Fibonacci Sequence and Difference Equations (Recurrence relations)

Created by Mr. Francis Hung on 20081104 Last updated: 2022-09-03

1. Fibonacci Sequence

Define the sequence: $F_1 = F_2 = 1$, when $n \ge 1$, $F_{n+2} = F_{n+1} + F_n$

Let $\alpha > \beta$ be the roots of $x^2 - x - 1 = 0$, then $\alpha + \beta = 1$, $\alpha\beta = -1$

$F_{n+2} = F_{n+1} + F_n \Rightarrow F_{n+2} - (\alpha + \beta)F_{n+1} + \alpha\beta F_n = 0$

$\Rightarrow F_{n+2} - \alpha F_{n+1} = \beta(F_{n+1} - \alpha F_n)$ ………(*)

Let $V_n = F_{n+1} - \alpha F_n$, for $n = 1, 2, \ldots, n - 1$

By (*) $V_{n+1} = \beta V_n$

$n = 1$, $V_2 = \beta V_1$

$n = 2$, $V_3 = \beta V_2 = \beta^2 V_1$

…………………….....

$n \to n - 1$, $V_n = \beta V_{n-1} = \ldots = \beta^{n-1} V_1$

$F_{n+1} - \alpha F_n = \beta^{n-1} V_1$ …….(1)

On the other hand (*) may be rewritten as $F_{n+2} - \beta F_{n+1} = \alpha(F_{n+1} - \beta F_n)$ ………(**)

Let $W_n = F_{n+1} - \beta F_n$, for $n = 1, 2, \ldots, n - 1$

By (**) $W_{n+1} = \alpha W_n$

$n = 1$, $W_2 = \alpha W_1$

$n = 2$, $W_3 = \alpha W_2 = \alpha^2 W_1$

……………………........

$n \to n - 1$, $W_n = \alpha W_{n-1} = \ldots = \alpha^{n-1} W_1$

$F_{n+1} - \beta F_n = \alpha^{n-1} W_1$ …….(2)

(2) – (1) $(\alpha - \beta)F_n = \alpha^{n-1}(F_2 - \beta F_1) - \beta^{n-1}(F_2 - \alpha F_1)$

$$F_n = \frac{\alpha^{n-1}(1-\beta) - \beta^{n-1}(1-\alpha)}{\alpha - \beta}$$

$$= \frac{\alpha^{n-1} - \beta^{n-1} - \alpha\beta(\alpha^{n-2} - \beta^{n-2})}{\alpha - \beta}$$

$$= \frac{\alpha^{n-1} - \beta^{n-1} + (\alpha^{n-2} - \beta^{n-2})}{\alpha - \beta} \quad (\because \alpha\beta = -1)$$

$$= \frac{\alpha^{n-1} + \alpha^{n-2} - (\beta^{n-1} + \beta^{n-2})}{\sqrt{5}} \quad \left(\because \alpha - \beta = \sqrt{(\alpha-\beta)^2 - 4\alpha\beta} = \sqrt{5}\right)$$

$$= \frac{\alpha^{n-2}(1+\alpha) - \beta^{n-2}(1+\beta)}{\sqrt{5}}$$

$$= \frac{\alpha^{n-2}(\alpha^2) - \beta^{n-2}(\beta^2)}{\sqrt{5}} \quad (\because \alpha > \beta \text{ are roots of } x^2 - x - 1 = 0,\ \alpha^2 = \alpha + 1,\ \beta^2 = \beta + 1)$$

$$= \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]$$

Define $R_n = \dfrac{F_n}{F_{n+1}} = \dfrac{\frac{1}{\sqrt{5}}\left(\alpha^n - \beta^n\right)}{\frac{1}{\sqrt{5}}\left(\alpha^{n+1} - \beta^{n+1}\right)}$

$$= \frac{\alpha^n - \beta^n}{\alpha^{n+1} - \beta^{n+1}}$$

$$= \frac{\alpha^n\left[1 - \left(\frac{\beta}{\alpha}\right)^n\right]}{\alpha^{n+1}\left[1 - \left(\frac{\beta}{\alpha}\right)^{n+1}\right]} = \frac{\left[1 - \left(\frac{\beta}{\alpha}\right)^n\right]}{\alpha\left[1 - \left(\frac{\beta}{\alpha}\right)^{n+1}\right]}$$

$$\lim_{n\to\infty} R_n = \lim_{n\to\infty} \frac{\left[1 - \left(\frac{\beta}{\alpha}\right)^n\right]}{\alpha\left[1 - \left(\frac{\beta}{\alpha}\right)^{n+1}\right]} = \frac{1}{\alpha} \quad \left(\because -1 < \frac{\beta}{\alpha} < 1, \lim_{n\to\infty}\left(\frac{\beta}{\alpha}\right)^n = 0\right)$$

$$= \frac{2}{1+\sqrt{5}} = \frac{\sqrt{5}-1}{2}$$

The Fibonacci sequence is defined inductively as follows:

$a_1 = a_2 = 1$, $a_n = a_{n-1} + a_{n-2}$ for $n > 2$.

