = 278 + 551(5832 + 11m) = 6061m + 3213710$

$3213710 = 530\times6061 + 1380$

The largest possible 4-digit integer satisfying the condition is $1380 + 6061 = 7441$.