Prove that $\sqrt{2}$ is irrational

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Suppose on the contrary that $\sqrt{2}$ is a rational number.

$\sqrt{2} = \frac{m}{n}$; where m, n are integers, $n \neq 0$ and m, n have no common factors.

$n\sqrt{2} = m$

$2n^2 = m^2 \cdots\cdots (*)$

$\because$ LHS is an even integer

$\therefore$ RHS is also an even integer

If m is odd, then $m = 2k + 1$, where k is an integer

RHS $= m^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$, which is an odd integer, impossible.

$\therefore$ m must not be an odd integer.

m is an even integer.

Let $m = 2p$, where p is an integer.

Sub. $m = 2p$ into (*): $2n^2 = (2p)^2$

$2n^2 = 4p^2$

$n^2 = 2p^2$

$\because$ RHS is an even integer

$\therefore$ LHS is also an even integer

If n is odd, then $n = 2a + 1$, where a is an integer

LHS $= n^2 = (2a + 1)^2 = 4a^2 + 4a + 1 = 2(2a^2 + 2a) + 1$, which is an odd integer, impossible.

$\therefore$ n must not be an odd integer.

n is an even integer.

Let $n = 2q$, where q is an integer.

$\because$ $m = 2p$ and $n = 2q$

$\therefore$ m and n have a common factor 2.

This contradicts to the fact that m and n have no common factors.

$\therefore$ Our assumption is wrong.

$\therefore$ $\sqrt{2}$ is an irrational number.