(a) Show that $\begin{pmatrix} a_{n+2} \\ a_{n+1} \end{pmatrix} = A\begin{pmatrix} a_{n+1} \\ a_n \end{pmatrix}$, where $A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$.

Hence prove that for all positive integral values of n, $\begin{pmatrix} a_{n+2} \\ a_{n+1} \end{pmatrix} = A^n\begin{pmatrix} a_2 \\ a_1 \end{pmatrix}$.

(b) Show that A satisfies the matrix equation $A^2 - A - I = \mathbf{0}$.

(c) Suppose when x^n is divided by $x^2 - x - 1$, the remainder is $r_1x + r_2$, where r_1, r_2 are real numbers,

show that: $r_1 = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]$, $r_2 = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{n-1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n-1}\right]$.

(d) Show that $A^n = r_1A + r_2I$. (where I is the identity matrix.)

Deduce that $a_{n+1} = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}\right]$.

(e) Show that $A^n = \begin{pmatrix} a_{n+1} & a_n \\ a_n & a_{n-1} \end{pmatrix}$ for $n > 1$ and hence deduce that $a_{n+1}a_{n-1} - a_n^2 = (-1)^n$.

(a) $\begin{cases} a_{n+2} = a_{n+1} + a_n \\ a_{n+1} = a_{n+1} \end{cases} \Rightarrow \begin{pmatrix} a_{n+2} \\ a_{n+1} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a_{n+1} \\ a_n \end{pmatrix}$

To prove $\begin{pmatrix} a_{n+2} \\ a_{n+1} \end{pmatrix} = A^n\begin{pmatrix} a_2 \\ a_1 \end{pmatrix}$, induction on n.

$n = 1$, done above.

Suppose $\begin{pmatrix} a_{k+2} \\ a_{k+1} \end{pmatrix} = A^k\begin{pmatrix} a_2 \\ a_1 \end{pmatrix}$.

$\begin{pmatrix} a_{k+3} \\ a_{k+2} \end{pmatrix} = A\begin{pmatrix} a_{k+2} \\ a_{k+1} \end{pmatrix} = A^{k+1}\begin{pmatrix} a_2 \\ a_1 \end{pmatrix}$. By MI, it is true for all positive integer n.

(b) $A^2 - A - I = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$= \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \mathbf{0}$

(c) Let $x^n = (x^2 - x - 1)Q(x) + r_1x + r_2$

$= (x - \frac{1+\sqrt{5}}{2})(x - \frac{1-\sqrt{5}}{2})\, Q(x) + r_1x + r_2$

Put $x = \frac{1+\sqrt{5}}{2} \Rightarrow \left(\frac{1+\sqrt{5}}{2}\right)^n = r_1\frac{1+\sqrt{5}}{2} + r_2$ …… (1)

Put $x = \frac{1-\sqrt{5}}{2} \Rightarrow \left(\frac{1-\sqrt{5}}{2}\right)^n = r_1\frac{1-\sqrt{5}}{2} + r_2$ …… (2)

(1) – (2) $\sqrt{5}\ r_1 = \left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n$

$r_1 = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]$

$\frac{(1)}{\left(\frac{1+\sqrt{5}}{2}\right)} - \frac{(2)}{\left(\frac{1-\sqrt{5}}{2}\right)}$ $r_2\left(\frac{1}{\frac{1+\sqrt{5}}{2}} - \frac{1}{\frac{1-\sqrt{5}}{2}}\right) = \left(\frac{1+\sqrt{5}}{2}\right)^{n-1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n-1}$

$\frac{2r_2}{1-5}\left(1-\sqrt{5}-1-\sqrt{5}\right) = \left(\frac{1+\sqrt{5}}{2}\right)^{n-1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n-1}$

$r_2 = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{n-1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n-1}\right]$

(d) $A^n = (A^2 - A - I)Q(A) + r_1A + r_2I = r_1A + r_2I$

$A^n\begin{pmatrix} a_2 \\ a_1 \end{pmatrix} = r_1A\begin{pmatrix} a_2 \\ a_1 \end{pmatrix} + r_2\begin{pmatrix} a_2 \\ a_1 \end{pmatrix}$

$\begin{pmatrix} a_{n+2} \\ a_{n+1} \end{pmatrix} = r_1\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a_2 \\ a_1 \end{pmatrix} + r_2\begin{pmatrix} a_2 \\ a_1 \end{pmatrix}$

$\Rightarrow a_{n+1} = r_1a_2 + r_2\, a_1 = r_1 + r_2$

$a_{n+1} = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right] + \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{n-1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n-1}\right]$

$a_{n+1} = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{n-1}\left(\frac{3+\sqrt{5}}{2}\right) - \left(\frac{1-\sqrt{5}}{2}\right)^{n-1}\left(\frac{3-\sqrt{5}}{2}\right)\right]$

$= \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}\right]$

(*e*) Induction on n. $n = 2$, $A^2 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^2 = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} a_3 & a_2 \\ a_2 & a_1 \end{pmatrix}$

$n = 3$, $A^3 = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} a_4 & a_3 \\ a_3 & a_2 \end{pmatrix}$

Suppose $A^k = \begin{pmatrix} a_{k+1} & a_k \\ a_k & a_{k-1} \end{pmatrix}$ and $A^{k+1} = \begin{pmatrix} a_{k+2} & a_{k+1} \\ a_{k+1} & a_k \end{pmatrix}$ for $k > 1$.

By (*b*), $A^2 - A - I = \mathbf{0}$

$\therefore A^{k+2} - A^{k+1} - A^k = \mathbf{0} \Rightarrow A^{k+2} = A^{k+1} + A^k = \begin{pmatrix} a_{k+2} & a_{k+1} \\ a_{k+1} & a_k \end{pmatrix} + \begin{pmatrix} a_{k+1} & a_k \\ a_k & a_{k-1} \end{pmatrix} = \begin{pmatrix} a_{k+3} & a_{k+2} \\ a_{k+2} & a_{k+1} \end{pmatrix}$

By induction, the result is true for all $n > 1$.

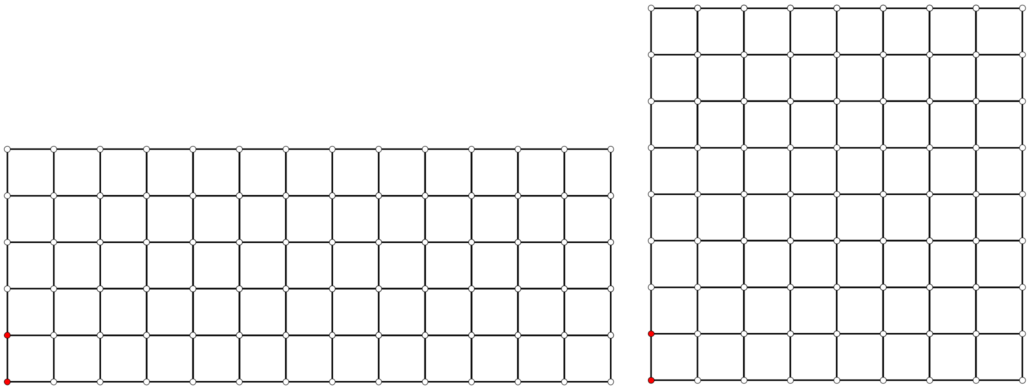
$\det(A^n) = \det(A)^n$

$\begin{vmatrix} a_{n+1} & a_n \\ a_n & a_{n-1} \end{vmatrix} = (-1)^n$

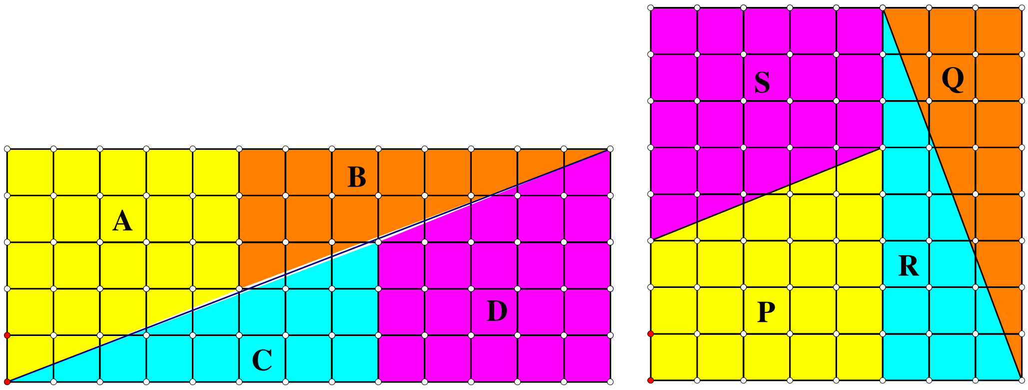
$a_{n+1}a_{n-1} - a_n^2 = (-1)^n$

In particular, put $n = 6$, $a_7a_5 - a_6^2 = 13\times5 - 8^2 = (-1)^6 = 1$.

We can draw two rectangles whose dimensions are 13×5 and 8×8 respectively so that:



If we dissect the two rectangles into small parts A, B, C, D and P, Q, R, S as shown:



It seems that $A = P$, $B = Q$, $C = R$, $D = S$ and the areas of the two rectangles appeared to be equal.

65 = 64?

Can you see that there is a white gap between the regions A, B C and D?

Exercise

Let $a_1 = 3$, $a_2 = 2$, $a_n = 2a_{n-1} - a_{n-2}$ for $n > 2$.

Express a_n in terms of n only.

Let $A = \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix}$. To prove $\begin{pmatrix} a_{n+2} \\ a_{n+1} \end{pmatrix} = A^n \begin{pmatrix} a_2 \\ a_1 \end{pmatrix}$, induction on n.

$n = 1$, $\begin{pmatrix} a_3 \\ a_2 \end{pmatrix} = \begin{pmatrix} 2a_2 - a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a_2 \\ a_1 \end{pmatrix}$

Suppose $\begin{pmatrix} a_{k+2} \\ a_{k+1} \end{pmatrix} = A^k \begin{pmatrix} a_2 \\ a_1 \end{pmatrix}$.

$\begin{pmatrix} a_{k+3} \\ a_{k+2} \end{pmatrix} = \begin{pmatrix} 2a_{k+2} - a_{k+1} \\ a_{k+2} \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a_{k+2} \\ a_{k+1} \end{pmatrix} = A \cdot A^k \begin{pmatrix} a_2 \\ a_1 \end{pmatrix} = A^{k+1} \begin{pmatrix} a_2 \\ a_1 \end{pmatrix}$.

By MI, it is true for all positive integer n.

Next, A satisfies the matrix equation: $A^2 - 2A + I = \mathbf{0}$

L.H.S. $= A^2 - 2A + I = \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix} - 2\begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 3 & -2 \\ 2 & -1 \end{pmatrix} - \begin{pmatrix} 4 & -2 \\ 2 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbf{0}$

Thirdly, we find the remainder when x^n is divided by $x^2 - 2x + 1$ for $n > 2$.

Let $x^n = (x^2 - 2x + 1)Q(x) + r_1x + r_2$

Put $x = 1$: $1 = r_1 + r_2$ ⋯⋯ (1)

Differentiate w.r.t. x: $nx^{n-1} = (x - 1)^2Q'(x) + 2(x - 1)Q(x) + r_1$

Put $x = 1$: $n = r_1$ ⋯⋯ (2)

Sub. (2) into (1): $r_2 = 1 - n$

$\therefore$ The remainder is $nx + 1 - n$

$A^n = (A^2 - 2A + I)Q(A) + nA + (1 - n)I$

$A^n \begin{pmatrix} a_2 \\ a_1 \end{pmatrix} = r_1A \begin{pmatrix} a_2 \\ a_1 \end{pmatrix} + r_2 \begin{pmatrix} a_2 \\ a_1 \end{pmatrix}$

$\begin{pmatrix} a_{n+2} \\ a_{n+1} \end{pmatrix} = n\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a_2 \\ a_1 \end{pmatrix} + (1 - n)\begin{pmatrix} a_2 \\ a_1 \end{pmatrix}$

$a_{n+1} = na_2 + (1 - n)a_1$

$\Rightarrow a_n = (n - 1)\cdot 2 + (1 - n + 1)\cdot 3 = 4 - n$

Example Given a_1 and a_2, $a_{n+2} = a_n + a_{n+1}$ for $n \geq 1$.
Prove that $a_1 + a_2 + a_3 + \cdots + a_n = a_{n+2} - a_2$ for $n \geq 1$
Proof: Induction on n.

$n = 1$, L.H.S. $= a_1 = (a_1 + a_2) - a_2 = a_3 - a_2 =$ R.H.S.

$\therefore$ It is true for $n = 1$.

Suppose $a_1 + a_2 + a_3 + \cdots + a_k = a_{k+2} - a_2$ for some $k \geq 1$

$a_1 + a_2 + a_3 + \cdots + a_k + a_{k+1} = (a_{k+1} + a_{k+2}) - a_2$ (by induction assumption)

$= a_{k+3} - a_2$

If it is true for $n = k$, then it is also true for $n = k + 1$.

By the principal of mathematical induction,

$a_1 + a_2 + a_3 + \cdots + a_n = a_{n+2} - a_2$ for all integers $n \geq 1$

In particular, evaluate the following expression:
4 + 5 + 9 + 14 + 23 + 37 + 60 + 97 + 157 + 254 + 411 + 665 + 1076 + 1741 + 2817 + 4558.

$a_1 = 4$, $a_2 = 5$, $a_3 = 9 = 4 + 5 = a_1 + a_2$, $a_4 = 14 = 5 + 9 = a_2 + a_3$, $\cdots$, $a_{n+2} = a_n + a_{n+1}$ for $n \geq 1$
$a_{16} = 4558$
$a_{17} = 2817 + 4558 = 7375$
$a_{18} = 4558 + 7375 = 11933$
$a_1 + a_2 + a_3 + \cdots + a_{16} = a_{18} - a_2 = 11933 - 5 = 11928$

2. Difference Equation (Recurrence relation)

2.1 Types of difference equations.

The general form of difference equation is $a_{r,n}\, u_{n+r} + a_{r,n-1}\, u_{n+r-1} + \dots + a_{r,1}\, u_{r+1} + a_{r,0}\, u_r = b_r$.

It is called a linear difference equation of order n ($a_{r,n} \neq 0$, $a_{r,0} \neq 0$).

e.g.1 $u_{r+2} - u_{r+1} + u_r = r$ is a linear difference equation of **order 2**.

Some example of difference equation which is **non-linear**.

e.g.2 $u_r\, u_{r-1} - a u_r - b u_{r-1} + c = 0$

If $b_r = 0$, the equation is said to be **homogeneous**.

e.g.3 $u_r - 2r\, u_{r-1} + k^2\, u_{r-2} = 0$. This is a linear homogeneous difference equation of order 2.

(e.g.1 is **non-homogeneous**.)

If $a_{r,n}$, $a_{r,n-1}, \dots$, $a_{r,1}$, $a_{r,0}$ are constants and independent of r, the equation is called a linear difference equation with **constant coefficients** of order n.

e.g.4 $u_r - A\, u_{r-1} + B = 0$.

This is a linear non-homogeneous difference equation with constant coefficients of order 1.

e.g.5 $u_r = 3\, u_{r-1} - 2\, u_{r-2}$

This is a linear homogeneous difference equation with constant coefficients of order 2.

2.2 First order linear homogeneous difference equations.

$u_{n+1} - a\, u_n = 0$

$u_n = a\, u_{n-1} = a^2\, u_{n-2} = a^3\, u_{n-3} = \dots\dots = a^{n-1}\, u_1$

Let $u_1 = Aa$, where A is a constant and independent of n.

$\therefore\ u_n = A\, a^n$

2.3 First order linear non-homogeneous difference equation.

$u_{n+1} - a\, u_n = b_n$

$u_n = a\, u_{n-1} + b_{n-1}$

$= a\,(a\, u_{n-2} + b_{n-2}) + b_{n-1}$

$= \dots\dots\dots\dots\dots\dots\dots\dots$

$= a^{n-1}\, u_1 + b_{n-1} + a\, b_{n-2} + \dots\dots + a^{n-2}\, b_1$

$= A\, a^n + b_{n-1} + a\, b_{n-2} + \dots\dots + a^{n-2}\, b_1$

In particular, if b_n is independent of n, let $b_n = b$.

$$u_n = \begin{cases} Aa^n + b \cdot \dfrac{a^{n-1} - 1}{a - 1} & \text{, if } a \neq 1 \\ A + (n-1)b & \text{, if } a = 1 \end{cases}$$

$$u_n = \begin{cases} A_1 a^n + B & \text{, if } a \neq 1 \\ A_1 + Bn & \text{, if } a = 1 \end{cases}.$$

2.4 Second order linear homogeneous difference equation $u_{n+2} - a\, u_{n+1} + b\, u_n = 0$

Define the **characteristic equation** to be $t^2 - at + b = 0$

Let α, β be the roots of the characteristics equation.

$\therefore u_{n+2} - (\alpha + \beta)\, u_{n+1} + \alpha\beta\, u_n = 0$ ....(*)

$(u_{n+2} - \alpha\, u_{n+1}) + \beta\, (u_{n+1} - \alpha\, u_n) = 0$ ....(1)

Let $v_n = u_{n+1} - \alpha\, u_n$

then $v_{n+1} = u_{n+2} - \alpha\, u_{n+1}$

$\therefore$ (1) becomes $v_{n+1} - \beta\, v_n = 0$, which is of type in section 2.

By the result in section 2, $v_n = A\, \beta^n$ .... (2)

(*) can also be written as $(u_{n+2} - \beta\, u_{n+1}) + \alpha\, (u_{n+1} - \beta\, u_n) = 0$ .... (3)

Let $w_n = u_{n+1} - \beta\, u_n$

(3) becomes $w_{n+1} - \alpha\, w_n = 0$

By the result of section 2, $w_n = B\, \alpha^n$ .... (4)

Rewrite (2) and (4): $\begin{cases} u_{n+1} - \alpha u_n = A\beta^n \\ u_{n+1} - \beta u_n = B\alpha^n \end{cases}$

Case 1: $\alpha \neq \beta$, solving (2) and (4), $u_n = A_1\alpha^n + B_1\beta^n$ for some constants A_1, B_1.

Case 2: $\alpha = \beta$, (2) becomes $u_{n+1} - \alpha\, u_n = A\, \alpha^n$, which is of type in section (3)

$u_n = A'\, \alpha^n + A(\alpha^{n-1} + \alpha\cdot\alpha^{n-2} + \ldots\ldots + \alpha^{n-2}\cdot\alpha)$

$= (A_1 n + B_1)\, \alpha^n$ for some constant A_1, B_1.

2.5 m^{th} order linear homogeneous difference equation with constant coefficients.

$a_m\, u_{m+r} + a_{m-1}\, u_{m+r-1} + \ldots + a_1\, u_{r+1} + a_0\, u_r = 0$

The characteristic equation is $a_m\, t^m + a_{m-1}t^{m-1} + \ldots + a_1\, t + a_0 = 0$ ...... (*), where $a_m \neq 0$, $a_0 \neq 0$

Suppose $\alpha_1, \alpha_2, \ldots\ldots, \alpha_m$ are the roots of (*)

Case 1: All roots are distinct, then $u_n = A_1\alpha_1{}^n + A_2\alpha_2{}^n + \ldots\ldots + A_m\alpha_m{}^n$

Case 2: $\alpha_1, \alpha_2, \ldots\ldots, \alpha_k$ are distinct ($k < m$)

β_1 of multiplicity $\ell_1 > 1$, β_2 of multiplicity $\ell_2 > 1$, ... , β_j of multiplicity $\ell_j > 1$

$k + \ell_1 + \ell_2 + \ldots\ldots + \ell_j = m$, then

$$u_n = A_1\alpha_1{}^n + A_2\alpha_2{}^n + \ldots\ldots + A_k\alpha_k{}^n$$
$$+\left(B_{1,\ell_1-1}n^{\ell_1-1} + \cdots + B_{1,1}n + B_{1,0}\right)\beta_1^n + \cdots + \left(B_{j,\ell_j-1}n^{\ell_j-1} + \cdots + B_{j,1}n + B_{j,0}\right)\beta_j^n$$

where all letters except $\alpha_1, \alpha_2, \ldots\ldots, \alpha_k, \beta_1, \beta_2, \ldots\ldots, \beta_j$ are constant.

The proof are outside the scope of text.

2.6 The solution for non-homogeneous difference equation are much difficult and so is omitted. We give an example to solve a non-linear one.

Advanced Level Pure Mathematics by S.L. Green p.408 Q16

(Reference: Techniques of Mathematics Analysis by C.J. Tranter p.33 ex.12)

Show that if $u_n u_{n+1} - a u_{n+1} - b u_n + c = 0$, for all positive integral values of n, the constants a, b, c being independent of n, then

$$v_{n+2} + (a - b) v_{n+1} + (c - ab) v_n = 0, \text{ where } u_n = \frac{v_{n+1}}{v_n} + a.$$

Hence find u_n if $a = 4$, $b = 1$, $c = 6$, $u_1 = 1$. Show that $u_n \to 2$ as $n \to \infty$.

Solution $u_n = \frac{v_{n+1}}{v_n} + a$. The difference equation becomes

$$\left(\frac{v_{n+1}}{v_n} + a\right)\left(\frac{v_{n+2}}{v_{n+1}} + a\right) - a\left(\frac{v_{n+2}}{v_{n+1}} + a\right) - b\left(\frac{v_{n+1}}{v_n} + a\right) + c = 0$$

$$\frac{v_{n+2}}{v_n} + a\frac{v_{n+2}}{v_{n+1}} + a\frac{v_{n+1}}{v_n} + a^2 - a\frac{v_{n+2}}{v_{n+1}} - a^2 - b\frac{v_{n+1}}{v_n} - ab + c = 0$$

$$\frac{v_{n+2}}{v_n} + a\frac{v_{n+1}}{v_n} - b\frac{v_{n+1}}{v_n} - ab + c = 0$$

$v_{n+2} + (a - b) v_{n+1} + (c - ab) v_n = 0$ Q.E.D.

$a = 4$, $b = 1$, $c = 6$

$v_{n+2} + 3 v_{n+1} + 2 v_n = 0$

Characteristic equation $(t + 2)(t + 1) = 0$

$v_n = A(-1)^n + B(-2)^n$

$$u_n = \frac{A(-1)^{n+1} + B(-2)^{n+1}}{A(-1)^n + B(-2)^n} + 4$$

$$u_n = 3 - \frac{B(-2)^n}{A(-1)^n + B(-2)^n}$$

$$= 3 - \frac{1}{1 + C\cdot 2^{-n}}, \text{ where } C \text{ is a constant.}$$

$$u_1 = 1 \Rightarrow 1 = 3 - \frac{1}{1 + C/2}$$

$$\frac{1}{2} = 1 + \frac{C}{2}$$

$C = -1$

$$u_n = 3 - \frac{1}{1 - 2^{-n}} = 2 - \frac{1}{2^n - 1}$$

as $n \to \infty$, $u_n \to 3 - 1 = 2$

Exercise

1. Find the general solution of $U_{n+2} - 3U_{n+1} + 2U_n = 0$.
2. Find the general solution of $U_{n+3} + 3U_{n+2} + 3U_{n+1} + 2U_n = 0$.
(This is a third order difference equation where all the roots of the characteristic equation are different.)
3. Find the general solution of $V_{n+1} - V_n = 5$.
4. Find the general solution of $W_{n+1} - 2W_n = 5$.
5. Using Q.1, Q.3 and Q.4, solve $U_{n+2} - 3U_{n+1} + 2U_n = 5$.
(This is second order non-homogeneous difference equation.)
6. Find the general solution of $4U_{n+2} + 4U_{n+1} + U_n = 0$
(In this case, the roots of the characteristic equation are equal.)
7. Using the method in Q.5, solve $4U_{n+2} + 4U_{n+1} + U_n = -1$.
(Can you solve $U_{n+2} + bU_{n+1} + cU_n = a$? Just memorizes the formulae you have derived, you need not to do it all again.)
8. Solve $U_{n+2} - 7U_{n+1} + 12U_n = 2n$.
(This is a second order difference equation, but the coefficients are not similar, use a method similar to Q.5)
9. Prove that the general solution of $U_{n+2} + U_{n+1} + U_n = 0$ is
$U_n = A\cos 120n° + B\sin 120n°$, where A and B may be complex.
(In this case, the roots are complex, use De-Movrie's Theorem)
10. Solve $U_{n+2} + U_n = n^2 + n$
(This is a second order non-homogeneous difference equation, and the roots of the characteristic equation are complex.)
11. Solve $U_{n+3} + U_{n+2} - U_{n+1} - U_n = 0$
(2 equal roots.)
12. Do Q.2 again, with the aids of Q.9.
(3 different roots.)
13. Solve $U_{n+3} - 6U_{n+2} + 12U_{n+1} - 8U_n = 0$
(3 equal roots, now can you guess a formula for an n^{th} order linear homogeneous difference equation?)

The above questions have no initial conditions. Now we come with initial conditions and the recurring series. The solutions are called particular solutions.

14. Solve $U_{n+3} - 3U_{n+2} + 2U_{n+1} = 0$, $U_1 = 3$, $U_2 = 4$.
(This is a 'second' order difference equation.)
15. Solve $U_{n+2} - 4U_{n+1} - 5U_n = 2n$, $U_1 = 0$, $U_2 = 1$.
(Complex roots, non-homogeneous.)
16. Solve $U_{n+2} - U_n = 0$, $U_1 = 1$, $U_2 = 0$.
17. Solve $U_{n+2} - 6U_{n+1} + 9U_n = 1$, $U_1 = 0$, $U_2 = 0$

18. Suppose U_n satisfies the second order recurrence relation and is the coefficient of x_n in the series: $1 + 5x + 9x^2 + 13x^3 + \ldots$

(*a*) Find the recurrence relation $U_{n+2} + a\,U_{n+1} + b\,U_n = 0$.

(*b*) Find the particular solution U_n.

(*c*) Prove that the series converges to $\dfrac{A+Bx}{1+ax+bx^2}$.

Find the restriction on x.

19. Do Q.18 again with the series: $1 + x + 2x^2 + 3x^3 + \ldots$

20. Advanced Level Pure Mathematics by S.L. Green p.408 Q14(*i*)

Find the n^{th} term of the recurring series, and the sum to infinity of the series:

$1 + 5x + 9x^2 + 13x^2 + \ldots\ldots..$

21. Advanced Level Pure Mathematics by S.L. Green p.408 Q15

The sequence u_1, u_2, u_3, ... is such that each term after the second is the sum of the two preceding terms. Find u_n, given $u_1 = 1$, $u_2 = -1$.

22. Advanced Level Pure Mathematics by S.L. Green p.408 Q17

In the series $u_1 + u_2 + u_3 + \ldots.. + u_n + \ldots\ldots$, $\quad u_n = x\,u_{n-1} + 2x^2\,u_{n-2}$

If $u_1 = 4$, $u_2 = 5x$, find the general term u_n and the sum of the first n terms of the series.

23. Techniques of Mathematics Analysis by C.J. Tranter p.35 Q1

Find the general solution of the difference equation $u_{r+2} + 3\,u_{r+1} - 4\,u_r = 0$, $r \ge 0$.

Find also the values of the constants in the general solution if $u_0 = 21$ and $u_1 = 1$.

24. Techniques of Mathematics Analysis by C.J. Tranter p.37 Q17

The numbers x_n satisfy the recurrence formula $x_{n+3} = \frac{1}{3}\left(x_n + x_{n+1} + x_{n+2}\right)$.

Prove that if $y_n = x_n + 2x_{n+1} + 3x_{n+2}$ then $y_{n+1} = y_n = y_1 = 6x$ (say).

Prove also that if $z_n = (x_n - x)^2 + 2(x_n - x)(x_{n+1} - x) + 3(x_{n+1} - x)^2$, then $z_{n+1} = \frac{1}{3}z_n = 3^{-n}\,z_1$.

25. Algebra by J.W. Archbold p.41 Q11

Show that $\displaystyle\sum_{r=0}^{n} r(r+1)\cdots(r-k+1) = \frac{n(n+1)\cdots(n+k)}{k+1}$.

Deduce that, if $a_{r+1} = \dfrac{a_r}{1+ra_r}$, then

$$\sum_{r=0}^{n}\frac{1}{a_r a_{r+2}} = \frac{n+1}{a_0^2} + \frac{n+1}{3a_0}\left(n^2+2n+3\right) + \frac{1}{20}(n-1)(n)(n+1)(n+2)(n+3).$$

26. Algebra by J.W. Archbold p.41 Q12

If $a_{r+1} = \dfrac{a_r}{a_r+1}$, then $\displaystyle\sum_{r=1}^{n}\frac{1}{a_r^{\,2}} = \frac{n}{a_0^2} + \frac{n(n+1)}{a_0} + \frac{n(n+1)(2n+1)}{6}$.

27. Algebra by J.W. Archbold p.41 Q13

If $f(r) = \frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{r}$, where r is a positive integer, prove that

$$\sum_{r=1}^{n}(2r+1)f(r)=(n+1)^2 f(n)-\frac{1}{2}n(n+1).$$

28. Algebra by J.W. Archbold p.41 Q14.

The Fibonacci sequence $u_1, \ldots, u_n, \ldots$ is defined by $u_1 = 1$, $u_2 = 2$ and $u_n = u_{n-1} + u_{n-2}$ for $n > 2$

Prove that (*i*) $u_n = \frac{\left(1+\sqrt{5}\right)^{n+1}-\left(1-\sqrt{5}\right)^{n+1}}{2^{n+1}\sqrt{5}}$,

(*ii*) $u_1 + \ldots + u_n = u_{n+2} - 2$,

(*iii*) $u_1 + \ldots + u_{2n-1} = u_{2n} - 1$,

(*iv*) $u_2 + u_4 + \ldots + u_{2n} = u_{2n+1} - 1$,

(*v*) $u_{2n} = u_n^2 + u_{n-1}^2$,

(*vi*) $u_{2n+1} = u_{n+1}^2 - u_{n-1}^2$.

Supplementary exercises: Techniques of Mathematics Analysis by C.J. Tranter p.37 Q13 – Q18

End of Exercise

Answers

1. $U_n = A + B\,2^n$

2. $U_n = A(-2)^n + B\frac{\left(-1+\sqrt{3}i\right)^n}{2^n} + C\frac{\left(-1-\sqrt{3}i\right)^n}{2^n}$

3. $V_n = A + 5n$

4. $W_n = B\,2^n - 5$

5. $U_n = A + B\,2^n - 5n$

6. $U_n = \left(-\frac{1}{2}\right)^n (A+Bn)$

7. $U_n = \left(-\frac{1}{2}\right)^n (A+Bn)-\frac{1}{9}$

General solution of $U_{n+2} + b\,U_{n+1} + c\,U_n = a$:

Let r and s be the roots of the characteristic equation.

Case 1 $r \neq s$, $r \neq 1$, $s \neq 1$

$U_n = A\,r^n + B\,s^n + \frac{a}{1+b+c}$

Case 2 $r \neq s$, $r \neq 1$, $s = 1$ then $r = c$

$U_n = A + B\,c^n + \dfrac{an}{1-c}$

Case 3 $r = s \neq 1$

$U_n = r^n(A + B\,n) + \dfrac{a}{1+b+c}$

Case 4 $r = s = 1$

$U_n = A + B\,n + \dfrac{an^2}{2}$

Note that in all cases, all the solutions have the general solution of the homogeneous equation. So you need to just memories the non-homogeneous part.

8. $U_n = A\,4^n + B\,3^n + \dfrac{n}{3} + \dfrac{5}{18}$

General solution of $U_{n+2} + b\,U_{n+1} + c\,U_n = 2\,n$.
Let r and s be the roots of the characteristic equation.
Suppose $r \neq s$, $r \neq 1$, $s \neq 1$

$U_n = A\,r^n + B\,s^n + \dfrac{2n}{1+b+c} - \dfrac{4+2b}{1+b+c}$

10. $U_n = A\cos 90n° + B\sin 90n° + \dfrac{n^2-n-1}{2}$

11. $U_n = A + (-1)^n(B + Cn)$

12. $U_n = A(-2)^n + B\cos 120n° + C\sin 120n°$

13. $U_n = 2^n(A + B\,n + C\,n^2)$

14. $U_n = 2 + 2^{n-1}$

15. Using the result of Q.8, $r = -1$, $s = 5$, then $U_n = \dfrac{(-1)^n}{12} + \dfrac{13\times 5^n}{240} - \dfrac{n}{4} + \dfrac{1}{16}$

16. $U_n = \dfrac{1}{2} - \dfrac{(-1)^n}{2}$

17. $U_n = 3^n\left(-\dfrac{5}{36} + \dfrac{n}{18}\right) + \dfrac{1}{4}$

18. (*a*) $U_{n+2} - 2\,U_{n+1} + U_n = 0$
(*b*) $U_n = -3 + 4n$
(*c*) $\dfrac{1+3x}{1-2x+x^2}$; $|x| < 1$

19. (*a*) $U_{n+2} - U_{n+1} - U_n = 0$
(*b*) $U_n = \dfrac{1}{\sqrt{5}}\left[\left(\dfrac{1+\sqrt{5}}{2}\right)^n - \left(\dfrac{1-\sqrt{5}}{2}\right)^n\right]$
(*c*) $\dfrac{1}{1-x-x^2}$; $|x| < \dfrac{\sqrt{5}-1}{2}$

20. $(4n-3)x^{n-1}$, $\dfrac{1+3x}{(1-x)^2}$, $-1 < x < 1$.

21. $\dfrac{1}{10}\left(5-3\sqrt{5}\right)\left(1+\sqrt{5}\right)^{n-1}\cdot 2^{-n+1} + \dfrac{1}{10}\left(5+3\sqrt{5}\right)\left(1-\sqrt{5}\right)^{n-1}\cdot 2^{-n+1}$.

22. $[3\cdot 2^{n-1} + (-1)^{n-1}]x^{n-1}$; $3[1-(2x)^n](1-2x)^{-1} + [1-(-x)^n](1+x)^{-1}$.

23. $u_r = A + B(-4)^r$, $A = 17$, $B = 4$.